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CONTENTS.

VOL. XX.

PAGE

On the term by term integration of an infinite series over an infinite range and the inversion of the order of integration in repeated infinite integrals. By S. Pollard, M.A., Trinity College, Cambridge. (Communicated by Prof. G. H. Hardy) 1

Note on Mr Hardy's extension of a theorem of Mr Pólya. By Edmund Landau. (Communicated by Prof. G. H. Hardy) 14

Studies on Cellulose Acetate. By H. J. H. Fenton and A. J. Berry . 16

An examination of Searle's method for determining the viscosity of very viscous liquids. By Kurt Molin, Filosofie Licentiat, Physical Institute, Technical College, Trondhjem. (Communicated by Dr G. F. C. Searle.) (Four figs. in Text) 23

Preliminary Note on Antennal Variation in an Aphis (Myzus ribis, *Linn.*). By Maud D. Haviland, Fellow of Newnham College. (Communicated by Mr H. H. Brindley) 35

The effect of a magnetic field on the Intensity of spectrum lines. By H. P. Waran, M.A., Government Scholar of the University of Madras. (Communicated by Professor Sir Ernest Rutherford.) (Plates I and II and one fig. in Text) 45

Further Notes on the Food Plants of the Common Earwig (Forficula auricularia). By H. H. Brindley, M.A., St John's College . . 50

Lagrangian Methods for High Speed Motion. By C. G. Darwin . . 56

A bifilar method of measuring the rigidity of wires. By G. F. C. Searle, Sc.D., F.R.S., University Lecturer in Experimental Physics. (Five figs. in Text) 61

The Rotation of the Non-Spinning Gyrostat. By G. T. Bennett, M.A., F.R.S., Emmanuel College, Cambridge 70

Proof of the equivalence of different mean values. By Alfred Kienast. (Communicated by Professor G. H. Hardy) 74

Notes on the Theory of Vibrations. (1) *Vibrations of Finite Amplitude.* (2) *A Theorem due to Routh.* By W. J. Harrison, M.A., Fellow of Clare College 83

Experiments with a plane diffraction grating. By G. F. C. Searle, Sc.D., F.R.S., University Lecturer in Experimental Physics. (Ten figs. in Text) 88

The Shadow Electroscope. By R. Whiddington, M.A., St John's College. (One fig. in Text) 109

Mathematical Notes. By Professor H. F. Baker and C. V. Hanumanta:

On the Hart circle of a spherical triangle 116

On a property of focal conics and of bicircular quartics 122

On the construction of the ninth point of intersection of two plane cubic curves of which eight points are given 131

PAGE

Mathematical Notes (continued):

On a proof of the theorem of a double six of lines by projection from four dimensions. (Three figs. in Text) 133

On transformations with an absolute quadric 145

On a set of transformations of rectangular axes. (One fig. in Text) . 147

On the generation of sets of four tetrahedra of which any two are mutually inscribed 155

On the reduction of homography to movement in three dimensions. (One fig. in Text) 158

On the transformation of the equations of electrodynamics in the Maxwell and in the Einstein forms 166

On the stability of periodic motions in general dynamics . . . 181

On the stability of rotating liquid ellipsoids 190

On the general theory of the stability of rotating masses of liquid . . 198

Sur le principe de Phragmén-Lindelöf. Par Marcel Riesz. With Note by G. H. Hardy 205

A note on the nature of the carriers of the Anode Rays. By G. P. Thomson, M.A., Fellow of Corpus Christi College 210

Proceedings at the Meetings held during the Session 1919—1920 . . 212

The Problem of Soaring Flight. By E. H. Hankin, M.A., Sc.D., late Fellow of St John's College, Cambridge, Chemical Examiner to Government, Agra, India. (Communicated by Mr H. H. Brindley.) With an Introduction by F. Handley Page, C.B.E., F.R.Aer.S. . 219

Preliminary Note on the Superior Vena Cava of the Cat. By W. F. Lanchester, M.A., King's College, and A. G. Thacker . . . 228

A Note on Vital Staining. By F. A. Potts, M.A., Trinity Hall. (One fig. in Text) 231

Preliminary Note on a Cynipid hyperparasite of Aphides. By Maud D. Haviland, Fellow of Newnham College. (Communicated by Mr H. H. Brindley) 235

A method of testing Triode Vacuum Tubes. By E. V. Appleton, M.A., St John's College. (Two figs. in Text) 239

The Rotation of the Non-Spinning Gyrostat. By Sir George Greenhill and Dr G. T. Bennett 243

On the representation of the simple group of order 660 *as a group of linear substitutions on* 5 *symbols.* By Dr W. Burnside, Honorary Fellow of Pembroke College 247

On the representation of algebraic numbers as a sum of four squares. By L. J. Mordell. (Communicated by Professor H. F. Baker) . . 250

On a Gaussian Series of Six Elements. By L. J. Rogers. (Communicated by Professor G. H. Hardy) 257

Note on Ramanujan's trigonometrical function $c_q(n)$, *and certain series of arithmetical functions.* By Professor G. H. Hardy 263

On the distribution of primes. By H. Cramér, Stockholm. (Communicated by Professor G. H. Hardy) 272

Note on the parity of the number which enumerates the partitions of a number. By Major P. A. MacMahon 281

Note on constant volume explosion experiments. By S. Lees, M.A., St John's College. (Two figs. in Text) 285

PAGE

On the Latent Heats of Vaporisation. By Eric Keightley Rideal, M.A., Trinity Hall 291

On the function $[x]$. By Viggo Brun (Dröbak, Norway). (Communicated by Professor G. H. Hardy) 299

A theorem concerning summable series. By Professor G. H. Hardy . . 304

Standing Waves parallel to a Plane Beach. By H. C. Pocklington, M.A., St John's College 308

The Origin of the Disturbances in the Initial Motion of a Shell. By R. H. Fowler and C. N. H. Lock. (One fig. in Text) 311

Tides in the Bristol Channel. By G. I. Taylor, F.R.S. (Four figs. in Text) 320

Experiments with Rotating Fluids. By G. I. Taylor, F.R.S. . . . 326

Experiments on focal lines formed by a zone plate. By G. F. C. Searle, Sc.D., F.R.S., University Lecturer in Experimental Physics. (Five figs. in Text) 330

The Tensor Form of the Equations of Viscous Motion. By E. A. Milne, B.A., Trinity College 344

Insect Oases. By C. G. Lamb, M.A. 347

A Note on the Hydrogen Ion Concentration of some Natural Waters. By J. T. Saunders, M.A., Christ's College 350

The Mechanism of Ciliary Movement. By J. Gray, M.A., Balfour Student, and Fellow of King's College, Cambridge. (Three figs. in Text) . 352

A Note on the Biology of the 'Crown-Gall' Fungus of Lucerne. By J. Line, M.A., Emmanuel College. (Seven figs. in Text). . . . 360

On some Alcyonaria in the Cambridge Museum. By Sydney J. Hickson, M.A., F.R.S., Professor of Zoology in the University of Manchester. (One fig. in Text) 366

The Influence of Function on the Conformation of Bones. By A. B. Appleton, M.A., Downing College. (Five figs. in Text) 374

Animal Oecology in Deserts. By P. A. Buxton, M.A., Fellow of Trinity College, Cambridge 388

Venational Abnormalities in the Diptera. By C. G. Lamb, M.A. (Fourteen figs. in Text) 393

The Cooling of a Solid Sphere with a Concentric Core of a Different Material. By Professor H. S. Carslaw. (Three figs. in Text) . 399

Symbolical Methods in the theory of Conduction of Heat. By Dr T. J. I'a. Bromwich, F.R.S. (Two figs. in Text) 411

On the effect of a magnetic field on the intensity of spectrum lines. By H. P. Waran, M.A., Government of India Scholar of the University of Madras. (Communicated by Professor Sir Ernest Rutherford, F.R.S.) (Three figs. in Text and Plate III) 428

On a property of focal conics and of bicircular quartics. By C. V. Hanumanta Rao, University Professor, Lahore. (Communicated by Professor H. F. Baker) 434

Convex Solids in Higher Space. By Dr W. Burnside, Honorary Fellow of Pembroke College 437

Note on the Velocity of X-ray Electrons. By R. Whiddington, M.A. (One fig. in Text) 442

A Laboratory Valve method for determining the Specific Inductive Capacities of Liquids. By R. Whiddington, M.A. (One fig. in Text) 445

PAGE

The Theoretical Value of Sutherland's Constant in the Kinetic Theory of Gases. By C. G. F. James, Trinity College, Cambridge. (Communicated by Mr R. H. Fowler.) (One fig. in Text) 447

On the Stability of the Steady Motion of viscous liquid contained between two rotating coaxal circular cylinders. By W. J. Harrison, M.A., Fellow of Clare College, Cambridge 455

The soaring flight of dragon-flies. By E. H. Hankin, M.A., Sc.D., Agra, India. (Three figs. in Text) 460

The Gluteal Region of Tarsius Spectrum. By A. B. Appleton. (Plate IV) 466

An unusual type of male secondary characters in the Diptera. By C. G. Lamb, M.A. (Four figs. in Text) 475

A Note on the Mouth-parts of certain Decapod Crustaceans. By L. A. Borradaile, M.A., Fellow and Tutor of Selwyn College, Cambridge, and Lecturer in Zoology in the University 478

An Apparatus for Projecting Spectra. By H. Hartridge . . . 480

Note on true and apparent hermaphroditism in sea-urchins. By J. Gray, M.A., Balfour Student, Cambridge University. 481

On Certain Simply-Transitive Permutation-Groups. By Dr W. Burnside, Honorary Fellow of Pembroke College 482

Proceedings at the Meetings held during the Session 1920—1921 . . 485

Index to the Proceedings with references to the Transactions . . 492

PLATES.

Plates I—III. To illustrate Mr Waran's papers 48, 433

Plate IV. To illustrate Mr Appleton's paper 474

CORRECTION.

Mordell, p. 250, line 5, after *conjugate numbers* insert *in the real conjugate fields.*

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

On the term by term integration of an infinite series over an infinite range and the inversion of the order of integration in repeated infinite integrals. By S. POLLARD, M.A., Trinity College, Cambridge. (Communicated by Prof. G. H. Hardy.)

[*Received* 1 January, 1920. *Read* 8 March, 1920.]

THE PROBLEM FOR INFINITE SERIES.

1. The problem to be solved is that of determining conditions under which the equation

$$\sum_{n=1}^{\infty} \int_a^{\infty} u_n(x)\,dx = \int_a^{\infty} \sum_{n=1}^{\infty} u_n(x)\,dx, \qquad (1)$$

is true. It is discussed in detail in Bromwich's *Infinite Series*, pp. 452–455, where various conditions are given. All these conditions will be found to involve uniform convergence, the fact being that the infinite integrals there considered are obtained as limits of Riemann integrals and, in the theory of the latter, considerations as to the validity of the equation

$$\lim_{m\to\infty} \int_a^b \sum_{n=1}^{m} u_n(x)\,dx = \int_a^b \sum_{n=1}^{\infty} u_n(x)\,dx, \qquad (2)$$

almost always involve uniform convergence. Thus conditions for term by term integration over an infinite range, being built up from the conditions for term by term integration over a finite range, involve uniform convergence.

Now the condition of uniform convergence is by no means a necessary one: it occurs because of the lack of power in the methods of the Riemann theory. Much wider conditions can be obtained by the use of the Lebesgue theory. It is the object of this paper to give these.

Conditions for passage to the limit under the sign of integration, the range of integration being finite.

2. We give, for the sake of reference, the two principal elementary conditions.

(C 1) *If* $u_n(x)$ *is positive for* $a \leqslant x \leqslant b$; $n = 1, 2, 3 \ldots$, *then if either side of* (2) *is finite the equation holds, and if either side is infinite both are.*

(C 2) *If* $|\sum_{n=1}^{\nu} u_n(x)| < \psi(x)$ *for* $a \leqslant x \leqslant b$, $\nu = 1, 2, 3, \ldots$, *where* ψ *is summable in* (a, b), *then both sides of* (2) *exist and are finite and equal**.

Resumé of theorems of double limits.

3. As the use of double limits is fundamental in the theory about to be developed, we give a short summary of the results required.

(α) *If the double limit* $\lim_{x \to \infty, y \to \infty} S_{x,y}$ *exists, and* $\lim_{x \to \infty} S_{x,y}$ *exists for all sufficiently large* y; *then* $\lim_{y \to \infty} (\lim_{x \to \infty} S_{x,y})$ *exists and is equal to the double limit. Similarly for the limit* $\lim_{x \to \infty} (\lim_{y \to \infty} S_{x,y})$.

(β) *If* $S_{x,y}$ *is increasing in* x *and* y, *and any one of*

$$\lim_{x \to \infty, y \to \infty} S_{x,y}, \quad \lim_{x \to \infty} (\lim_{y \to \infty} S_{x,y}), \quad \lim_{y \to \infty} (\lim_{x \to \infty} S_{x,y})$$

exist; then all three exist and are equal.

(γ) *If* $S_{x,y}$ *can be expressed as the difference of two functions* $S'_{x,y}$, $S''_{x,y}$ *each of which is increasing in* x *and* y *and*

$$\lim_{x \to \infty, y \to \infty} (S'_{xy} + S''_{xy})$$

exists and is finite; then

$$\lim_{x \to \infty, y \to \infty} S_{x,y}, \quad \lim_{x \to \infty} (\lim_{y \to \infty} S_{x,y}), \quad \lim_{y \to \infty} (\lim_{x \to \infty} S_{x,y})$$

all exist and are finite and equal.

The condition (γ) is especially convenient when

$$S_{x,y} = \int_a^x \int_b^y f(\xi, \eta)\, d\xi d\eta.$$

* De la Vallée Poussin, *Cours d'analyse infinitésimale*, t. I., 3rd Ed., p. 264, theorems III and II.

For if
$$\lim_{x\to\infty, y\to\infty} \int_a^x \int_a^y |f(\xi, \eta)|\, d\xi d\eta$$
exists and is finite, then $S_{x,y}$ satisfies the condition of (γ). We have in fact
$$S_{x,y} = S'_{x,y} - S''_{x,y},$$
where
$$S'_{x,y} = \int_a^x \int_b^y |f(\xi, \eta)|\, d\xi d\eta,$$
$$S''_{x,y} = \int_a^x \int_b^y \{|f(\xi, \eta)| - f(\xi, \eta)\}\, d\xi d\eta,$$
and both $S'_{x,y}$, $S''_{x,y}$ are increasing in x and y and have a finite double limit—the former by hypothesis and the latter because
$$0 \leqslant |f(\xi, \eta)| - f(\xi, \eta) \leqslant 2|f(\xi, \eta)|.$$

NOTE. The above results still hold when either or both of the variables x, y take only positive integral values.

DEFINITION OF INFINITE INTEGRALS.

4. Let $f(x)$ be any function which is summable in (a, X) for all X greater than a.

If
$$\lim_{X\to\infty} \int_a^X f(x)\, dx,$$
where the integral is taken in the sense of Lebesgue, exists and is finite, we say that
$$\int_a^\infty f(x)\, dx$$
converges and attribute to it the value of the limit.

This definition is evidently consistent with and more general than that usually given, where $f(x)$ is assumed to be integrable in Riemann's sense in (a, X). It has the special advantage of not being restricted to functions which are bounded in every (a, X). And we lose nothing by adopting it, as the two theorems on which the theory of infinite integrals rests, the first and second mean value theorems, are still true when we abandon the restriction that $f(x)$ is to have a Riemann integral and make only the assumption that $f(x)$ is summable*.

GENERAL THEOREMS.

5. I. *If the double limit* $\lim_{m\to\infty, X\to\infty} \int_a^X \sum_{n=1}^m u_n(x)\, dx$ *exists and*

(a) $\int_a^\infty u_n(x)\, dx$ *converges for all* n,

* *Ibid.* t. II. 2nd Ed., p. 53.

(*b*) $\sum_{n=1}^{\infty} u_n(x)$ *converges for* $X \geqslant a$,

(*c*) $\lim_{m\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx, \quad \int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx,$

exist and are equal for all X*; then both sides of* (1) *exist and are equal.*

Proof. Write $\int_a^X \sum_{n=1}^{m} u_n(x)\,dx = S_{m,X},$

and let $\lim_{m\to\infty,\, X\to\infty} S_{m,X} = S.$

Since $\int_a^\infty u_n(x)\,dx$ converges for all n

$$\lim_{X\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx, \text{ i.e. } \lim_{X\to\infty} S_{m,X},$$

exists for all m. Hence

$$\lim_{m\to\infty} (\lim_{X\to\infty} S_{m,X}) = S \qquad (3),$$

by (α).

In virtue of (*b*) and (*c*)

$$\lim_{m\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx$$

exists and is equal to $\int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx.$

Thus $\lim_{m\to\infty} S_{m,X}$ exists for $X \geqslant a$.

Hence $\lim_{X\to\infty} (\lim_{m\to\infty} S_{m,X})$ exists and is equal to S.

Taking $\lim_{m\to\infty} S_{m,X}$ in the form $\int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx$, we see that

$$\int_a^\infty \sum_{n=1}^{\infty} u_n(x)\,dx = S. \qquad (4)$$

And (3) and (4) give us our theorem.

II. *If* $\sum_{n=1}^{\infty} |u_n(x)|$ *converges for* $x \geqslant a$ *and the double limit*

$$\lim_{m\to\infty,\, X\to\infty} \int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx$$

exists and is finite; then without further condition both sides of (1) *exist and are finite and equal.*

Proof. By (γ), if $\lim_{n\to\infty,\, X\to\infty} \int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx$ exists, so does

$$\lim_{m\to\infty,\, X\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx.$$

Also $$S_{m,X} = \int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx \leqslant S,$$

for all X and m. But $S_{m,X}$ increases with X for each m. Hence

$$\lim_{X\to\infty} \int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx$$

exists for each m and is less than S.

And therefore

$$\lim_{X\to\infty} \int_a^X |u_n(x)|\,dx = \lim_{X\to\infty} \left\{ \int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx - \int_a^X \sum_{n=1}^{m-1} |u_n(x)|\,dx \right\},$$

exists for each m, *i.e.* $\int_a^\infty |u_n(x)|\,dx$ and therefore $\int_a^\infty u_n(x)\,dx$ converges for each m. This is (a) of (I).

Again, $S_{m,X}$ increases with m for each X.

Hence $\lim_{m\to\infty} S_{m,X}$ exists and is finite for each X. So from

(C 1) $$\int_a^X \sum_{n=1}^{\infty} |u_n(x)|\,dx$$

is finite. Thus $\sum_{n=1}^{\infty} |u_n(x)|$ is summable in (a, X). But

$$\left|\sum_{n=1}^{m} u_n(x)\right| \leqslant \sum_{n=1}^{m} |u_n(x)| \leqslant \sum_{n=1}^{\infty} |u_n(x)|,$$

and so by (C 2)

$$\lim_{m\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx, \quad \int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx,$$

exist and are finite and equal. This is (c) of (I). Now (b) of (I) is satisfied by hypothesis.

Thus all the conditions of (I) are satisfied and so both sides of (1) exist and are equal.

Deductions from the general theorems.

6. A. *If* $$u_n(x) = \phi(x) f_n(x),$$

where $$\sum_{n=1}^{\infty} f_n(x) \text{ converges for } x \geqslant a,$$

$$\left|\sum_{n=1}^{\nu} f_n(x)\right| < G, \text{ for } x \geqslant a \text{ and all } \nu,$$

and $\int_a^\infty |\phi(x)|\,dx$ *converges; then both sides of* (1) *exist and are finite and equal.*

B. *If either of*

$$\sum_{n=1}^{\infty} \int_a^\infty |u_n(x)|\,dx, \quad \int_a^\infty \sum_{n=1}^{\infty} |u_n(x)|\,dx,$$

exist and are finite; then both sides of (1) *exist and are finite and equal.*

C. *If*

$$\lim_{m\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx, \quad \int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx,$$

exist and are finite and equal, and

$$\Sigma \int_a^X u_n(x)\,dx$$

converges uniformly for $a \leqslant x$, *and each*

$$\int_a^\infty u_n(x)\,dx$$

converges; then both sides of (1) *exist and are finite and equal.*

D. *If* $|\sum_{n=1}^{\nu} u_n(x)| < \psi_X(x)$ *for* $a \leqslant x \leqslant X$ *and all* ν, *where* ψ_X *is summable in* (a, X), *and*

$$\Sigma \int_a^X u_n(x)\,dx$$

converges uniformly for $a \leqslant X$, X *being arbitrary, and each*

$$\int_a^\infty u_n(x)\,dx$$

converges; then both sides of (1) *exist and are finite and equal.*

D is a special case of C obtained by making use of (C 2). A, B, D may be regarded as generalisations of theorems A—C, pp. 452–455 of Bromwich's *Infinite Series*.

Proofs. A. If $m' > m$,

we have

$$\sum_{n=m+1}^{m'} f_n(x) = \sum_{n=1}^{m'} f_n(x) - \sum_{n=1}^{m} f_n(x),$$

and therefore

$$|\sum_{n=m+1}^{m'} f_n(x)| \leqslant |\sum_{n=1}^{m'} f_n(x)| + |\sum_{n=1}^{m} f_n(x)| \leqslant 2G.$$

Hence

$$\left|\int_X^{X'} \sum_{n=m+1}^{m'} u_n(x)\,dx\right| \leqslant \int_X^{X'} \left|\sum_{n=m+1}^{m'} \phi(x) f_n(x)\right| dx$$
$$\leqslant \int_X^{X'} |\phi(x)| \sum_{n=m}^{m'} |f_n(x)|\,dx$$
$$\leqslant 2G\int_X^{X'} |\phi(x)|\,dx.$$

Now, given any positive number ϵ, we can, since $\int_a^\infty |\phi(x)|\,dx$ converges, find X_0 such that

$$\int_X^{X'} |\phi(x)|\,dx < \epsilon$$

for $X, X' > X_0$. Hence

$$\left|\int_X^{X'} \sum_{n=m+1}^{m'} u_n(x)\,dx\right| < \epsilon$$

for $X, X' > X_0$ and all m, m'. Thus the double limit

$$\lim_{m\to\infty,\, X\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx$$

exists. Further

$$\left|\sum_{n=1}^{\nu} u_n(x)\right| \leqslant |\phi(x)| \left|\sum_{x=1}^{\nu} u_n(x)\right| \leqslant G|\phi(x)|,$$

and $G|\phi(x)|$ is summable in (a, X) for all X greater than a. Thus by (C 2)

$$\lim \int_a^X \sum_{n=1}^{m} u_n(x)\,dx, \quad \int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx$$

exist and are equal and finite.

All the conditions of (I) are now satisfied, and our theorem follows.

B. If we write

$$S_{m,X} = \int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx,$$

then $\lim\limits_{m\to\infty,\, X\to\infty} S_{m,X}$ exists and is either finite or positive infinity.

In the first case our theorem follows at once by (II).

In the second case, both the repeated limits

$$\lim_{m\to\infty} (\lim_{X\to\infty} S_{m,X}), \quad \lim_{X\to\infty} (\lim_{m\to\infty} S_{m,X}),$$

are infinite. Suppose now that

$$\sum_{n=1}^{\infty} \int_a^\infty |u_n(x)|\,dx$$

exists and is finite. Then $\lim_{m\to\infty}(\lim_{x\to\infty} S_{m,X})$ exists and is finite, and we get a contradiction. And if

$$\int_a^\infty \sum_{n=1}^{\infty} |u_n(x)|\,dx$$

exists and is finite, then so does

$$\int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx$$

for all X greater than a. Hence as in theorem II

$$\int_a^X \sum_{n=1}^{\infty} |u_n(x)|\,dx = \lim_{m\to\infty} \int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx,$$

and it follows that

$$\int_a^\infty \sum_{n=1}^{\infty} |u_n(x)|\,dx = \lim_{X\to\infty}(\lim_{n\to\infty} S_{m,X}),$$

and we again get a contradiction.

Thus the first case alone is possible, and this is the case in which our theorem is true.

C. Write $$\int_a^X u_n(x)\,dx = g_n(X),$$

$$\sum_{n=1}^{m} g_n(X) = S_{m,X}.$$

Since $\sum_{n=1}^{\infty} g_n(X)$ converges uniformly for $a \leqslant X$, given $\epsilon > 0$ we can find N_0 such that

$$|\sum_{n=N+1}^{\infty} g_n(X)| < \epsilon, \qquad (X \geqslant a,\ N \geqslant N_0).$$

Thus $$|S_{m,X} - \sum_{n=1}^{\infty} g_n(X)| < \epsilon, \quad (X \geqslant a,\ m \geqslant N_0).$$

Hence if $\lim_{X\to\infty} \sum_{n=1}^{\infty} g_n(X)$ exists and is finite, so does $\lim_{m\to\infty,\,X\to\infty} S_{m,X}$, and the two are equal. Now

$$\begin{aligned}|\sum_{n=1}^{\infty} g_n(X') - \sum_{n=1}^{\infty} g_n(X'')| &\leqslant |\sum_{n=1}^{N} g_n(X') - \sum_{n=1}^{N} g_n(X'')| \\ &\quad + |\sum_{n=N+1}^{\infty} g_n(X')| + |\sum_{n=N+1}^{\infty} g_n(X'')| \\ &\leqslant |\sum_{n=1}^{N} g_n(X') - \sum_{n=1}^{N} g_n(X'')| + 2\epsilon.\end{aligned}$$

But since $\lim_{X\to\infty} g_n(X)$ $(= \int_a^\infty u_n(x)\,dx)$ exists and is finite for

each n, so does $\lim_{X\to\infty} \sum_{n=1}^{N} g_n(X)$ and we can find X_0 such that

$$\left| \sum_{n=1}^{N} g_n(X') - \sum_{n=1}^{N} g_n(X'') \right| < \epsilon, \qquad (X', X'' \geqslant X_0).$$

Hence $$\left| \sum_{n=1}^{\infty} g_n(X') - \sum_{n=1}^{\infty} g_n(X'') \right| < 3\epsilon, \qquad (X', X'' \geqslant X_0)$$

and therefore, by the general principle of convergence

$$\lim_{X\to\infty} \sum_{n=1}^{\infty} g_n(X)$$

exists. Thus $\lim_{m\to\infty,\, X\to\infty} S_{m,X}$ exists. The other conditions of (I) are satisfied by hypothesis and our theorem follows.

The problem for infinite integrals.

7. We have to determine conditions under which the equation

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy = \int_b^\infty dy \int_a^\infty f(x, y)\, dx \qquad (5)$$

is true. The methods adopted above apply almost without change and we get conditions almost identical with those already given. We quote them without proof, as the proofs can be made up immediately on the lines of those already given.

As regards the nature of $f(x, y)$, we assume throughout that $f(x, y)$ is summable in the region

$$(a \leqslant x \leqslant X,\ b \leqslant y \leqslant Y),$$

for all $X \geqslant a$, $Y \geqslant b$; so that, by Fubini's theorem*, the repeated integrals $\int_a^X dx \int_b^Y f(x, y)\, dy$, $\int_b^Y dy \int_a^X f(x, y)\, dx$ exist and are equal to the double integral.

General theorems.

8. I′. *If the double limit* $\lim_{X\to\infty,\, Y\to\infty} \int_a^X \int_b^Y f(x, y)\, dx dy$ *exists and*

(*a*) $\int_a^\infty f(x, y)\, dx$, *converges for* $y \geqslant b$,

(*b*) $\int_b^\infty f(x, y)\, dy$, *converges for* $x \geqslant a$,

(*c*) $\lim_{Y\to\infty} \int_a^X dx \int_b^Y f(x, y)\, dy$, $\int_a^X dx \int_b^\infty f(x, y)\, dy$,

* De la Vallée Poussin, *Intégrales de Lebesgue* etc., p. 53.

exist and are finite and equal for $X \geqslant a$ *; then both sides of* (5) *exist and are equal.*

II′. *If the double limit* $\lim_{X\to\infty,\, Y\to\infty} \int_a^X \int_b^Y |f(x, y)|\, dx dy$ *exists and is finite; then without further condition both sides of* (5) *exist and are finite and equal.*

Deductions from the general theorems.

9. A′. *If* $f(x, y) = \phi(x)\,\theta(x, y)$,

where $\left|\int_b^Y \theta(x, y)\, dy\right| < G$ *for* $x \geqslant a$, $y \geqslant b$,

$\int_b^\infty f(x, y)\, dy$ *converges for* $x \geqslant a$,

and $\int_a^\infty |\phi(x)|\, dx$ *converges; then both sides of* (5) *exist and are finite and equal.*

B′. *If either of*

$$\int_a^\infty dx \int_b^\infty |f(x, y)|\, dy, \quad \int_b^\infty dy \int_a^\infty |f(x, y)|\, dx,$$

exist and are finite; then both sides of (5) *exist and are finite and equal*.*

C′. *If* $\lim_{Y\to\infty} \int_a^X dx \int_b^Y f(x, y)\, dy$, $\int_a^X dx \int_b^\infty f(x, y)\, dy$,

exist and are finite and equal, and

$$\int_b^\infty dy \int_a^X f(x, y)\, dx$$

converges uniformly for $a \leqslant X$, *and*

$$\int_a^\infty f(x, y)\, dx$$

converges for $y \geqslant b$; *then both sides of* (5) *exist and are finite and equal.*

* This is de la Vallée Poussin's theorem. See Bromwich, *Infinite Series*, p. 457. The hypothesis given by Bromwich to the effect that *both* the integrals

$$\int_a^\infty f(x, y)\, dx, \quad \int_b^\infty f(x, y)\, dy,$$

are convergent is unnecessary, the existence of one (the one necessary to the existence of the repeated integral) is sufficient. That of the other is implied by the existence of the double limit, see Note 2.

D′. *If* $\left|\int_b^Y f(x, y)\,dy\right| \leqslant \psi_X(x)$ *for* $a \leqslant x \leqslant X,\ b \leqslant Y,$

where ψ_X *is summable in* (a, X), *and*

$$\int_b^\infty dy \int_a^X f(x, y)\,dx$$

converges uniformly for $a \leqslant X$, X *being arbitrary, and*

$$\int_a^\infty f(x, y)\,dx$$

converges for $y \geqslant b$; *then both sides of* (5) *exist and are finite and equal.*

10. *Note* 1. Results B are especially valuable, as they are easy to remember and convenient to apply. The power of the Lebesgue theory is shewn very clearly here in that by using it we are enabled to make the hypothesis which ensures the existence of the double limit* ensure also the passage to the limit under the sign.

Note 2. It is well to be precise as to the meaning of the word "exists" as used in connection with repeated Lebesgue integrals.

Suppose $f(x, y)$ is measurable in x, y in the rectangle

$$\begin{pmatrix} a \leqslant x \leqslant X \\ b \leqslant y \leqslant Y \end{pmatrix}.$$

We know that the function $f(x, y)$ considered as a function of x, is measurable in (a, X) for each y in (b, Y) a set of zero measure being excepted. It may not, however, be summable in (a, X), *i.e.*

$$\int_a^X f(x, y)\,dx$$

may not exist, for all y concerned. But, if $f(x, y)$ is summable over the rectangle, *i.e.*, if the double integral

$$\int_a^X \int_b^Y f(x, y)\,dx\,dy$$

exists; then it can be shewn that

$$\int_a^X f(x, y)\,dx$$

exists for all values of y in (b, Y), save possibly those of a set of measure zero.

* The existence of $\sum_{n=1}^{\infty} \int_a^\infty |u_n(x)|\,dx$ implies the existence of the double limit by (γ) of § 3; and in addition, by the use of (C 1) on $|u_n(x)|$, it will be found to imply the validity of the passage to the limit under the sign.

Now in the Lebesgue theory the integral of any summable function over a set of zero measure is zero, and consequently we may neglect a set of measure zero without affecting the value of the integral. Hence when we are faced with the problem of finding the value of a function which is indefinite or infinite at the points of a set of measure zero, we simply neglect these points and find the value of the integral over the residue. This is taken to be the value of the integral over the original set.

With the above convention it is true that, if

$$\int_a^X \int_b^Y f(x, y)\, dxdy$$

exists, so does

$$\int_b^Y dy \int_a^X f(x, y)\, dxdy,$$

although there may be points in (b, Y) at which the single integral

$$\int_a^X f(x, y)\, dx$$

does not exist.

It is always to be understood in dealing with repeated Lebesgue integrals (finite or infinite) that the inner integrals need only exist at all the points of the range of integration of the outer integral save those of a set of measure zero.

Let us apply the foregoing remarks to theorem B′. Suppose

$$\int_b^Y dy \int_a^X |f(x, y)|\, dx$$

exists. Then we know that

$$\lim_{X\to\infty,\, Y\to\infty} \int_a^X \int_b^Y |f(x, y)|\, dxdy$$

exists. It follows that

$$\int_a^X dx \int_b^Y |f(x, y)|\, dy,$$

considered as a function of Y, is bounded as Y tends to infinity. Thus

$$\lim_{Y\to\infty} \int_a^X dx \int_b^Y |f(x, y)|\, dy$$

exists and is finite. It follows that

$$\int_b^\infty |f(x, y)|\, dy$$

converges at all the points of (a, X) save possibly those of a set of

measure zero, because if it did not the above limit would be infinite; and so, for our purposes

$$\int_a^X dx \int_b^\infty |f(x, y)| \, dy$$

exists.

Our convention has enabled us to infer the existence of the inner integrals from the existence of the double limit.

Note 3. A thorough treatment on different lines of the subject of this paper will be found in two papers by Prof. W. H. Young:

(1) "On the change of order of integration in an improper repeated integral," *Trans. Camb. Phil. Soc.*, XXI. p. 361.

(2) "The application of expansions to definite integrals," *Proc. Lond. Math. Soc.*, IX. (1910), p. 463.

In this paper we content ourselves with giving simple generalisations of well-known results with proofs depending on comparatively elementary theorems. There is no attempt to obtain comprehensive results.

Note on Mr Hardy's extension of a theorem of Mr Pólya. By EDMUND LANDAU. (Communicated by Prof. G. H. Hardy.)

[*Received* 10 *December* 1919. *Read* 26 *January* 1920.]

In a recent note in these *Proceedings** Mr Hardy has established an improved form of a theorem of Mr Pólya, viz.:

Suppose that $g(x)$ *is an integral function, and* $M(r)$ *the maximum of* $|g(x)|$ *for* $|x| \leqslant r$. *Suppose further that* $g(x)$ *is an integer for* $x = 0, 1, 2, 3, \ldots$, *and that*

$$M(r) = o(2^r).$$

Then $g(x)$ *is a polynomial.*

As Mr Hardy remarks at the beginning of his note, it is sufficient (after the analysis given already by Mr Pólya), to prove the one formula

$$n!\, 2^{2n} \int_{-\pi}^{\pi} \frac{d\theta}{\prod_{s=1}^{n} (2n - s\cos\theta)} = O(1).$$

Mr Hardy's proof of this formula may be replaced by the following shorter proof.

Since

$$n!\, 2^{2n} \frac{1}{\prod_{s=1}^{n} (2n-s)} = n!\, 2^{2n} \frac{(n-1)!}{(2n-1)!} = 2 \cdot 2^{2n} \frac{n!\, n!}{2n!} = O(\sqrt{n}),$$

it is enough to prove

$$\int_{-\pi}^{\pi} \psi(\theta, n)\, d\theta = O\left(\frac{1}{\sqrt{n}}\right),$$

where

$$\psi(\theta, n) = \prod_{s=1}^{n} \left(\frac{2n-s}{2n - s\cos\theta}\right).$$

Now

$$1 - \cos\theta = \frac{\theta^2}{2} - \frac{\theta^4}{24} + \ldots \geqslant \frac{\theta^2}{2} - \frac{\theta^4}{24} \geqslant \frac{\theta^2}{2} - \frac{10\theta^2}{24} = \frac{\theta^2}{12},$$

for $-\pi \leqslant \theta \leqslant \pi$, and

$$\frac{1}{1+y} = e^{-y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \ldots} \leqslant e^{-y + \frac{1}{2}y^2} \leqslant e^{-y + \frac{1}{2}y} = e^{-\frac{1}{2}y},$$

* Vol. XIX. (1919), pp. 60–63.

for $0 \leqslant y \leqslant 1$. Hence

$$\frac{1-\eta}{1-\eta\cos\theta} = \frac{1}{1+\frac{\eta}{1-\eta}(1-\cos\theta)} \leqslant \frac{1}{1+\eta(1-\cos\theta)}$$

$$\leqslant e^{-\frac{1}{2}\eta(1-\cos\theta)} \leqslant e^{-\frac{1}{24}\eta\theta^2},$$

for $0 \leqslant \eta \leqslant \frac{1}{2}$, $-\pi \leqslant \theta \leqslant \pi$; and so

$$\psi(\theta, n) = \prod_{s=1}^{n} \frac{1-\frac{s}{2n}}{1-\frac{s}{2n}\cos\theta} \leqslant e^{-\frac{\theta^2}{48n}\sum_{s=1}^{n} s} = e^{-\frac{1}{96}(n+1)\theta^2} \leqslant e^{-\frac{1}{96}n\theta^2},$$

for $-\pi \leqslant \theta \leqslant \pi$ and $n = 1, 2, 3, \ldots$. Therefore

$$\int_{-\pi}^{\pi} \psi(\theta, n)\, d\theta \leqslant \int_{-\infty}^{\infty} e^{-\frac{1}{96}n\theta^2} d\theta = O\left(\frac{1}{\sqrt{n}}\right).$$

GÖTTINGEN, 4 *December* 1919.

Studies on Cellulose Acetate. By H. J. H. Fenton and A. J. Berry.

[*Read* 8 *March* 1920.]

The enormous demand for cellulose acetate and the serious shortage of acetone and certain other materials used in the manufacture of aeroplane dopes during the war originated a systematic research on cellulose acetate, especially as regards the behaviour of this material towards solvents and its chemical properties generally. The research has been pursued in a number of directions, the most important of which have been (*a*) substitutes for acetone as solvents, (*b*) the preparation of cellulose acetate and a study of the influence of the mode of preparation on the properties of the resulting product, and (*c*) the analytical chemistry of cellulose acetate. Most of our experiments, especially those relating to aeroplane dopes were necessarily of a technical character, but as a few results of general chemical interest have been obtained in the course of the work, we have thought it desirable to give a brief account of them in the present communication.

Solvents.

At the time of the difficulty caused by the serious shortage of acetone we were urged to discover efficient substitutes for this solvent for use in aeroplane dopes. It should, in passing, be observed that the properties of acetone make it an ideal solvent: its conveniently low boiling point, rapid solvent action on cellulose acetate, non-poisonous character, and, in normal times, cheap and abundant supply. All other liquids which have so far been suggested show a deficiency in some one or other of these particulars.

In August, 1917, we suggested that in case of emergency the three following solvents might be employed, viz. *acetaldehyde*, *acetonitrile*, and *nitrobenzene* with certain additions. Quite early in the investigation (October, 1916) we suggested *acetic acid* and *ethyl formate* as solvents. We also suggested the use of *cyclohexanone* and of beechwood *creosote* as substitutes for tetrachloroethane or benzyl alcohol as high boiling solvents. We were never informed whether these solvents were actually employed. It is remarkable that at considerably later dates, patents have been taken out for the use of both acetaldehyde and cyclohexanone as dope constituents. (British Patent 131647, July 4th, 1918 (acetaldehyde) and *Ibid.* 130402, February 15th, 1918 (Cyclohexanone).)

Our experiments have demonstrated that the destructive effect of acids upon fabrics is dependent on the strength of the acid in the physico-chemical sense. Hitherto it had been supposed that esters were objectionable as dope constituents on account of the possibilities of free acids resulting from hydrolysis. This, however, we found not to be the case. As far as weak acids only are concerned, tensile strength determinations gave excellent results; and fabrics doped with acetic acid as the principal solvent compared most favourably with others.

In our experiments a large number of liquids have been examined, not only from the purely practical point of view, but also from a desire to obtain if possible some information with regard to possible relationships between the nature of the liquid and its solvent action. It is of course impossible to define strictly the solubility of cellulose acetate in any given solvent owing to the colloidal nature of the products. The term "positive" is used in the following lists to imply that the liquid named has the property of gelatinizing cellulose acetate and subsequently converting it into a clear homogeneous "sol" without the aid of heat. All the results were obtained with a sample of the material which yields 54 per cent. of acetic acid on cold alkaline saponification.

Positive.

Liquid ammonia, liquid sulphur dioxide, liquid hydrogen cyanide, acetaldehyde, benzaldehyde, salicylaldehyde, acetone, methyl ethyl ketone, suberone, acetonitrile, propionitrile, formic acid, acetic acid, butyric acid, formamide, ethyl formate, ethyl oxalate, ethyl malonate, ethyl acetoacetate, aniline, phenylhydrazine, ortho-toluidine, piperidine, pyridine, tetrachloroethane, nitrobenzene*, nitromethane, cyclohexanone, guaiacol, chloroform*.

Although cellulose acetate is insoluble in water and in absolute ethyl alcohol, a mixture of these two liquids dissolves it freely on boiling. On cooling, however, precipitation takes place almost completely.

Negative.

Liquid air, liquid ethylene, liquid nitrous oxide, liquid hydrogen sulphide, benzene, toluene, turpentine, carbon disulphide, carbon tetrachloride, alcohol, ether, ethyl chloride, acetal, dimethyl acetal, nickel carbonyl, and many other liquids.

No general conclusion can be drawn as regards the chemical nature of a liquid and its solvent action on cellulose acetate. It is,

* Nitrobenzene requires certain additions. Chloroform had only a partial solvent action on this specimen of the material.

however, worthy of note that there appears to be some relation (with undoubted exceptions) between the dielectric constant and solvent action.

Influence of methods of preparation upon the properties of cellulose acetate.

The materials obtained by acetylating cellulose with acetic anhydride diluted with acetic acid in presence of various catalysts such as concentrated sulphuric acid, ferric sulphate, ortho toluidine bisulphate, may show considerable variations in properties depending upon the temperature, length of time of acetylation, and numerous other factors. When cellulose is acetylated and the product at once precipitated by water, it is nearly insoluble in acetone. Various methods have been adopted in order to convert the product so obtained into an acetone-soluble modification. The most widely used of these methods is that of Miles. This consists in heating the acetic acid solution of the cellulose acetate with water in rather greater quantity than that required to combine with the residual acetic anhydride. Sodium acetate may also be added to react with the catalyst if still present. The results are usually supposed to be due to chemical hydration.

In our experiments, cellulose was acetylated under the influence of various catalysts, and the effect of treatment by the Miles process was subjected to a critical examination. The most marked effects of this process are the changes in solubility in acetone and chloroform, most cellulose acetates being soluble in chloroform and insoluble in acetone before the treatment. This change in physico-chemical properties was found to be accompanied by a fall in the acetyl number. In one case the untreated cellulose acetate with an acetyl number of 60·9, yielded a product after the Miles process carried out at 100° for 48 hours with an acetyl number of 46·7. In another case when the treatment was carried out at the same temperature for 23 hours, the acetyl number fell from 60·5 to 50·4. The specific gravity of the cellulose acetate is also greatly reduced after the treatment. The influence on the heat test is not well marked but the decomposition point appears to be lowered somewhat.

In our view these results are to be ascribed to partial hydrolysis of the cellulose esters, not to hydration as is commonly supposed*. Apart from the diminution of the acetyl number already mentioned, we have carried out a series of experiments which have demonstrated that cellulose acetate does not form a hydrate. These

* Our view that the effect of the Miles process is essentially hydrolytic and not due to chemical hydration has been expressed subsequently by Ost (*Zeitsch. angew. Chem.* 1919, xxxii, 66, 76, and 82).

experiments originated in connexion with our determinations of the water contained in commercial samples of cellulose acetate. As is well known, the water is readily expelled by exposure of the material over concentrated sulphuric acid in a desiccator or by heating to 100°. It has frequently been supposed that the approximately constant proportion of 5 or 6 per cent. of water usually met with indicates a definite hydrate. In order to obtain positive information on this point, we determined the pressure-concentration relationship in the manner originally adopted by van Bemmelen in his well known researches on silicic acid (*Zeitsch. anorg. Chem.* 1896, XIII. 233). Weighed quantities of the material were exposed in a series of exhausted desiccators over sulphuric acid of various determined concentrations, and the corresponding vapour pressures were found by reference to Landolt and Börnstein's tables. The weights were found to be constant after 24–48 hours, and the pressure concentration relationship showed that no chemical hydration occurs. The phenomenon is to be regarded as one of adsorption, probably with subsequent diffusion, and is precisely similar to the absorption of water by cellulose itself. (Compare Masson and Richards (*Proc. Roy. Soc.* 1906, LXXVIII. 421), Trouton and Pool (*Ibid.* 1906, LXXVII. 292) and Travers (*Ibid.* 1906, LXXVIII. 21, and 1907, LXXIX. 204).)

Characterization and Analysis of cellulose acetate.

In the technical analysis of cellulose acetate, it is usual to examine the product by the heat test, solubility, acidity, and viscosity of the solutions, in addition to the determinations of acetyl (as acetic acid), copper reducing power, water, ash, and impurities. We have made an exhaustive investigation of various methods of carrying out these determinations, especially of the acetyl number, and have also carried out many ultimate analyses for carbon and hydrogen in some commercial specimens of the material.

The methods of determining the *acetyl* group may be classified under the two heads of *alkaline saponification* and *acid hydrolysis*. In the former the substance is saponified by excess of standard alkali, either at the ordinary temperature or at some higher temperature, and the excess of alkali determined by titration. In the latter, the substance is hydrolysed by strong acid, usually sulphuric or phosphoric, and the resulting acetic acid separated by steam distillation (Ost), or alcohol is added and the resulting ethyl acetate distilled off and collected in excess of standard alkali (Green and Perkin). The following is a summary of the principal results obtained in our experiments.

(1) *Cold alkaline saponification* (Ost and Katayama, *Zeitsch. angew. Chem.* 1912 (25), 1467). A known weight of the substance

is soaked with alcohol, then a measured volume of normal alkali is added and allowed to stand for 24 hours. The excess of alkali is then determined by standard acid. The mean result was 54 per cent. of acetic acid calculated for the dry substance.

(2) *Cold alkaline saponification* (Böeseken, van der Berg and Kerstjens, *Rec. Trav. Chim.* 1916, xxxv. 320). The substance is treated with strong aqueous potash for one or two days. A measured excess of normal hydrochloric acid is then added, the liquid then boiled for a moment to expel carbon dioxide and the resulting solution titrated with baryta water. The mean result calculated as above was 53·5 per cent. of acetic acid.

(3) *Hot alkaline saponification* (Barthelemy, *Moniteur Scientifique*, 1913 (3), II. 549). In this method the saponification is effected by heating the substance with normal soda for about 16 hours at 85°. The excess of alkali is then determined by titration with standard acid. Several experiments were made in which the conditions were subjected to considerable variations as regards length of heating and amount of excess of alkali. The extreme variations in the acetyl number calculated as above were 60·0 and 62·1 per cent.

(4) *Hot alkaline saponification* (Green and Perkin, *Trans. Chem. Soc.* 1906, 812). The saponification is carried out at the boiling point with semi-normal alcoholic soda and the excess of alkali titrated by standard acid. Our experiments yielded results of 60 per cent. of acetic acid, the extreme variations being 58·2 and 61·9 per cent. These numbers are in agreement with those of Green and Perkin (*loc. cit.*).

It is evident that the methods of hot alkaline saponification invariably yield results which are considerably higher than those obtained by cold saponification. There can be little doubt that the higher results are due to the action of alkali on the regenerated cellulose. Support to this contention was obtained by digesting two equal weights of filter paper with 50 c.c. of normal soda for two days, one at the ordinary temperature, the other at 85°. In the former case no alkali was consumed, while the heated product showed a loss of nearly 2 c.c. of normal alkali on titration.

(5) *Acid hydrolysis* (Ost, *loc. cit.*). The substance is first digested with 50 per cent. (by volume) sulphuric acid. After 24 hours the liquid is diluted considerably and the acetic acid separated by steam distillation, and titrated with baryta water. In our experiments phosphoric acid was substituted for sulphuric acid in order to avoid error due to possible formation of sulphur dioxide. The results varied from 51·5 to 55·0 per cent. of acetic acid.

(6) *Acid hydrolysis* (A. G. Perkin, *Trans. Chem. Soc.* 1905, 107). In this method the cellulose acetate is treated with ethyl alcohol and sulphuric acid, and the resulting ethyl acetate distilled into

excess of standard alkali. The ester is then saponified and the excess of alkali determined by titration. In our experiments phosphoric acid was used instead of sulphuric acid for the reason already mentioned. The results varied from 52·2 to 54·4 per cent. of acetic acid.

In our opinion, preference should be given to the method of cold alkaline saponification of Ost. Not only are the results more uniform, but they agree well with those obtained by acid hydrolysis. The latter methods are exceedingly tedious to carry out. We have also carried out some experiments with the use of hot baryta water as a saponifying agent and subsequent gravimetric determination of the barium, the results averaging 57–58 per cent. of acetic acid.

The materials met with in commerce known as cellulose acetate are most probably mixtures or solid solutions of various acetates, not definite chemical individuals. If, however, it were desired to represent cellulose acetate as a chemical individual, the results of our analyses of a number of specimens do not correspond with the formula of the triacetate $C_6H_7O_2(OCOCH_3)_3$ which is commonly supposed. They agree better with the formula of a pentacetyl derivative of $C_{12}H_{20}O_{10}$ and still better with that of a heptacetyl compound of $C_{18}H_{30}O_{15}$.

Thus

	Carbon	Hydrogen	Acetic acid
$C_6H_7O_2(OCOCH_3)_3$ requires	50·0	5·5	62·1 per cent.
$C_{12}H_{15}O_5(OCOCH_3)_5$,,	49·4	5·6	56·0 ,,
$C_{18}H_{23}O_8(OCOCH_3)_7$,,	49·2	5·64	53·8 ,,

Our most reliable results average carbon 49·2, hydrogen 5·5, and acetic acid 54 per cent.

Certain authors have stated that sodium ethylate may be used for the determination of acetyl in cellulose acetates. In investigating this reaction, we were surprised to find that ethyl acetate was always produced along with a yellow sodium derivative of cellulose. Quantitative experiments were performed in which the ethyl acetate was distilled into an excess of standard sodium hydroxide, and after saponification determined with standard acid. The residue was washed with alcohol to remove the unaltered sodium ethylate and this solution was titrated with standard acid. The residue was then treated with water to decompose the sodium compound and titrated also. It was found that the quantity of acetic acid converted into ethyl acetate to that becoming sodium acetate appears to depend to some extent on the proportion of sodium ethylate employed. The results can be explained, if the average commercial cellulose acetates are represented by the formula $C_{12}H_{15}O_5(OCOCH_3)_5$, by the equation:

$$C_{12}H_{15}O_5(OCOCH_3)_5 + C_2H_5ONa + 4C_2H_5OH = C_{12}H_{19}O_9ONa + 5CH_3COOC_2H_5$$

which may be taken to represent the main reaction.

In support of this, the yellow sodium compound from a similar experiment, after thorough washing with alcohol, was digested for several hours in a reflux apparatus with excess of methyl iodide, and the methoxy group in the resulting product determined by Zeisel's method. The result obtained was 9·2 per cent. of methoxyl in agreement with that calculated for the formula $C_{12}H_{19}O_9OCH_3$.

The adsorption of basic dyestuffs by cellulose acetate.

Certain dyestuffs, such as gentian violet are adsorbed in considerable quantities from aqueous solution by cellulose acetate, the solid being coloured blue. Cellulose, it is true, also adsorbs the dye, but to a much smaller extent, and the solid becomes violet. This property may be utilized to identify unaltered cellulose in commercial preparations of cellulose acetate. Methyl orange gave negative results, but methyl red was adsorbed in considerable quantity, the solid becoming red. Free dimethylaminoazobenzene gave negative results, but the hydrochloride of this base was strongly adsorbed, the solid cellulose acetate assuming a pinkish yellow colour and the colour of the aqueous solution being almost completely discharged.

The authors desire to express their grateful thanks to Mr J. W. H. Oldham, M.A., of Trinity College, for much valuable assistance in connexion with this investigation. Mr Oldham has also carried out a large number of experiments on the influence of the mode of preparation upon the resulting properties of cellulose acetate, and it is hoped that his results when completed may form the subject of a future communication.

An examination of Searle's method for determining the viscosity of very viscous liquids. By Kurt Molin, Filosofie Licentiat, Physical Institute, Technical College, Trondhjem. (Communicated by Dr G. F. C. Searle.)

[*Read* 9 February 1920.]

§ 1. The determination of the coefficient of internal friction in very viscous liquids has been the object of measurements by many different methods. A review of these will be found in Reiger*. A number of more recent methods are given by Kohlrausch†, and among them is a method of Searle's‡. An examination of this method is the object of the present paper.

In his paper, "A simple viscometer for very viscous liquids," Dr Searle‡ gives an account of a viscometer he has constructed. The method consists in causing a vertical cylinder to rotate within a coaxal cylinder containing liquid, and in determining the angular velocity of the inner cylinder for a known value of the driving couple. The couple is produced by the weights of two loads acting on a drum by two threads. The time, T seconds, of one revolution of the cylinder is found, and the length, l cm., of the inner cylinder immersed in the liquid is observed.

Newton's statement is that

$$f = -\eta \frac{dV}{dn}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where f is the force per unit area which acts against the direction of motion and at right angles to the normal, n, to the surface, dV/dn is the velocity gradient, and η is the coefficient of viscosity. In this statement the motion of the liquid is supposed to take place parallel to a fixed plane. Treating the liquid as incompressible, and modifying (1), by substituting the rate of shearing for dV/dn, so as to suit the case of rotation, we obtain the following formula:

$$\eta = \frac{gD\,(a^2 - b^2)}{8\pi^2 a^2 b^2}\left(\frac{MT}{l}\right) = C\left(\frac{MT}{l}\right).$$

Here D is the effective diameter of the drum, a and b are the radii of the cylinders, and M is the mass of each of the two loads, which are required to move the inner cylinder with the constant angular velocity Ω, such that $2\pi/\Omega = T$.

* R. Reiger, *Ann. d. Phys.*, 19, p. 985, 1906.
† F. Kohlrausch, *Lehrbuch d. praktischen Physik*, xii. Aufl., p. 268.
‡ G. F. C. Searle, *Proc. Cambridge Phil. Soc.*, 16, p. 600, 1912.

The angular velocity of the liquid about the axis of the cylinders, at a distance r from the axis, is given by

$$\omega = \frac{2\pi}{T} \cdot \frac{b^2}{a^2 - b^2}\left(\frac{a^2}{r^2} - 1\right).$$

When $r = b$, the radius of the inner rotating cylinder,

$$\omega = \Omega = 2\pi/T,$$

and when $r = a$, the internal radius of the outer fixed cylinder, $\omega = 0$. This problem was first treated, not quite accurately, by Newton. The above results were given substantially by Stokes*, and are also given by Lamb† and by Searle‡.

The rate of shearing, $r\,d\omega/dr$, varies somewhat as r increases from b to a, as is shown by the formula

$$r\frac{d\omega}{dr} = -\frac{2\pi}{T} \cdot \frac{2a^2b^2}{(a^2 - b^2)\,r^2}.$$

We have only taken into account the friction between the coaxal cylindrical layers of the liquid and not the friction between the horizontal layers in proximity to the bottom surface of the movable cylinder, and have not considered the conditions that arise near that surface. In practice, only the lower end of the rotating cylinder is exposed to viscous action; Dr Searle makes an allowance for this end by writing

$$\eta = C \cdot \frac{MT}{l + k}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where k is the length by which the height, l, of the liquid, in the simple theory, must be increased, in order that the increase of couple shall correspond to the viscous action in proximity to the end surface and the edge of the rotating cylinder.

Dr Searle gives a graphical method of determining k. The values of MT are plotted against l, and he says, "It will be found that the points lie on a straight line, which cuts the axis of l at a distance k from the origin." Dr Searle adds "If the corresponding total load hung from each thread be M grammes, it will be found, on repeating the observation with various loads, that MT is constant for a given level of liquid. This result confirms the fundamental assumption that the viscous stress at each point is proportional to the rate of shearing of the liquid."

* G. G. Stokes, *Brit. Ass. Report*, p. 539, 1898.
† H. Lamb, *Hydrodynamics*, Third Ed., p. 546, 1906.
‡ G. F. C. Searle, *loc. cit.*, p. 602. Compare C. Brodman, *Wied. Ann.*, 45, p. 163, 1892.

§ 2. In my experiments I used Dr Searle's viscometer, as supplied by Messrs W. G. Pye and Co., Cambridge*. I determined the viscosity of treacle, as Dr Searle refers to a determination of η for that liquid. I found $2b = 3{\cdot}74$ cm., $2a = 5{\cdot}01$ cm., and $D = 1{\cdot}95$ cm. Since $g = 982$ cm. sec.$^{-2}$ at Trondhjem, the constant C has the value

$$C = 3{\cdot}070 \pm 0{\cdot}035.$$

From the data given by Dr Searle, I find for the constant of the instrument used by him, $C_s = 3{\cdot}153$.

In my instrument the rate of shearing for radius r is given by

$$r\frac{d\omega}{dr} = -\frac{2\pi}{T}\cdot\frac{15{\cdot}80}{r^2}.$$

§ 3. To examine how MT depends upon M, when l is kept constant, six series of observations were taken with six values of l varying from 10·0 to 2·15 cm., and in each series M was made to vary from 5 to 205 grammes.

Since the viscosity of highly viscous substances diminishes very rapidly as the temperature increases, as was shown by Reiger† and by Glaser‡ for values of η of the magnitudes $4{\cdot}8 \times 10^5$ to $67{\cdot}2 \times 10^6$, and by Ladenburg§ for $\eta = 1{\cdot}3 \times 10^3$, great care must be taken to keep the temperature constant. The apparatus was, therefore, placed in a thermostat with electric temperature regulation, and a very constant temperature of 19·8° C. was maintained. The apparatus was left in the thermostat for 24 hours before the measurements were begun, and, during the short time a rotation trial was in progress, only the outer wooden door of the thermostat was opened, since one could see into the thermostat through the inner glass door. The final measurements were all carried out in the course of a day; the observations were made at intervals of about 10 minutes, so that the unavoidable disturbances of temperature, due to the manipulations, might have time to disappear.

In other respects the measurements were carried out in accordance with Dr Searle's‖ instructions. The revolutions were timed by aid of a stop-watch and the times were taken for different numbers of revolutions with odd numbers up to 9, as well as the average time for one revolution. As no decrease in the time of a single revolution could be noticed as the rotation continued, the divergences from the mean lying within the limits of the errors

* Catalogue of Scientific Apparatus manufactured by W. G. Pye and Co., List No. 120, p. 39, 1914.

† R. Reiger, *loc. cit.*, p. 998.

‡ H. Glaser, *Ann. d. Phys.*, 22, p. 719, 1907.

§ R. Ladenburg, *Ann. d. Phys.*, 22, p. 309, 1907.

‖ G. F. C. Searle, *loc cit.*, p. 603.

of observation, there was no observable acceleration. We may conclude that, even for the greatest values of M, the viscosity of the liquid remained sensibly constant, in spite of the fact that some potential energy was converted into heat.

The values of T found in these experiments are given in Table 1.

TABLE 1.

Time, in seconds, of one revolution of cylinder.

M grm.	$l=10{\cdot}0$ cm.	$l=8{\cdot}45$ cm.	$l=7{\cdot}65$ cm.	$l=5{\cdot}50$ cm.	$l=3{\cdot}30$ cm.	$l=2{\cdot}15$ cm.
5						129·0
7				114·6		50·7
10	120·4	108·7	100·2	71·3	44·7	
12	93·3	85·0	77·0	54·3		
15	71·5	65·3	59·0	42·3	26·6	19·5
20	52·8	47·0	41·7	29·7		
25	41·1	35·9	32·2	23·6	14·5	11·1
30	33·3	29·6	26·4	19·0		
35	28·8	24·8	22·2	16·3	10·0	
40	25·1	21·7	19·2	14·0		
45	22·0	19·0	17·1	12·3		
55	17·9	15·2	13·6	10·1	6·3	4·6
65	14·9	12·6	11·4	8·3		
75	12·8	10·9	9·8	7·2	4·5	3·4
105	9·1	7·7	7·0	5·1		
155	6·1	5·2	4·8	3·5		
205	4·6	3·9	3·6			

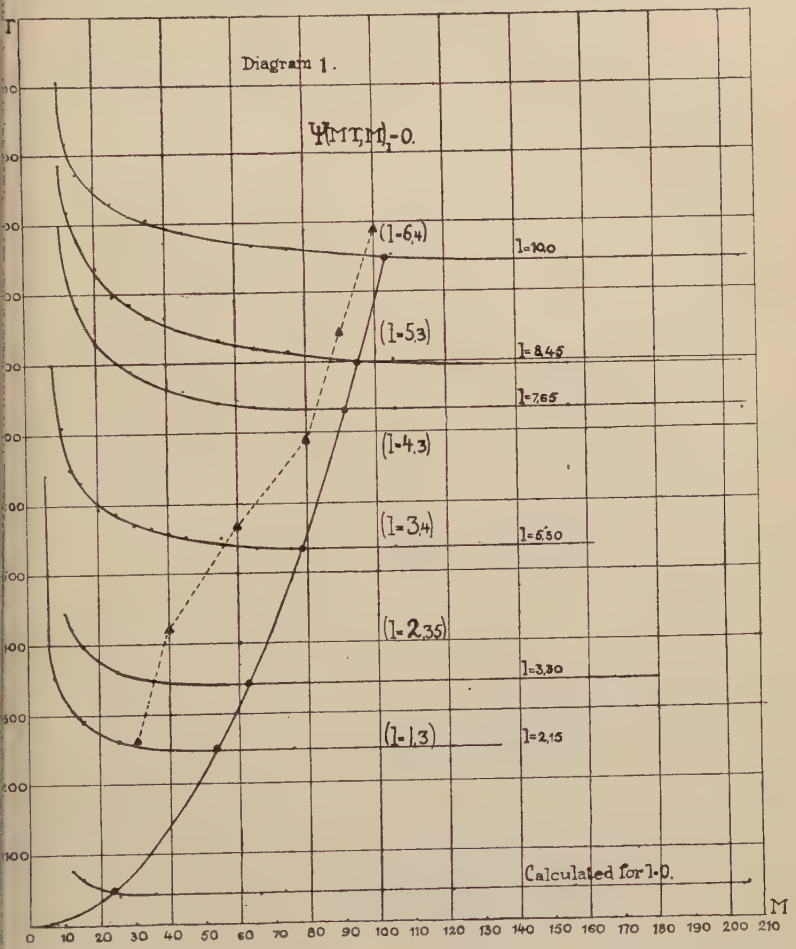
The results have been plotted in the form of six curves each for one value of l, as in Diagram 1. The curves are represented in the form $\Psi\,(MT, M)_{l=\text{const.}} = 0$.

From the diagram it is clear that the function $\Psi\,(MT, T)_l = 0$ does not represent a family of straight lines parallel to the M-axis, and that each of the six curves has a hyperbolic appearance. When M approaches a certain lower limit M_0, MT tends to infinity. The area covered by the group of curves can be divided by a parabolic boundary curve into two departments, in one of which MT is sensibly constant for a given value of l.

§ 4. I have, further, examined how MT depends upon l, when M is kept constant, and have found that the function

$$F\,(MT, l)_{M=\text{const.}} = 0$$

presents, not a single straight line*, but a family of approxiately straight lines. Each line can be represented by the equation $T = \alpha l + \beta$. For this group of curves $\partial (MT)/\partial l$ tends to a definite alue as M increases, i.e. the curves approach a certain border line



which is comparable with Searle's straight line. The coefficients α and β have been calculated for each line by the method of least squares†, using the formulae

$$\alpha = \frac{\Sigma l \,.\, \Sigma MT - 6\Sigma l \,.\, \Sigma lMT}{(\Sigma l)^2 - 6\Sigma l^2}, \quad \beta = \frac{\Sigma l \,.\, \Sigma lMT - \Sigma MT \,.\, \Sigma l^2}{(\Sigma l)^2 - 6\Sigma l^2},$$

* G. F. C. Searle, *loc. cit.*, p. 604.
† F. Kohlrausch, *Lehrbuch d. praktischen Physik*, p. 13, 1914.

the various observations being regarded as having equal weights. The values of α and β have been thus calculated for seven different lines, and the results are given in Table 2.

TABLE 2.

Values of α, β and k.

M grm.	α	β	k cm.
12	105·86	78·40	0·740
15	102·64	65·61	0·639
20	99·29	53·54	0·539
35	108·80	47·79	0·439
65	90·78	43·83	0·482
75	90·27	43·63	0·483
105	89·79	45·47	0·506

When $l = 0$, then $MT = \beta$, and Table 2 shows how β varies with M. The curve thus extrapolated for $l = 0$ is marked "Calculated for $l = 0$" in Diagram 1.

When $MT = 0$, we have $k = |\, l \,| = |\, \beta/\alpha \,|$, where k is the correction for the lower end of the rotating cylinder.

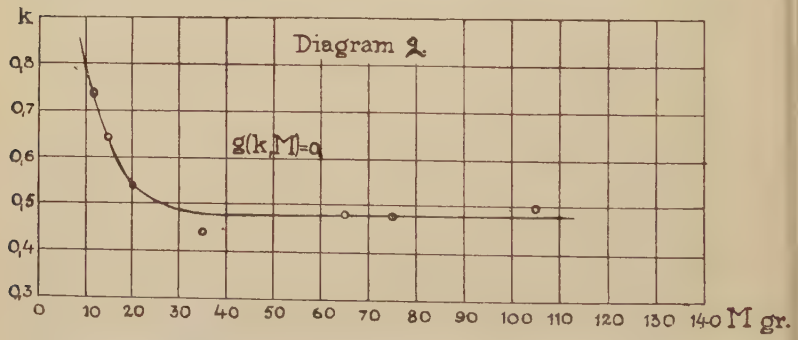


Diagram 2 shows how k depends upon M.

The facts here recorded show that equation (2) should be replaced by

$$\eta = C \frac{M_i T}{l + k_i}, \qquad \text{.....(3)}$$

where k_i is the value of k corresponding to the load M_i.

If the value of k_i corresponding to M_i is read off from the curve of Diagram 2, the viscosity η can be calculated by equation (3). The values of k found from Diagram 2 have been used in forming Table 3.

TABLE 3.

Values of $M_iT/(l + k_i)$.

M grm.	l=10·0 cm.	l=8·45 cm.	l=7·65 cm.	l=5·50 cm.	l=3·30 cm.
10	111·7	118·0	118·5	113·4	109·5
12	104·2	111·0	110·0	104·5	
15	101·2	108·0	106·8	103·2	101·0
20	99·3	104·5	102·0	98·6	
25	97·9	100·1	99·1	98·5	95·1
30	96·5	99·0	97·2	95·4	
35	95·6	97·0	95·6	94·7	92·5
40	95·0	97·0	94·5	93·7	
45	94·5	96·0	94·3	92·5	
55	93·3	94·0	91·5	92·4	91·6
65	92·4	92·0	90·7	89·7	
75	92·0	91·7	90·4	89·7	
105	91·1	90·2	90·4	90·1	
155	90·3	89·2	90·6	89·6	
205	90·0	89·3	89·3		

From Table 3 it appears that the area in Diagram 1 in which equation (3) holds good is restricted to that part of the diagram to which the parabolic boundary curve is convex. From the values of MT derived from Table 1 and plotted in Diagram 1, the equation of the parabola is found to be $M^2 = 11{\cdot}26\,(MT)$. I have not been able to give the parabola any definite physical interpretation, and it ought to be regarded as representing a diffuse limit region. But it is only when we pay regard to this, that we obtain values of η differing from each other by amounts lying within the limits of experimental error*. To make a comparison with the values of M and l which Dr Searle has used, I have, in Diagram 1, plotted (the broken line) his values of MT† (strictly speaking, MT/C, which are comparable in magnitude with my values of MT) against M.

Dr Searle has pointed out to me that the effect shown in Diagram 1 might conceivably be due to pivot friction. I have carefully considered this possibility. Before the liquid was put into the apparatus, I adjusted the pivots so that the rotation due

* Compare G. F. C. Searle, *loc. cit.*, Table II, p. 606.

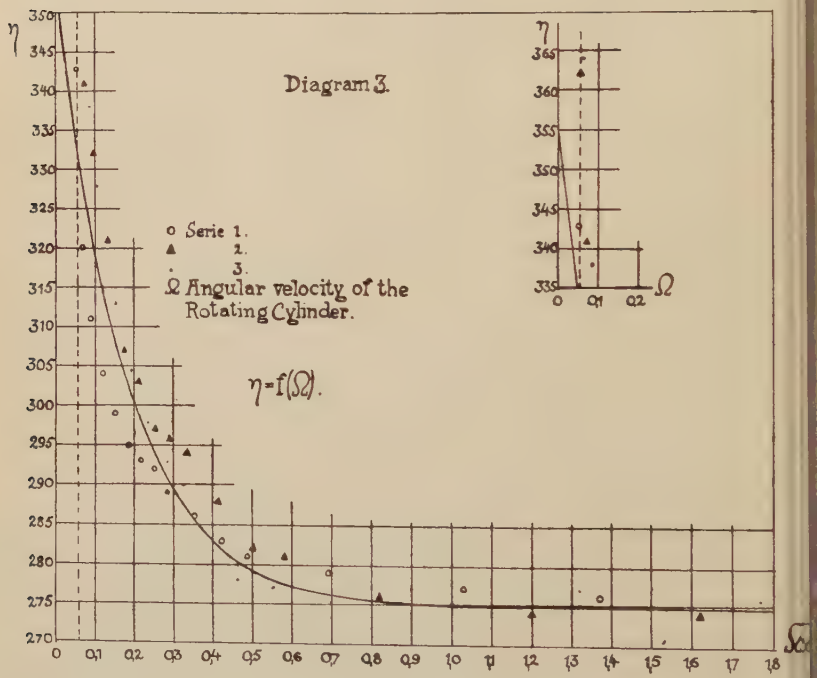
† Calculated from Table 1, G. F. C. Searle, *loc. cit.*, p. 605.

to the weights of the two empty pans (5 grm. each) was so rapid that I was hardly able to measure, for instance, $3T$ by using a stop watch. I have, therefore, not been able to take account of any pivot friction. This cause of error would, at any rate, produce effects much smaller than those actually found.

§ 5. From the results for $M = 205$ grm. given in Table 1 we find the mean value

$$\eta = 274{\cdot}7 \text{ dyne sec. cm.}^{-2},$$

for the temperature of + 19·8° C. To show how η depends upon



the angular velocity $\Omega = 2\pi/T$, the values of η and Ω, obtained from the first three series, have been plotted in Diagram 3. The curve drawn among the plotted points suggests that the relation between η and Ω can be expressed in the form

$$\eta = 274{\cdot}7 + \phi \exp(-\lambda\Omega^x).$$

To find the constants ϕ, λ and x, I considered the equation

$$\log_e(\eta - 274{\cdot}7) = \log_e \phi - \lambda\Omega^x. \quad \ldots\ldots\ldots\ldots(4)$$

When the values of $\log_\epsilon(\eta - 274{\cdot}7)$ were plotted against Ω, the curve was roughly a straight line. Hence x may be taken as unity, and thus the number of constants to be found is reduced to two. By the method of least squares, I obtained $\log_\epsilon \phi = 4{\cdot}375$ and $\theta = 5{\cdot}694$, and thus

$$\eta_{19{\cdot}8} = 274{\cdot}7 + 79{\cdot}44\,\epsilon^{-5{\cdot}694\,\Omega}. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Equation (5) expresses the results of the observations when Ω exceeds 0·1, but not for smaller values of Ω.

§ 6. Experiments carried out at different temperatures showed that the curves representing the function

$$\Psi(MT, M)_{l=\text{const.}} = 0$$

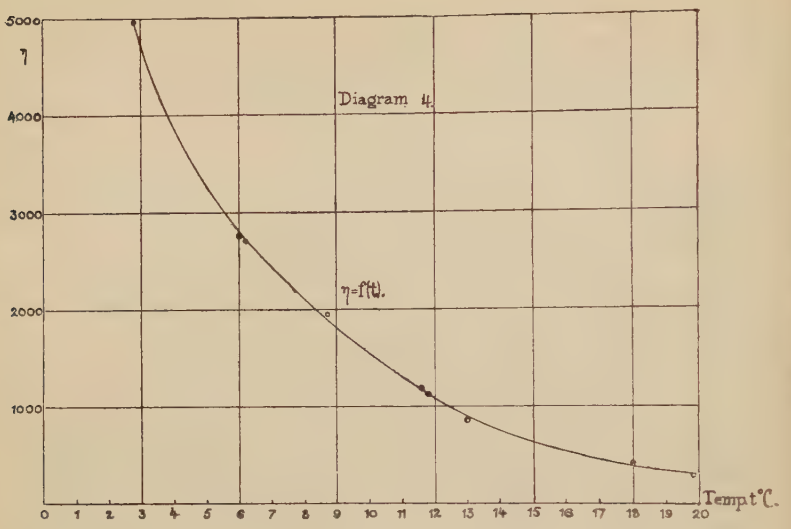
are of the same character as those given in Diagram 1. Table 4 gives the values of η found for various temperatures. In these experiments l was 10·0 cm.; and, at each temperature, six different loads were used, in order that I might be able to decide with certainty that the values of M, used in calculating the value of η for each temperature, lay in the area to the right of the parabolic boundary line of Diagram 1. The same value of k, viz. the limiting value 0·48 cm. shown in Diagram 2, was used in calculating the

TABLE 4.

Values of η at various temperatures.

Temp. $t°$ C.	η Dyne sec. cm.$^{-2}$	Temp. $t°$ C.	η Dyne sec. cm.$^{-2}$
19·8	274·7	8·75	1950
18·0	415	6·2	2700
13·0	860	6·0	2750
11·8	1140	2·8	4970
11·6	1200		

various values of η. These values are not claimed to be exact. In these experiments it was very difficult to keep the temperature constant during each series of observations, and thus a determination of k for each temperature was out of the question. From the curve of the function $\eta = f(t)$, shown in Diagram 4, it follows that $|\, d\eta/dt \,|$ rises rapidly as η increases; this tallies with what was said above.



§ 7. I thought it would be interesting to compare the results given by Searle's method with those obtained by Poiseuille's method. The utility of the latter method for very viscous liquids* is proved by the investigations of Kahlbaum and Räber† for values of η in the neighbourhood of 40, and by Ladenburg‡ for $\eta = 1{\cdot}3 \times 10^3$. Fausten§ has found that the length of the discharge tube must exceed 45 cm., if the simple Poiseuille formula

$$\eta = \pi g h R^4 \rho^2 t / 8 L m$$

is to represent actual facts. In the formula

h = Height of liquid corresponding to difference of pressure between ends of tube.

R = Internal radius of tube. L = Length of tube.

ρ = Density of liquid (= $1{\cdot}4103 \pm 0{\cdot}0003$ grm. cm.$^{-3}$ at 19·8° C.).

m = Mass of liquid discharged. t = time of discharge.

For shorter tubes, Hagenbach's* correction must be employed; otherwise the value obtained for η will be too high. As the liquid flows out into the air in an even jet, it carries kinetic energy with it; in order to allow for this, the value of η given by Poiseuille's

* H. Glaser, *Erlangen Diss.*, 1906.
† G. W. A. Kahlbaum and S. Räber, *Acta Ac. Leop.*, 84, p. 204, 1905.
‡ R. Ladenburg, *Ann. d. Phys.*, 22, p. 298, 1907.
§ A. Fausten, *Bonn. Diss.*, 1906.

formula must be multiplied, according to Hagenbach*, by a correcting factor slightly less than unity. As the thermostat could only accommodate tubes shorter than 45 cm., Hagenbach's correction was calculated, but was found to be negligible. Ladenburg† points out that both Hagenbach's and Couette's corrections to Poiseuille's formula can be entirely ignored for liquids such that η is of the magnitude $1{\cdot}3 \times 10^3$.

The discharge vessel consisted of a wide glass cylinder; through the bottom of this was bored a hole through which the discharge tube was connected with the interior of the cylinder. The whole apparatus was placed in the thermostat and the same temperature, 19·8° C., was maintained as was used in the earlier experiments. When a tube whose internal radius was about 0·26 cm. was used, the liquid did not issue in a continuous jet but in drops. The values obtained for η are given in Table 5. The mean value is $\eta = 271{\cdot}1$. The value obtained by Searle's method, viz. 274·7, differs from that obtained by Poiseuille's method by 1·3 per cent.; the agreement may be regarded as good.

TABLE 5.

Values of η by Poiseuille's method.

R cm.	l cm.	h cm.	m grm.	t sec.	η
0·3168	46·48	49·36 49·93	54·421 53·568	1790 1757	269·9 272·3

§ 8. The influence of the base of the rotating cylinder can be eliminated, without determining k, by using the relation ‡

$$\eta = C \cdot \frac{M_1T_1 - M_2T_2}{l_1 - l_2} = C\gamma,$$

provided that the points corresponding to M_1T_1 and M_2T_2 lie to the right of the parabolic boundary line in Diagram 1. If we put $l_1 = 10{\cdot}0$ cm., we obtain the results given in Table 6.

* F. Kohlrausch, *Lehrbuch d. praktischen Physik*, pp. 264—269, 1914.
† R. Ladenburg, *loc. cit.*, p. 298.
‡ Compare C. Brodman, *loc. cit.*, p. 163.

Table 6.

Values of γ.

MT	944	804	734	537	342	251
l	10·0	8·45	7·65	5·50	3·30	2·15
γ		90·2	89·3	90·5	90·0	88·4

When the various values are given the same weight, the mean value of γ is 89·7, and then $\eta = 275{\cdot}4$.

§ 9. Diagram 3 and formula (5) show that η cannot be regarded as independent of Ω unless Ω exceed a certain value, in this case 0·9. Since Ω is related to the rate of shearing $r\,d\omega/dr$, according to the formula

$$r\frac{d\omega}{dr} = -\Omega \cdot \frac{15{\cdot}80}{r^2},$$

it follows that η is a function of the rate of shearing. Hence, the assumption on which formula (1) is based, viz. that η is independent of the rate of shearing, seems to be unjustifiable for small values of the rate of shearing, at least in the case of the highly viscous liquid used in these experiments.

Preliminary Note on Antennal Variation in an Aphis (Myzus ribis, *Linn.*). By Maud D. Haviland, Fellow of Newnham College. (Communicated by Mr H. H. Brindley.)

[*Read* 8 March 1920.]

In 1918, during an investigation of the life-history of the Red Currant Aphis, *Myzus ribis*, Linn., it was observed that considerable variation occurred in the antennae of the winged parthenogenetic females; and the evidence pointed to the conclusion that this variation was induced by the food[3]. Antennal variation in certain Aphididae has been studied by Warren[5], Kelly[4], Ewing[2] and Agar[1]. Warren's experiments on *Hyalopterus trirhodus* showed some diminution of the correlation co-efficient in passing back from parent to grandparent. Kelly, for *Aphis rumicis*, considered that somatic variations of the parents were not inherited by the offspring. Ewing, who bred eighty-seven generations of *Aphis avenae*, concluded that the variations were not transmitted to the offspring.

Agar found some evidence of a partial inheritance of individual variations in *Macrosiphum antherini*, but he showed that this might be due to causes other than true inheritance.

Myzus ribis is a common pest of red currant bushes. The sucking of the aphides upon the leaves tends to cause red galls or blisters, within which the plant lice continue to feed and reproduce. The fifth and sixth antennal segments of the winged parthenogenetic females normally bear two sense organs of unknown function—one on the distal third of Seg. v., the other on the proximal third of Seg. vi. It was observed in 1918 that, in individuals reared on red blistered leaves, these sensoria were placed comparatively close to the articulation of Segs. v. and vi. On the other hand, if the aphides were fed upon green unblistered leaves, the sensoria were placed further away from the articulation.

For the sake of brevity, the first type of antenna will be referred to hereafter as the Red (or R) type, and the second as the Green (or G) type; but every degree of transition may exist between the two extreme types.

The experiments of 1918 were incomplete, and were conducted with a polyclonal population. They were repeated in 1919 with a monoclonal population, but the results are still far from being conclusive owing to the small numbers available in some generations. Only the winged forms show the required character. The production of these forms is probably governed by environmental factors which at present are imperfectly understood, and, for some

reason, in the population used in 1919, it was unusually low. It is hoped to repeat and extend the range of the experiments in 1920.

The character chosen is the distance between the sensoria of antennal segments v. and vi. and the articulation of these two segments, expressed as the percentage of the width of the head between the eyes. The ratios are shown separately for each segment, with a dividing line to represent the articulation.

Thus $\frac{19}{8}$ denotes that $\frac{\text{Seg. vi.} = 19\% \text{ of the head-width}}{\text{Seg. v.} = 8\% \text{ of the head-width}}$.

Each generation is designated by combinations of two letters: R (= red leaves) and G (= green leaves) and numerals, which express its complete ancestry. Thus R_2G_2 denotes the fourth generation from the fundatrix of the population, and the F_1 generation after transference to Green leaves after two consecutive generations on Red blistered leaves. In the transferred generations, the aphides were removed to the new environment when less than twelve hours old. The individuals for transference were selected wholly at haphazard. Thus, if a brood mother R_2 gave birth to four young in the day, two were transferred to red blistered leaves, and two to green leaves, and so on in equal numbers from day to day.

The pure Red (RRR, etc.) lines, and pure Green (GGG) lines were used as controls. The latter unfortunately became extinct in the third (G_3) generation. Hence for later generations the next longest unbroken line on green leaves (R_2G_2, etc.) had perforce to be taken as the control, though as it had been fed for the first two generations upon red leaves, it cannot be regarded as wholly satisfactory. In Table 1, the curves of error of the ratios of generations R_2, R_4 and R_2G_4 are shown. R_2 is the common ancestral generation. The mode of the curve of R_4 tends to shift to the left, *i.e.* the ratios of the antennal segments to the head-width are smaller. For the sake of clearness, in the graph only the curve of R_4 is shown, but those of R_3, R_5 and R_6, though with a smaller number of individuals, are almost identical with it. The curves of the ratios of R_2G_1 and R_2G_2 are very similar to their red controls. The R_2G_3 generation produced very few winged individuals, but these indicate a somewhat greater range of variation in Seg. vi. The curve of R_2G_4, as shown in the graph, has a marked tendency to shift to the right, indicating that the ratio of the antennal joints to head-width has increased, and this tendency is maintained in the succeeding generations, R_2G_5 and R_2G_6. The position in the generation series does not account for the change in the antennal structure, for the modes for the six Red generations are nearly identical.

So far we have considered only the modes. The mean ratios of the different generations are dealt with in the succeeding tables.

Table 2 shows the mean ratios of the successive generations in four lines of descent, including the red and green controls. The extinction of the green control line was unfortunate, and in future experiments it will be very desirable to obtain a pure green line. At present the explanation that suggests itself of the variation of the $R_2G_2\ldots$ line is that the influence of red feeding persists for at least two, and probably three generations after removal to different food, and this is somewhat confirmed by the $R_4G_1\ldots$ etc. line.

Tables 3, 3*a*, 4, 4*a* and 5, 5*a*, give the effect of transference upon the mean ratios of the first, second, and third generations respectively, and below each is an analysis of the ratio of each segment, indicating its increase or decrease over previous generations and the controls.

Examination of the figures seems to show that the ratios of the first generation after transference vary irrespectively of the parental ratio.

In transference to Red, the ratio of Seg. v. increases over that of the parental ratio, but in Seg. vi. it decreases (Table 3). In transference to Green, the results for both segments are quite inconclusive as regards the parental ratio (Table 3*a*). In the second generation after transference to Red, the results are likewise inconclusive for both segments (Table 4). After transference to Green, the ratio of Seg. v. shows a tendency to rise above, and Seg. vi. a tendency to fall below, the parental and grandparental ratios (Table 4*a*).

In the third generation after transference to Red, the ratio of Seg. v. rises above the ancestral ratios, and that of Seg. vi. falls (Table 5). After transference to Green, the ratio of Seg. v. rises above those of the ancestral generations, and that of Seg. vi. rises in one case and falls in the other (Table 5*a*).

These results are inconclusive, but examination of the control ratios shows that, with occasional exceptions, the ratio of a generation with a mixed ancestry tends to rise above that of the Red control, but remains below that of the Green. Many more experiments in transference are required, and a much larger number of individuals must be examined before any conclusion can be reached; but at present the evidence suggests that the antennae of *Myzus ribis* are modified according to the food supplied, and that the effect induced by feeding in one generation is discernible in the succeeding three or four generations. It is difficult otherwise to explain the difference between the ratios of R_6 and R_2G_4, and between R_6 and R_4G_3, which, translated into the terms of human relationship, would be third cousins, and first cousins once removed, respectively, for all were produced by parthenogenesis, and, except for the food, reared side by side under identical environmental conditions.

TABLE 1. *Curves showing the ratio of the distance of the sensoria from the articulation of antennal Segments V and VI to the width of head. The lower curves refer to the fifth, and the upper to the sixth segment.*

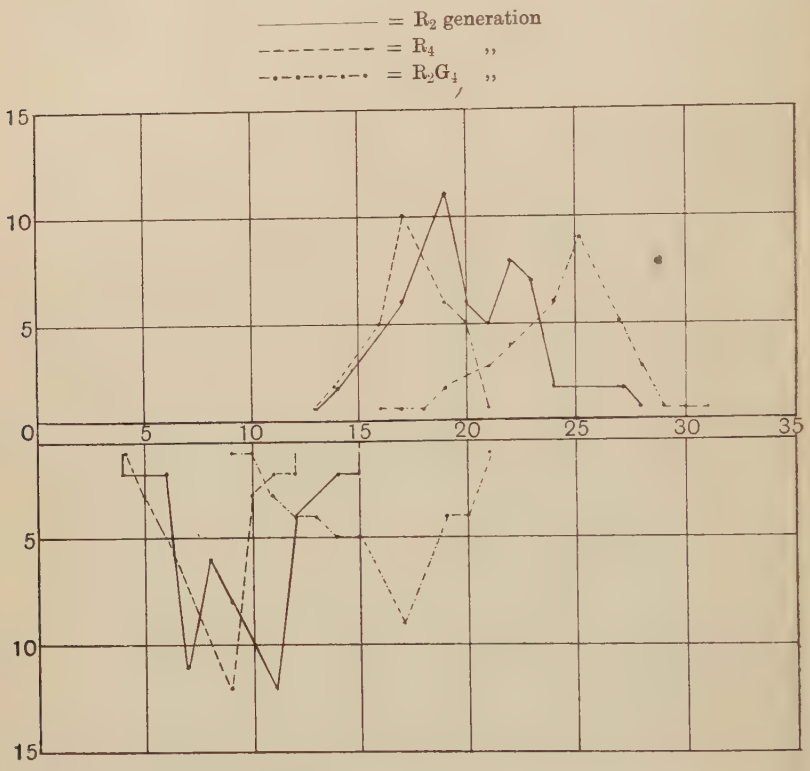


TABLE 2. *Mean ratios of the successive generations of the lines, G_2 ..., R_3 ..., R_4G_1 ..., R_2G_1 ..., and $R_2G_3R_1$ ....*

II	III	IV	V	VI	VII	VIII
$G_2\ \frac{22}{8}$	$G_3\ \frac{25}{19}$	—	—	—	—	—
$R_2\ \frac{20}{9}$	$R_3\ \frac{18}{8}$	$R_4\ \frac{17}{8}$	$R_5\ \frac{14}{9}$	$R_6\ \frac{18}{8}$	—	—
,, ,,	,, ,,	,, ,,	$R_4G_1\ \frac{22}{7}$	$R_4G_2\ \frac{19}{9}$	$R_4G_3\ \frac{29}{15}$	—
,, ,,	$R_2G_1\ \frac{19}{7}$	$R_2G_2\ \frac{18}{10}$	$R_2G_3\ \frac{17}{10}$	$R_2G_4\ \frac{24}{15}$	$R_2G_5\ \frac{26}{18}$	$R_2G_6\ \frac{25}{15}$
,, ,,	,, ,,	,, ,,	,, ,,	$R_2G_3R_1\ \frac{20}{10}$	$R_2G_3R_2\ \frac{17}{7}$	—

TABLE 3. *Mean ratios of the first generation transferred from Green leaves to Red blisters, with an analysis below.*

\+ = increase over ancestral ratio
\− = decrease from ,, ,,
0 = identical with ,, ,,

Generation Ratio	Parental Generation Ratio	Red Control Ratio	Green Control Ratio
$G_1R_1\ \frac{21}{8}$	G no winged forms	$R_2\ \frac{20}{9}$	$G_2\ \frac{22}{8}$
$G_2R_1\ \frac{21}{10}$	$G_2\ \frac{22}{8}$	$R_3\ \frac{18}{8}$	$G_3\ \frac{25}{19}$
$R_2G_1R_1\ \frac{17}{9}$	$R_2G_1\ \frac{19}{7}$	$R_4\ \frac{17}{8}$	$R_2G_2\ \frac{18}{10}$
$G_1R_2\ \frac{19}{8}$	$G_1R_1\ \frac{21}{8}$	$R_3\ \frac{18}{8}$	$G_3\ \frac{25}{19}$
$R_4G_1R_1\ \frac{19}{8}$	$R_4G_1\ \frac{22}{7}$	$R_6\ \frac{18}{8}$	$R_2G_4\ \frac{24}{15}$
$R_2G_3R_1\ \frac{20}{10}$	$R_2G_3\ \frac{17}{10}$	$R_6\ \frac{18}{8}$	$R_2G_4\ \frac{24}{15}$
$R_2G_4R_1\ \frac{21}{11}$	$R_2G_4\ \frac{24}{15}$	$R_6\ \frac{18}{8}$	$R_2G_5\ \frac{26}{18}$

Segment V

Generation	Variation from Parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_1	no winged forms	−	0
G_2R_1	+	+	−
$R_2G_1R_1$	+	+	−
G_1R_2	0	0	−
$R_4G_1R_1$	+	0	−
$R_2G_3R_1$	0	+	−
$R_2G_4R_1$	−	+	−

Segment VI

Generation	Variation from Parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_1	no winged forms	+	−
G_2R_1	−	+	−
$R_2G_1R_1$	−	0	−
G_1R_2	−	+	−
$R_4G_1R_1$	−	+	−
$R_2G_3R_1$	0	+	−
$R_2G_4R_1$	−	+	−

TABLE 3*a*. *Mean ratios of the first generation transferred from Red blisters to Green leaves, with analysis as in Table* 3.

Generation Ratio	Parental Generation Ratio	Green Control Ratio	Red Control Ratio
R_1G_1 $\frac{19}{8}$	R no winged forms	G_2 $\frac{22}{8}$	R_2 $\frac{20}{9}$
R_2G_1 $\frac{19}{7}$	R_2 $\frac{20}{9}$	G_3 $\frac{25}{19}$	R_3 $\frac{18}{8}$
$G_1R_1G_1$ $\frac{19}{9}$	G_1R_1 $\frac{21}{8}$	G_3 $\frac{25}{19}$	R_3 $\frac{18}{8}$
R_3G_1 $\frac{18}{8}$	R_3 $\frac{18}{8}$	G_3 $\frac{25}{19}$	R_4 $\frac{17}{8}$
$G_1R_2G_1$ $\frac{19}{9}$	G_1R_2 $\frac{19}{8}$	R_2G_2 $\frac{18}{10}$	R_4 $\frac{17}{8}$
R_4G_1 $\frac{22}{7}$	R_4 $\frac{17}{8}$	R_2G_3 $\frac{17}{10}$	R_5 $\frac{14}{9}$

Segment V

Generation	Variation from Parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_1G_1	no winged forms	0	−
R_2G_1	−	−	−
$G_1R_1G_1$	+	−	+
R_3G_1	0	−	0
$G_1R_2G_1$	+	−	+
R_4G_1	−	−	−

Segment VI

Generation	Variation from Parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_1G_1	no winged forms	−	−
R_2G_1	−	−	+
$G_1R_1G_1$	−	−	+
R_3G_1	0	−	+
$G_1R_2G_1$	0	+	+
R_4G_1	+	+	+

TABLE 4. *Mean ratios of the second generation after transference from Green leaves to Red blisters, with analysis as in Table 3.*

Generation Ratio	Parental Generation Ratio	Grand-parental Generation Ratio	Red Control Ratio	Green Control Ratio
G_1R_2 $\frac{19}{8}$	G_1R_1 $\frac{21}{8}$	G_1 no winged forms	R_3 $\frac{18}{8}$	G_3 $\frac{25}{19}$
G_2R_2 $\frac{18}{11}$	G_2R_1 $\frac{21}{10}$	G_2 $\frac{22}{8}$	R_4 $\frac{17}{8}$	G_3 $\frac{25}{19}$
$R_2G_3R_2$ $\frac{17}{7}$	$R_2G_3R_1$ $\frac{20}{10}$	R_2G_3 $\frac{17}{10}$	R_6 $\frac{18}{8}$	R_2G_5 $\frac{26}{18}$

Segment V

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_2	0	no winged forms	0	−
G_2R_2	+	+	+	−
$R_2G_3R_2$	−	−	−	−

Segment VI

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_2	−	no winged forms	+	−
G_2R_2	−	−	+	−
$R_2G_3R_2$	−	0	−	−

TABLE 4*a*. *Mean ratios of the second generation after transference from Red blisters to Green leaves, with analysis as in Table* 3.

Generation Ratio	Parental Generation Ratio	Grand-parental Generation Ratio	Green Control Ratio	Red Control Ratio
$R_2G_2\ \frac{18}{10}$	$R_2G_1\ \frac{19}{7}$	$R_2\ \frac{20}{9}$	$G_3\ \frac{25}{19}$	$R_4\ \frac{17}{8}$
$R_4G_2\ \frac{19}{9}$	$R_4G_1\ \frac{22}{7}$	$R_4\ \frac{17}{8}$	$R_2G_4\ \frac{24}{15}$	$R_6\ \frac{18}{8}$

Segment V

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_2G_2	+	+	−	+
R_4G_2	+	+	−	+

Segment VI

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_2G_2	−	−	−	+
R_4G_2	−	+	−	+

TABLE 5. *Mean ratios of the third generation after transference from Green leaves to Red blisters, with analysis as in Table 3.*

Generation Ratio	Parental Generation Ratio	Grand-parental Generation Ratio	Great-grand-parental Generation Ratio	Red Control Ratio	Green Control Ratio
G_1R_3 $\frac{18}{9}$	G_1R_2 $\frac{19}{8}$	G_1R_1 $\frac{21}{8}$	no winged forms	R_4 $\frac{17}{8}$	G_3 $\frac{25}{19}$

Segment V

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Great-grand-parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_3	+	+	no winged forms	+	−

Segment VI

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Great-grand-parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_3	−	−	no winged forms	+	−

TABLE 5*a*. *Mean ratios of the third generation after transference from Red blisters to Green leaves, with analysis as in Table* 3.

Generation Ratio	Parental Generation Ratio	Grand-parental Generation Ratio	Great-grand-parental Generation Ratio	Green Control Ratio	Red Control Ratio
R_2G_3 $\frac{17}{10}$	R_2G_2 $\frac{18}{10}$	R_2G_1 $\frac{19}{7}$	R_2 $\frac{20}{9}$	G_3 $\frac{25}{19}$	R_5 $\frac{14}{9}$
R_4G_3 $\frac{29}{15}$	R_4G_2 $\frac{19}{9}$	R_4G_1 $\frac{22}{7}$	R_4 $\frac{17}{8}$	R_2G_5 $\frac{26}{18}$	R_6 $\frac{18}{8}$

Segment V

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Great-grand-parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_2G_3	0	+	+	−	+
R_4G_3	+	+	+	−	+

Segment VI

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Great-grand-parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_2G_3	−	−	−	−	+
R_4G_3	+	+	+	+	+

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The effect of a magnetic field on the Intensity of spectrum lines. By H. P. Waran, M.A., Government Scholar of the University of Madras. (Communicated by Professor Sir Ernest Rutherford.)

[*Read* 8 March 1920.]

[Plates I and II.]

Since the discovery of the Zeeman effect the main attention has been directed to the detailed study of the phenomenon of the small change of wave length suffered by a monochromatic radiation in a magnetic field. The question whether a magnetic field affects the spectrum as a whole has not received much attention.

While working on the Zeeman effect with a mercury discharge tube run by an induction coil as the source, a small portion of the capillary tube being subjected to a magnetic field of about 5000 c.g.s. units as shown in Fig. 1, the light was observed to suffer a change in intensity and also in colour opposite the pole pieces when the field was thrown on. A spectroscopic examination revealed the existence of some selective changes in the spectrum in addition to the increased brilliancy of the general spectrum. It was also noticed that the changes taking place varied considerably with the pressure, at a low pressure the tube showing little change visually but greater changes in the general spectrum. Attention was concentrated on the latter.

In the case of mercury which was the first spectrum investigated, the tube, containing a trace of residual air at very low pressure, gave the principal mercury lines, viz.:

5790·66, 5769·6, 5460·7, 4916·0, 4358·34

and the principal hydrogen lines

6563, 4861·5 and 4340·7.

On applying the magnetic field, however, marked changes were observed, including a new set of lines at

5426, 5679, 5872 and 5889,

and a very strong red line at 6152, brought out prominently by the field. Mercury lines have been recorded at these wave lengths and these lines brought out are probably due to mercury. The behaviour of the line 6152 was very remarkable. It was invisible under ordinary conditions but showed up brilliantly in the magnetic field, the effect being practically instantaneous. Exhausting the tube still further and increasing the current through the tube to about 5 m.a. Four faint lines appeared at wave lengths

6234, 6152, 6123 and 6072,

and corresponding to these wave lengths mercury lines are recorded by* Stiles, Eder, Valenta, Arons and Hermann. But Arons and

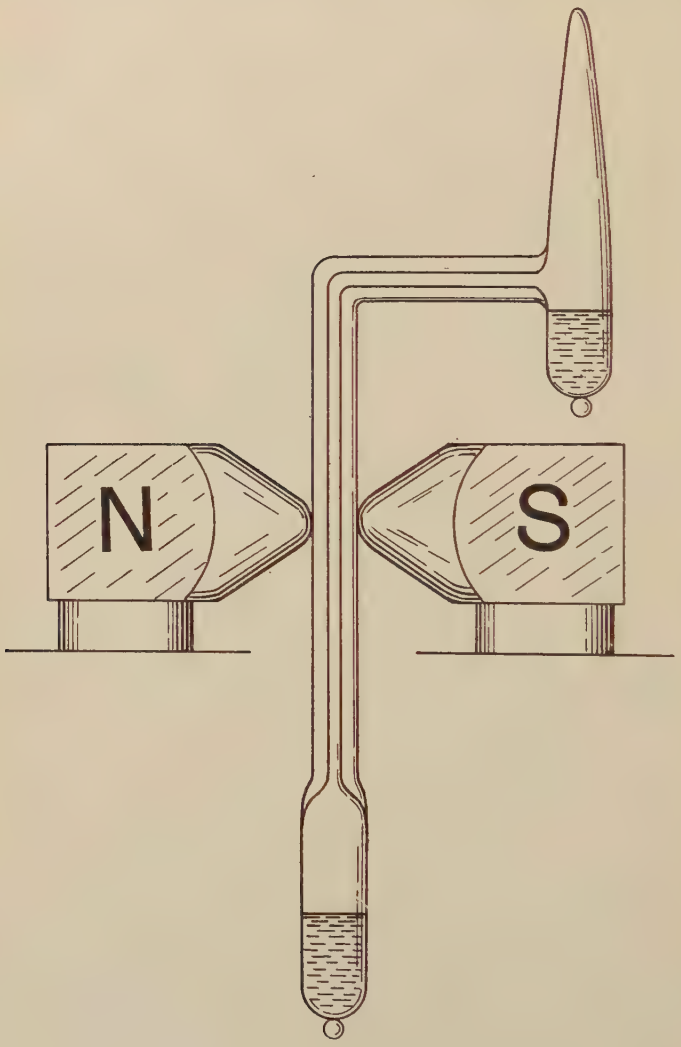

Fig. 1.

Hermann have not recorded the line 6152, while Stiles records it as of equal intensity with the line 6234. Eder and Valenta have not observed the latter lines at all, but record the line 6152 as

* Kayser, *Handbuch der Spectroscopie*, Band v. p. 538.

one of very great intensity. Examining the effect of the magnetic fields on these four lines, it is very interesting to note that the line 6152 alone increases about five times in brilliancy while the others if they suffer any change at all, decrease in intensity. It is also interesting to note that this line 6152 seems to be the same line that becomes so greatly enhanced when the tube contains a trace of helium as observed by* Collie. It seems very difficult to excite this line unless at least a trace of helium is present in the apparatus and at this stage it is not possible to suggest any explanation of its abnormal behaviour.

In addition to these very prominent changes there are also many minor changes, among which is the disappearance of a faint trace of continuous spectrum, as well as of some of the nebulous bands and lines, the remaining lines being quite sharp on a dark background.

The abnormal behaviour of the mercury spectrum in the visible region (the ultra violet spectrum has not yet been investigated) suggested the study of other spectra and the spectrum of helium was next examined.

The discharge tube contained hydrogen and a slight trace of mercury vapour as impurity and the hydrogen lines and the prominent mercury lines were also visible. The effect of the magnetic field in this case was to enhance the helium lines very considerably, leaving the hydrogen lines practically unaffected or even slightly reduced in intensity. In this spectrum there were also a few faint lines not yet identified definitely which remain quite unaffected by the magnetic field. In the further study of the helium spectrum, the gas was contained in a separate tube from which any small quantity of it could be introduced into the discharge tube. At a pressure of 1 mm. of mercury the addition of a small trace of helium produced no perceptible effect on the spectrum of residual air which showed the prominent hydrogen lines and the nitrogen bands, but no trace of any of the helium lines. But on switching on the magnetic field, the helium lines flashed out prominently and disappeared again as soon as the field was turned off. The effect is shown in the accompanying photographs (Plates I and II). In a plate taken with a greater percentage of helium the lines are visible without the magnetic field, but a great enhancement of these lines with the field is evident, and a dense new line at 4933·4 A.U. is also noticed which has not yet been definitely identified.

The spectrum of neon was also studied, and in a tube kindly lent to me by Dr Aston, there was a trace of hydrogen also present, showing the three principal hydrogen lines. Here also the effect of the field was to enhance very considerably the neon lines,

* *Proc. Roy. Soc.* 71, 25, 1902.

leaving the hydrogen lines comparatively unaffected, so that by a casual examination of the spectrum the hydrogen and the neon lines can be distinguished from one another.

The oxygen spectrum is rather difficult to excite when mixed with other gases. Yet a mixture of hydrogen, oxygen and a trace of helium was tried with success and here again the monatomic helium lines were brougIt out by the magnetic field, leaving the diatomic oxygen and hydrogen lines comparatively unaffected as shown in the photographs.

From these experiments the natural inference follows that in a mixture of the monatomic and diatomic gases, the monatomic gases alone seem to be selectively affected in a peculiar way resulting in their spectrum lines alone being very considerably enhanced or brought out prominently even when not visible at all previously. By this method minute traces of the monatomic gases when mixed with other diatomic gases can be detected. On this view we might also explain the abnormal mercury line 6152 and others as due to the radiation from the monatomic atom while the other lines may be classified as belonging to the molecule.

Examining the spectrum of the atmospheric air at low pressure in this way the effect of the magnetic field is to bring out new lines which are not present without the magnetic field, as shown in the photographs. As far as their wave lengths have been determined, though one or two of them fit in fairly well with lines catalogued as belonging to oxygen and nitrogen, yet there are others which are difficult to identify while the absence of other stronger lines of oxygen and nitrogen make even these two or three fits inconclusive.

Another interesting point noted in these experiments is the varying degrees of enhancement under the influence of the field for lines belonging to the same element helium. Preston has shown that the Zeeman effect is of the same magnitude for lines belonging to the same series, but differs in different series. Similarly we might expect the degree of enhancement of the lines in the magnetic field to depend on the series to which the line belongs.

The exact nature of this phenomena and the mechanism of the reaction that brings about these novel changes in the general spectrum is not yet definitely known and it is not desirable to attempt an explanation until the study of the spectrum has been extended to the ultra violet.

The current in the tube was usually about 3 m.a. and the effect of the field was to decrease the current by about 20 to 30 per cent. The changes of intensity observed cannot be attributed to this since the reduction of the current by a spark gap in series only brings about a proportionate decrease in brilliancy of the general spectrum.

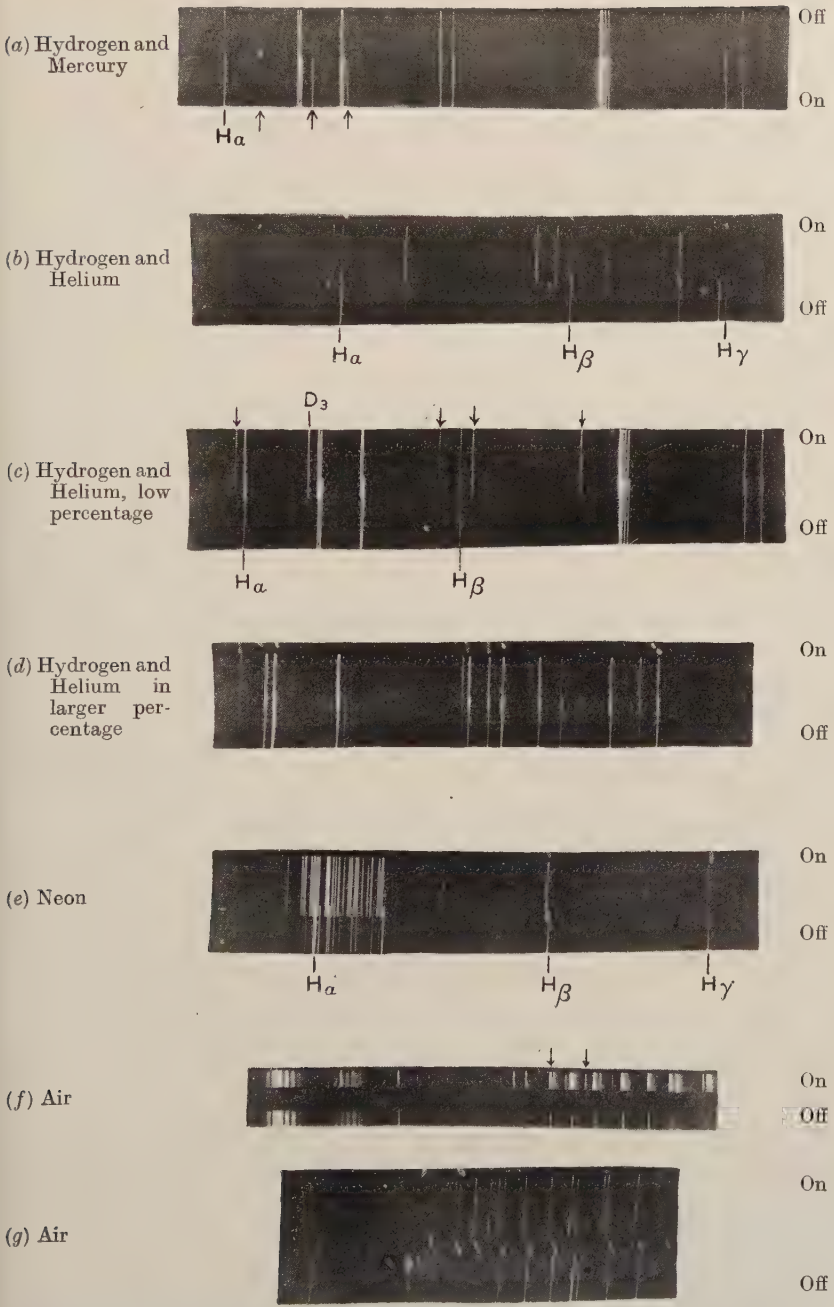

Fig. 2. Photographs showing the enhancing effect of the field. The small lateral shift is due to the camera slider, and in (*a*) the mercury line 6152 is indicated by the dot, while in the other cases the lines that newly turn up are indicated by the arrows.

It may be of interest to note that in solar spectroscopy the spectrum of the sunspots is found to differ in many respects from that of the photosphere, considerable numbers of enhanced lines occurring in the sunspot spectrum. The existence of a powerful magnetic field in sunspots has been demonstrated by the Zeeman effect and possibly the differences in the spectrum of the sunspot and the photosphere may be attributed to this new effect of the magnetic field on the spectrum.

The further study of this effect and the examination of other spectra are in progress.

CAVENDISH LABORATORY,
CAMBRIDGE.

Further Notes on the Food Plants of the Common Earwig (Forficula auricularia). By H. H. Brindley, M.A., St John's College.

[*Read* 8 March 1920.]

In a paper published in the *Proceedings of the Cambridge Philosophical Society*, xix, Part 4, July 1918, p. 170, I recorded certain observations in August and September, 1917, on the food plants of the Common Earwig, with the view of obtaining more exact information than was then available as to the damage likely to be done by this species in a flower or kitchen garden. The paper also epitomised recent literature on the subject, a consideration of which had revealed a considerable amount of diversity and want of exact information as to the favourite food plants of earwigs in the British Isles. The observations made by myself were on earwigs kept in captivity in connection with a statistical enquiry as to the variation of the forcipes which is still in progress. The observations in 1917 were on earwigs from St Mary's, Isles of Scilly, and those recorded in the present paper were made in the second half of the year 1918 on a collection from the Bass Rock, which swarms with earwigs. The animals were all adults and were kept in large glass dishes bedded with sand slightly damped occasionally. Earwigs remain healthy in a soaked substratum if the ventilation is good, but in captivity in a warm room without circulation of air they suffer heavy mortality from fungoid attack, as I have already recorded (*Proc. Camb. Phil. Soc.*, xvii, Part 4, Feb. 1914, pp. 335–338). The fungus appears to be usually *Entomophthora forficulae* (Picard, *Bull. Soc. Étude Vulg. Zool. Agric. Bordeaux*, Jan.–Apr., 1914, pp. 25, 37, 62). The importance of ventilation and of normal temperature is well illustrated by the far fewer fungoid attacks and the low mortality when the new Insect House belonging to the Cambridge Zoological Laboratory became available in 1919. It is at present too early to say how far an improvement is obtainable in the survival of eggs and young which it is hoped to rear in the spring in normal outside temperatures in the Insect House. Earwigs offer a great contrast to cockroaches as regards desire for water; the latter thrive in captivity for months in a warm room on food which is entirely dry, while earwigs certainly visit water to drink, as I have seen in both the captive and wild conditions. I have previously recorded (*Proc. Zool. Soc. Lond.*, Nov. 1897, p. 913) how *Stylopyga orientalis* in captivity seems to pay no attention to a damp sponge when that is the only source of moisture. We have however to bear in mind that the Common Cockroach is probably an

immigrant from warmer countries of the East. The earwigs under observation during the past three summers had no animal food save that afforded by those which died. In order to obtain information as to preference for one kind of plant above another they were usually given three different species, taken haphazard, at a time, for a period of two or more days.

In the following summary the observations of 1917 and 1918, with a few made in 1919, are combined. The dates when the different foods were given are noted, as in the latter part of September, when the animals tend to become lethargic, and in the succeeding two months the desire for food is much lessened, even in the artificial temperature of a laboratory. The capital letters after the names of the plants indicate those which were given at the same time, and the numbers appended indicate the preference exhibited by the earwigs: *e.g.* in food group M, M^1 was attacked more than M^2, M^2 more than M^3; in group F, F^1 after two plants indicates that they seemed to be attacked equally, and more readily than F^2: while in group Q, Q^0 indicates that the plant offered was not attacked at all. Similarly for the other groups.

24–26 Aug. '17. Alkanet, Blue (*Anchusa* sp.) C^1: *leaves* not attacked; petals gnawed considerably.

27–29 Aug. '17. Anemone, White Japanese (*Anemone japonica*) D^2: *leaves* not attacked; *petals* eaten moderately.

1–23 Sept. '17. Apple (*Pyrus Malus*) F^2: rather unripe *fruit* with skin whole was not attacked, but when cut across was gnawed moderately: 24–28 Sept. '18, *leaves* holed.

24–28 Sept. '18. Artichoke, Jerusalem (*Helianthus tuberosus*) M^1: *leaves* holed and edges gnawed down to midrib; *tuber*, cross slice attacked vigorously and its buds also devoured.

20 Sept.–5 Oct., 3–17 Nov. '18. Asparagus (*Asparagus officinalis*) O^1, T^0: *leaves* gnawed a little; *fruit* not attacked.

26–31 Aug. '18. Aster, Mauve China (*Callistephus chinensis*) H^2: *leaves* not attacked; *petals* and *flower buds* much eaten.

6–11 Sept. '18. Aster, Pink China (*Callistephus chinensis*): *leaves* slightly nibbled; *petals* much eaten; *flowers* used as a refuge.

15–20 Sept. '18. Balm, Pale Mauve (*Melissa officinalis*) J^1: *leaves* not attacked; *petals* of buds devoured.

22–23 Aug. '17. Bean, Dwarf (*Phaseolus vulgaris*) B^3: *leaves* nibbled very slightly.

30–31 Aug. '17. Bean, Scarlet Runner (*Phaseolus multiflorus*) E^3: *leaves*, *flowers* and *pods* apparently neglected: 16–18 Oct. '18, *leaves* holed a good deal and edges gnawed down to veins.

20–28 Oct. '18. Beard Tongue, Scarlet (*Pentstemon* sp.) R^0: *leaves* and *flowers* not attacked.

22–23 Aug. '17. Beet (*Beta vulgaris*) B^1: *leaves* much attacked, especially the petioles, which were opened out and their pith devoured.

20–24 Sept. '18. Bell Flower, White (*Campanula* sp.) K^1: *leaves* not touched; *petals* completely devoured.

31 Aug.–6 Sept. '18. Bindweed, Common (*Convolvulus* sp.): *leaves* much holed.

11–13 Sept. '18. Blackberry (*Rubus fruticosus*): ripe *fruit* well gnawed.

30–31 Aug. '17. Cabbage, Garden (*Brassica oleracea capitata*) E^1: *leaves* gnawed down to midrib and veins and ends of veins eaten off.
2–5 Oct. '18. Canterbury Bell, Blue (*Campanula medium*) N^1: *leaves* and *petals* well devoured.
6–7 Sept. '18. Carrot (*Daucus Carota*): *root* not attacked where covered by skin, but cut end was much gnawed.
6–11 Sept. '18. Celery (*Apium graveolens*) H^1: *leaves* holed and their edges gnawed.
29 Sept.–3 Oct. '18. Cherry (*Prunus* [*Cerasus*] sp.) M^3: *leaves* not attacked.
20–23 Oct. '18. Chickweed (*Stellaria media*) R^1: edges of *leaves* gnawed slightly.
31 Aug.–6 Sept. '18. Chrysanthemum, Garden (*Chrysanthemum indicum*): *flower buds* used as refuge, tips of *petals* apparently somewhat nibbled: 31 Aug.–6 Sept. '18, purple variety: edges of *leaves* much nibbled; *flower buds* used as refuge, tips of petals apparently somewhat nibbled: 31 Aug.–6 Sept. '18, white variety: *leaves* not attacked; *petals* much eaten.
20–24 Sept. '18. Clematis, White (*Clematis* sp.) K^1: *leaves*, a few eaten off at ends and edges gnawed here and there; *flowers* entirely devoured.
23–27 Oct. '18. *Cluvia miniata* (Natal): *leaves* not attacked; *petals* gnawed a little along edges.
15–20 Sept. '18. Cornflower (*Centaurea Cyanus*) J^1: *leaves* well eaten, only midrib left; *flowers* entirely devoured.
29 Sept.–3 Oct. '18. Cups and Saucers (*Cobaea scandens*) M^2: *petals* nibbled a little.
27 Oct.–3 Nov. '18. Dandelion (*Taraxacum officinale*): *petals* of ray florets entirely devoured.
26–31 Aug. '18. Elephant's Ear, Pink (*Begonia* sp.): *leaves* much gnawed along edges and also holed; *flowers* thoroughly devoured.
2–5 Oct. '18. Fern, Male (*Lastraea filis-mas*) O^0: *leaves* not attacked.
15–20 Sept. '18. Feverfew (*Pyrethrum* sp.) J^1: *leaves* gnawed down to midrib; *flowers* apparently not attacked.
21–28 Sept. '18. Fig (*Ficus Carica*): *leaves* not attacked; *fruit* neglected when whole, but cross section was well gnawed.
7–15 Oct. '18. Fox-glove (*Digitalis purpurea*) P^1: *leaves* holed.
6–11 Sept. '18. Fuchsia, Crimson Garden (*Fuchsia* sp.) H^3: neither *leaves* or *flowers* were attacked.
28 Sept.–2 Oct. '18. Geranium, Scarlet (*Geranium* sp.) L^2: *petals* eaten a little.
20–24 Sept. '18. Gesnera, Orange and Pink (*Gesnera* sp.) K^1: *leaves* not attacked; *petals* entirely devoured.
24–26 Aug. '17. Golden Rod (*Solidago* sp.) C^3: *leaves* gnawed at edges here and there; *flowers* apparently not attacked.
2–5 Oct. '18. Gooseberry (*Ribes grossularia*) O^0: *leaves* not attacked.
11–15 Sept. '18. Hawthorn (*Crataegus oxycantha*) I^0: neither *leaves* or *flowers* were attacked.
24–31 Aug. '18. Hollyhock, Dark Crimson (*Althaea rosea*): *leaves* not attacked; *flower buds* used as refuge, *petals* apparently eaten to some extent.
10–20 Aug. '18. Honeysuckle (*Lonicera* sp.) G^2: *leaves* not attacked; *fruit* gnawed considerably.
7–20 Oct. '18. Hydrangea, Pink (*Hydrangea* sp.) Q^0: neither *leaves* or *flowers* were attacked.
7–15 Oct. '18. Larkspur, Garden variety (*Delphinium* sp.) Q^1: *leaves* gnawed thoroughly down to midrib.
3–6 Nov. '18. Leek (*Allium porrum*) T^1: *leaves* gnawed deeply towards base.
6–15 Sept. '18. Lettuce, Cabbage (*Lactuca sativa*): *stem* abundantly gnawed and bored; *leaves* of "heart" entirely devoured.
7–27 Oct. '18. Lupin (*Lupinus polyphyllus*) S^2: *leaves* gnawed to some extent.

3–17 Nov. '18. Mallow (*Malvus ? sylvestris*): *leaves* holed and edges gnawed down to veins.

23 Oct.–17 Nov. '18. Marguerite, White-rayed (*Chrysanthemum leucanthemum*) S^1, U^2: *petals* of ray florets well gnawed.

20–21 Aug. '17. Marrow, Vegetable (*Cucurbita ovifera*) A^1: *leaves* thoroughly devoured.

20–21 Aug. '17. Michaelmas Daisy (*Aster* sp.) A^3, N^3: *leaves* hardly touched, if at all; *flowers* also neglected.

11–15 Sept. '18. Mignonette (*Reseda odorata*): *leaves* gnawed down to midrib; *flowers* attacked but slightly or not at all.

16–18 Sept. '18. Mint (*Mentha* sp.): *leaves*, edges and ends nibbled; *flowers* entirely devoured.

20–23 Oct. '18. Navew (*Brassica campestris*) R^1: *leaves* holed and edges gnawed a little; *petals* moderately attacked.

3–17 Nov. '18. Nettle (*Urtica dioica*) U^1: *leaves* well gnawed down to veins.

31 Aug.–6 Sept. '18. Onion (*Allium Cepa*) L^0: *inflorescence* used as refuge, but apparently not eaten.

7–15 Oct. '18. Pansy (*Viola tricolor*) P^1: *leaves* nibbled slightly.

10–20 Aug. '18. Parsley, Garden (*Carum Petroselinum*) G^2: *inflorescence* nibbled moderately.

29 Sept.–3 Oct. '18. Peach (*Prunus* [*Amygdalus*] sp.) N^2: *leaves* gnawed moderately.

28 Sept.–2 Oct. '18. Periwinkle, Blue (*Vinca* sp.) L^1: *leaves* and *petals* gnawed moderately.

22–23 Aug. '17. Phlox, White (*Phlox Drummondi*) B^2: *leaves* apparently not attacked; *petals* much gnawed and *pollen* found in gut of earwigs.

1–3 Sept. '17. Plum (*Prunus communis*) F^1: *fruit* well eaten.

23–31 Aug. '18. Poppy, Garden (*Papaver* sp.): dried *fruits* very popular as refuges; some were holed to obtain entrance.

1–18 Sept. '17. Potato (*Solanum tuberosum*) F^1: *tuber* in skin was neglected, but slices were thoroughly gnawed.

28–29 Aug. '17, 20–23 Oct. '18. Primrose, Evening, yellow variety (*Oenothera* sp.) D^1: *leaves* not attacked; *petals* eaten thoroughly; *pods* neglected.

7–15 Oct. '18. Privet (*Ligustrum vulgare*) Q^2: *leaves* holed and edges gnawed; *fruits* not attacked.

20–21 Aug. '17. Radish, Horse (*Raphanus sativus*) A^2: *leaves* nibbled slightly.

27–29 Aug. '17. Raspberry (*Rubus idaeus*) D^0: *leaves* not attacked, but earwigs assembled in crowds on their hairy undersides.

22–28 Sept. '18. Red hot poker (*Kniphofia* sp.) : cut end of *stem* gnawed; *leaves* and *petals* not attacked.

11–15 Sept. '18. Rest-harrow (*Ononis* sp.) I^0: apparently neither *leaves* or *flowers* were attacked.

30–31 Aug. '17. Rhubarb (*Rheum officinale*) E^2: *leaves* well gnawed.

24–26 Aug. '17. Rose, White garden variety (*Rosa* sp.) C^1: *leaves* not attacked; *petals* devoured.

7–10 Oct. '18. St John's Wort (*Hypericum* sp.) P^1: *leaves* holed and their edges gnawed; *flower buds* not attacked.

31 Aug.–6 Sept. '18. Scabious, Crimson Garden (*Scabiosa atro-purpurea*): *leaves* much holed; *flowers* apparently not attacked.

23–27 Oct. '18. Scotch Kale (*Brassica oleracea acephala*) S^3: *leaves* holed a very little; curled margins a favourite refuge.

10–24 Aug. '18. Sea Kale (*Brassica oleracea acephala*) G^1: *leaves* holed and gnawed away from edges to between veins.

6–11 Sept. '18. Snapdragon, Scarlet (*Antirrhinum* sp.): *leaves* gnawed moderately; *petals* apparently holed to some extent, also used as refuge.

23–30 Oct. '18. Sow thistle (*Sonchus oleaceus*): *leaves* holed slightly; *flower buds* not attacked.
3–17 Nov. '18. Strawberry (*Fragaria vesca*) U^3: *leaves* holed a little.
31 Aug.–6 Sept. '18. Tomato (*Lycopersicum esculentum*): *leaves* and ripe *fruit* gnawed thoroughly.
14–15 Sept. '18. Valerian, Red Garden (*Valeriana* sp.): edges of *leaves* gnawed moderately; *petals* entirely devoured.
21–24 Aug. '18. Vervain, Blue (*Verbena* sp.): *leaves* nibbled slightly, hairy undersides used for assembling; *petals* entirely devoured.
24–31 Aug. '18. Vetch, Mauve and White garden varieties (*Vicia* sp.): *leaves* attacked very slightly, if at all; *petals* entirely devoured.
23 Oct.–3 Nov. '18. Violet, Single and Double garden varieties (*Viola* sp.): *leaves* holed and edges gnawed moderately.
3–17 Nov. '19. Wartweed (*Euphorbia helioscopia*) T^2: edges of *leaves* gnawed very slightly.
15–18 Sept. '18. Wormwood (*Artemisia* sp): *leaves* not attacked.

These observations are of course subject to the drawback that in captivity animals which normally feed daily may take unusual food with apparent eagerness because no other is available: but the above record probably indicates normal preferences over a certain range of common plants, and also that some are disliked by earwigs; thus Wartweed was left entirely untouched for many days in the absence of any other food, the animals attacking potato tuber ravenously as soon as this was substituted. It seems natural that such stiff and dry foliage leaves as those of Raspberry, Hawthorn, and Cherry, should escape attack, and there is no doubt that the more succulent leaves are preferred. The list of plants affords some information which may facilitate the destruction of earwigs when they become a pest by the indications obtained as to plants which are popular as refuges, and also by the mode in which the attack on leaves is made; thus, some leaves seem to be attacked by holing as well as by gnawing along the edges, and others only by the latter method. There is no doubt that earwigs have preferences among the common plants of a flower or vegetable garden, and that if numerous they are likely to become a pest. In certain cases, as for instance, chrysanthemums, the actual damage done seems to be exaggerated by common report.

Since the epitome of recent literature on the subject in my previous paper (*Proc. Camb. Phil. Soc.*, XIX, Part 4, 1918, p. 170) was written, *The Review of Applied Entomology* has recorded attacks on beets and sugar-beets in Denmark sufficiently serious to obtain mention by Lind and others in their Report on Agricultural Pests in 1915 (*Beretning fra Statens Forsogsvirksomhed i Plantekultur*, Copenhagen, 1916, pp. 397–423).

As regards the carnivorous habit of *F. auricularia*, lean roast mutton without other food was given for several days to the earwigs under observation in 1918 and was gnawed sparingly, while

mutton suet substituted for it was eaten readily and extensively. In the *Journal of the Bombay Natural History Society*, xxvi, No. 2, May, 1919, p. 688, F. P. Connor records an unnamed earwig at Amara catching moths in its forcipes and in one case nibbling its prey. F. Maxwell Lefroy (*Indian Insect Life*, p. 52) remarks: "The function of the forcipes is a mystery that will be cleared up only when their food habits and general life are better understood." They are very possibly "frightening" as well as defensive organs. Pemberton (*Hawaiian Planters' Record*, Honolulu, xxi, No. 4, Oct. 1919, pp. 194–221) mentions the benefit to cane fields arising from the destruction of the leaf-hopper parasite *Perkinsiella optabilis* by the black earwig *Chelisoches morio*.

The importance of nocturnal observations on the feeding habits of *Forficula auricularia* to a satisfactory understanding of the economic effects of this insect in gardens, urged in my previous paper, may be referred to again.

Lagrangian Methods for High Speed Motion. By C. G. Darwin.

[*Read* 8 March 1920.]

1. In the later developments of Bohr's* spectrum theory, it is necessary to calculate the orbits of electrons moving with such high velocities that there is a sensible increase of mass. The selection of the orbits permitted by the quantum theory almost necessitates the treatment of such problems by Hamiltonian methods. Working on these lines Sommerfeld† and others have calculated with a very high degree of success those spectra which involve the motion of a single electron. But the application of the Hamiltonian function involves a knowledge of the momentum corresponding to any generalized coordinate, and in the formulation of most problems the momenta are not known *a priori* but must be calculated from the corresponding velocities. In other words the formation of the Hamiltonian function must in general be preceded by that of the Lagrangian. An exception occurs in precisely the problems referred to above; for, the electromagnetic theory furnishes directly values for the momentum and kinetic energy of a moving electron in terms of its velocity, and the velocity can be eliminated between them so as to obtain the Hamiltonian function. But in even slightly more complicated cases this simple relation is destroyed—thus the problem of a single electron in a constant magnetic field can only be solved by introducing the artificial conception of rotating axes —and in general it will be necessary to follow the direct course of finding the Lagrangian function in terms of the generalized velocities, and then deducing from it the momenta and the Hamiltonian function in the usual way.

If more than one particle is in motion another difficulty enters. For the interaction of two moving particles depends on a set of retarded potentials and the effect of the retardation is readily seen to be of the same order as the increase of mass with velocity. The calculation of the retardation can only be carried out by expansion and so the results are only approximate. This is not surprising since the methods of conservative dynamics cannot apply to such effects as the dissipation of energy by radiation, effects inevitably required by the electromagnetic theory, though they do not occur in actuality. We can also see from the fact that these radiation terms are of the order of the inverse cube of the velocity of light, that it will be useless to expand beyond the inverse square.

* N. Bohr, *Kgl. Dan. Wet. Selsk.*, 1918.
† A. Sommerfeld, *Ann. Phys.*, vol. 51, p. 1, 1916.

2. We first consider the motion of a single electron in an arbitrary electric and magnetic field varying in any manner with the time and position. If m is the mass for low velocities, the momentum is known to be mv/β, where $\beta = \sqrt{1 - v^2/c^2}$. Starting from this we have quasi-Newtonian equations of motion of the type

$$\frac{d}{dt}\left\{\frac{m}{\beta}\cdot\dot{x}\right\} = F_x \qquad \text{.....(2·1).}$$

The force F_x is given from the field $\mathbf{E}$, $\mathbf{H}$ as the vector $e\mathbf{E} + \frac{e}{C}[\mathbf{v}, \mathbf{H}]$, where $\mathbf{v}$ is the velocity vector of the particle's motion. $\mathbf{E}$ and $\mathbf{H}$ can be expressed in terms of the scalar and vector potentials in the form $\mathbf{E} = -\operatorname{grad}\phi - \frac{1}{C}\frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{H} = \operatorname{curl}\mathbf{A}$.

Then if $\mathbf{r}_1$ is the vector x, y, z we have as the vector equation of motion

$$\frac{d}{dt}\left\{\frac{m_1}{\beta_1}\dot{\mathbf{r}}_1\right\} = -e_1 \operatorname{grad}\phi - \frac{e_1}{C}\frac{\partial \mathbf{A}}{\partial t} + \frac{e_1}{C}[\dot{\mathbf{r}}_1, \operatorname{curl}\mathbf{A}] \quad \text{...(2·2),}$$

where $\beta_1 = \sqrt{1 - \dot{\mathbf{r}}_1^2/C^2}$.

Let q be any one of three generalized coordinates representing the position of the particle. Take the scalar product of (2·2) by $\frac{\partial \mathbf{r}_1}{\partial q}$. Then since $\frac{\partial \mathbf{r}_1}{\partial q} = \frac{\partial \dot{\mathbf{r}}_1}{\partial \dot{q}}$, we have

$$\left(\frac{\partial \mathbf{r}_1}{\partial q}, \frac{d}{dt}\left\{\frac{m_1}{\beta_1}\dot{\mathbf{r}}_1\right\}\right) = \frac{d}{dt}\left\{\frac{m_1}{\beta_1}\left(\frac{\partial \mathbf{r}_1}{\partial q}, \dot{\mathbf{r}}_1\right)\right\} - \frac{m_1}{\beta_1}\left(\frac{\partial \dot{\mathbf{r}}_1}{\partial q}, \dot{\mathbf{r}}_1\right)$$

$$= \frac{d}{dt}\left\{\frac{m_1}{\beta_1}\frac{\partial}{\partial \dot{q}}\left(\tfrac{1}{2}\dot{\mathbf{r}}_1^2\right)\right\} - \frac{m_1}{\beta_1}\frac{\partial}{\partial q}\left(\tfrac{1}{2}\dot{\mathbf{r}}_1^2\right)$$

$$= \mathfrak{D}_q\left(-m_1 C^2 \beta_1\right),$$

where $\mathfrak{D}_q = \frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}$ the Lagrangian operator.

Again $$-e_1\left(\frac{\partial \mathbf{r}}{\partial q}, \operatorname{grad}\phi\right) = -e_1\frac{\partial \phi}{\partial q} = e_1 \mathfrak{D}\phi.$$

The remainder can be reduced to

$$\frac{e_1}{C}\left(\dot{\mathbf{r}}_1, \frac{\partial \mathbf{A}}{\partial q}\right) - \frac{e_1}{C}\left(\frac{\partial \mathbf{r}_1}{\partial q}, \frac{d\mathbf{A}}{dt}\right) \qquad \text{.....(2·3),}$$

where $$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\dot{x} + \frac{\partial \mathbf{A}}{\partial y}\dot{y} + \frac{\partial \mathbf{A}}{\partial z}\dot{z}$$

and so is the total change of **A** at the moving particle. (2·3) can be reduced to $-\frac{e_1}{C}\mathfrak{D}_q(\dot{\mathbf{r}}_1, \mathbf{A})$.

Thus the whole equation of motion can be derived from a Lagrangian function

$$L = -m_1C^2\beta_1 - e_1\phi + \frac{e_1}{C}(\dot{\mathbf{r}}_1, \mathbf{A}) \qquad \text{.....(2·4).}$$

This is valid for any fields of force including explicit dependence of ϕ and **A** on the time. The first term in L, which reduces to the kinetic energy for low velocities, differs from it in general. It is very closely connected with the "world line" of the particle.

3. To treat of the case where several moving particles interact we shall start by supposing that there is a second particle present undergoing a constrained motion so that its coordinates are imagined to be known functions of the time. The same will then be true of the potentials it generates. The motion of e_1 will then be governed by (2·4) if ϕ and **A** are expressed in terms of the motion of e_2. These potentials are given by

$$\phi = \frac{e_2}{r + (\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)/C}, \qquad \mathbf{A} = \frac{e_2}{C}\frac{\dot{\mathbf{r}}_2}{r + (\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)/C} \text{...(3·1).}$$

In these expressions $r^2 = (\mathbf{r}_2 - \mathbf{r}_1)^2$ and the values are to be *retarded* values. If the time of retardation be calculated and the result substituted in (3·1) we obtain

$$\phi = \frac{e_2}{r} + \frac{e_2}{2C^2}\left\{\frac{\dot{\mathbf{r}}_2^2 + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r} - \frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)^2}{r^3}\right\}, \quad \mathbf{A} = \frac{e_2}{C}\frac{\dot{\mathbf{r}}_2}{r} \text{...(3·2),}$$

where now $\mathbf{r}_1$, $\mathbf{r}_2$ refer to the same instant of time. ϕ is an approximation valid to C^{-2}, but the value of **A** has only been found to the degree C^{-1} on account of the further factor C^{-1} in (2·4) which is to multiply it. Then substituting in (2·4) we obtain

$$L = -m_1C^2\beta_1 - \frac{e_1e_2}{r} - \frac{e_1e_2}{2C^2}\left\{\frac{\dot{\mathbf{r}}_2^2 + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1) - 2(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r} - \frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)^2}{r^3}\right\} \qquad \text{.....(3·3).}$$

The equations of motion are unaffected by adding to L the expression $-m_2C^2\beta_2 + \frac{d}{dt}\frac{e_1e_2}{2C^2}\frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r}$. The first is a pure function of the time and so contributes no terms to the equations of motion. The second contributes nothing because for any function f we have

$$\mathfrak{D}\left(\frac{d}{dt}f(q_1, q_2, \ldots, t)\right) \equiv 0.$$

The new form of L then reduces to

$$L = -m_1C^2\beta_1 - m_2C^2\beta_2 - \frac{e_1e_2}{r} + \frac{e_1e_2}{2C^2}\left\{\frac{(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)\,(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r^3}\right\} \quad \ldots\ldots(3{\cdot}4).$$

From the complete symmetry of this form the roles of e_1 and e_2 may be interchanged. Further from the covariance of the operator $\mathfrak{D}$ for point transformations, both may be included in the dynamical system, so that if q is any generalized coordinate involving both $\mathbf{r}_1$ and $\mathbf{r}_2$, the equations of motion will be of the form $\mathfrak{D}_qL = 0$.

For the sake of consistency, as the last term in (3·4) is only an approximation valid to C^{-2}, the first two should be expanded only to this power. The first term will give

$$-m_1C^2 + \tfrac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{8C^2}m_1\dot{\mathbf{r}}_1^4.$$

Generalizing our result to the case of any number of particles in any external field we have

$$L = \Sigma\tfrac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \Sigma\frac{1}{8C^2}m_1\dot{\mathbf{r}}_1^4 - \Sigma e_1\phi + \Sigma\frac{e_1}{C}(\dot{\mathbf{r}}_1\mathbf{A}) - \Sigma\Sigma\frac{e_1e_2}{r_{12}} + \Sigma\Sigma\frac{e_1e_2}{2C^2}\left\{\frac{(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r_{12}} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)\,(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r_{12}^3}\right\} \ldots(3{\cdot}5).$$

The double summations are taken counting each pair once only.

4. The transition to the Hamiltonian now follows the ordinary rules. We find momenta $p = \dfrac{\partial L}{\partial \dot{q}}$ and solve for the $\dot{q}$'s in terms of the p's. This can be done in spite of the cubic form of the equations in the q's by use of the approximation in powers of C. The Hamiltonian function will then be $H = \Sigma p\dot{q} - L$ and the equations of motion will be the canonical equations $\dot{q} = \dfrac{\partial H}{\partial p}$, $\dot{p} = -\dfrac{\partial H}{\partial q}$. If $\mathbf{p}_1$ be the momentum corresponding to $\mathbf{r}_1$, the Hamiltonian in these coordinates will be

$$H = \Sigma\frac{\mathbf{p}_1^2}{2m_1} - \Sigma\Sigma\frac{\mathbf{p}_1^4}{8C^2m_1^3} + \Sigma e_1\phi - \Sigma\frac{e_1}{Cm_1}(\mathbf{p}_1, \mathbf{A}) + \Sigma\Sigma\frac{e_1e_2}{r_{12}} - \Sigma\Sigma\frac{e_1e_2}{2C^2m_1m_2}\left\{\frac{(\mathbf{p}_1, \mathbf{p}_2)}{r_{12}} + \frac{(\mathbf{p}_1, \mathbf{r}_2 - \mathbf{r}_1)\,(\mathbf{p}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r_{12}^3}\right\} \ldots(4{\cdot}1).$$

All the applications of general dynamics, such as the Hamilton Jacobi partial differential equation, follow from this. As in ordinary dynamics, many problems can be conveniently solved in the La-

grangian form. The solution will usually depend on finding integrals corresponding to coordinates which do not occur explicitly in L and if ϕ and $\mathbf{A}$ do not involve the time explicitly there is also the energy integral. This has the form

$$\Sigma\tfrac{1}{2}m\dot{\mathbf{r}}_1^2 + \Sigma\tfrac{3}{8}\frac{m_1}{C^2}\mathbf{r}_1^4 + \Sigma e_1\phi + \Sigma\Sigma\frac{e_1e_2}{r_{12}}$$
$$+ \Sigma\Sigma\frac{e_1e_2}{2C^2}\left\{\frac{(\dot{\mathbf{r}}_1\mathbf{r}_2)}{r} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)\,(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r^3}\right\} = \text{const.}\ldots.(4{\cdot}2).$$

This completes the development of the method. Its direct applications are naturally somewhat limited, since, even with the large order terms only, there are comparatively few problems that are soluble. A problem of some interest that can be solved completely is the motion of two attracting particles, where their masses have a finite ratio*.

* A discussion of this problem by the present writer will be found in *Phil. Mag.*, Vol. 39, p. 537 (1920), together with a somewhat fuller account of the general theory.

A bifilar method of measuring the rigidity of wires. By G. F. C. Searle, Sc.D., F.R.S., University Lecturer in Experimental Physics.

[*Read* 3 May 1920.]

§ 1. *Introduction.* In this method the couple due to the torsion of two similar wires is balanced against the couple due to the load carried by the wires and arising from bifilar action.

The method is hardly suitable for accurate measurements of rigidity, but, as an exercise in the use of a bifilar suspension, it has proved useful at the Cavendish Laboratory.

§ 2. *Bifilar couple.* We first consider two light flexible strings. Let the strings AB, CD, each l cm. in length, hang from two fixed points A, C, which are at a distance $2a_1$ cm. apart in a horizontal plane. The lower ends B, D of the strings are attached to a rigid body of mass M grm., the points B, D being $2a_2$ cm. apart. The centre of gravity of the body is symmetrical with regard to B and D and thus the tensions of the strings are equal. The line BD will then be horizontal. If, now, a couple, whose axis is vertical, is applied to the body, the body will be in equilibrium when the couple due to the obliquity of the strings balances the applied couple*.

In Fig. 1, A', B', C', D' are the projections of A, B, C, D on a horizontal plane. In our symmetrical case, $A'C'$, $B'D'$ bisect each other in O. When the body has turned through θ radians from the zero position, in which the strings are in the vertical plane through $A'C'$, then $B'D'$ will make an angle θ with $A'C'$. Let ON be the perpendicular from O on $A'B'$. Let the tension in each string be T dynes.

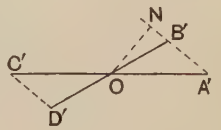

Fig. 1.

If the vertical distance of BD below AC is h cm., the vertical component of the tension is Th/l, and the horizontal component is $T.A'B'/l$. Since the weight of the body equals the sum of the vertical components,

$$Mg = 2Th/l.$$

The horizontal component of the tension at B acts along a line whose projection is $A'B'$, and hence its moment about the vertical

* For the general theory of the bifilar suspension, see Maxwell, *El. and Mag.*, Vol. II, § 459; A. Gray, *Absolute Measurements in El. and Mag.*, Vol. I, p. 242; Kohlrausch, *Physical Measurements* (1894), p. 226.

axis through O is $T.A'B'.ON/l$. Since the moment due to the two tensions equals that of the applied couple, G dyne-cm.,

$$G = 2T.A'B'.ON/l.$$

But $A'B'.ON$ is twice the area $OA'B'$ and thus is $a_1a_2 \sin\theta$. We thus obtain

$$\frac{G}{Mg} = \frac{A'B'.ON}{h} = \frac{a_1a_2 \sin\theta}{h}.$$

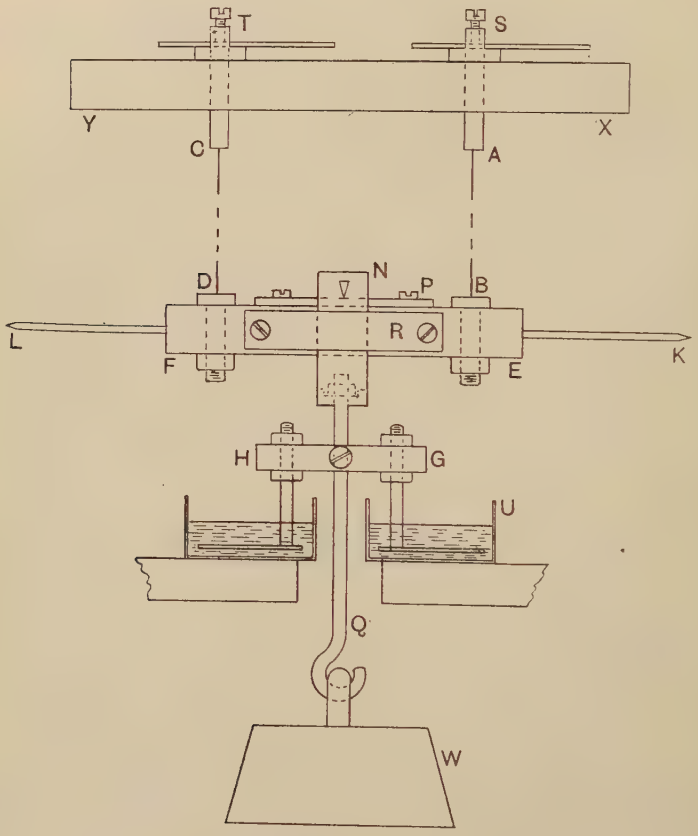

Fig. 2.

Since $h = \{l^2 - A'B'^2\}^{\frac{1}{2}}$, we see that, when $A'B'$ is small compared with l, we may put $h = l$, and so obtain

$$G = \frac{a_1a_2 \sin\theta}{l}.Mg. \qquad \text{.....(1)}$$

In the examples of § 7, h never differed from l by as much as 1 in 4000.

§ 3. *Apparatus.* This is shown diagrammatically in Fig. 2. The wires are soldered into torsion heads S, T, which pass through a board XY held in a firm support.

The lower ends of the wires are soldered into screws which pass through "clearing" holes in the bar EF, and are secured with nuts. The heads of the screws are made with "flats" to fit a spanner. Before the screws are secured to EF, the torsion heads are set to zero; the screws are then secured to EF so that, when the bar is only subject to the action of the wires and of gravity, the flats on *both* screws have the same directions as when the wires hung freely.

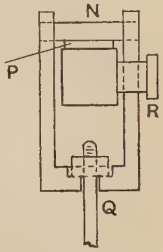

Fig. 3.

The distance BD is, as near as may be, equal to AC.

The load is carried by a knife-edge forming part of the link N, Figs. 2, 3. The knife-edge rests in a V-groove in a plate, P, fixed to EF by screws passing through slots. By adjusting P, the tensions can be equalised; the notes emitted by the wires when plucked have the same pitch when the tensions are equal.

A weight W (a few kilogrammes) is suspended by the rod Q from the link N. A slot in the lower cross-piece of N allows Q to be put into place; the nut drops into a recess. The weight should be so attached to Q that it cannot turn about a vertical axis relative to Q with any freedom; otherwise it will be difficult to reduce the system to rest.

The bar may be fitted with two pointers K, L, and the readings of their *tips* are taken on two horizontal scales. These scales are adjusted to be perpendicular to KL when the torsion heads read zero. If K_0L_0 is the *straight line* through the zero positions of the *tips* and K, L are the *tips* when the bar has turned through θ, Fig. 4 shows that

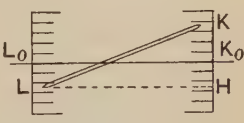

Fig. 4.

$$\sin\theta = \frac{KH}{LK} = \frac{y_1 + y_2}{p}, \qquad \text{.....(2)}$$

where $y_1 = KK_0$, $y_2 = LL_0$ and $p = LK$, the whole length of the pointer system.

The deflexion of the bar is best observed optically. A metal strip R is screwed to EF, packing pieces being interposed to allow the link N free movement, and a plane mirror is fixed to R. The deflexion can be observed by aid of a telescope and scale, or of a lamp and scale. It is, however, simpler to employ a goniometer such as those which have been in constant use at the Cavendish Laboratory for several years. A description of the instrument and

the method of using it for experiments of this type will be found in *Proc. Camb. Phil. Soc.*, XVIII, p. 31, or in the author's *Experimental Harmonic Motion*, p. 35. The goniometer measures the tangents of angles.

The motion of the suspended system, as so far described, being only slightly damped, it is consequently not easy to reduce the system to rest, and the vibrations of the building add to the difficulty. A simple damping device is therefore used. An annulus of thin sheet metal is carried by the bar GH, which is clamped to the rod Q. The annulus is immersed in motor lubricating oil or other highly viscous liquid contained in the annular trough U, which rests on the table. The rod Q passes through a hole in the table. By adjusting the height of GH, the annulus can be brought close to the bottom of the trough, and then the motion is so highly damped that the system is practically immune to vibrations of the floor or the table.

If the wires are overstrained by turning the heads through too large angles, the wires will no longer be vertical when the heads read zero, and it will be necessary to readjust the screws in the bar EF. To prevent overstrain, and at the same time to allow the heads to be turned through π in either direction from their zeros, a *movable* safety device is used. A metal disk, about 1 cm. in diameter, can turn freely about its centre on a screw by which it is attached to the board XY (Fig. 2). A vertical pin is fixed excentrically in the disk, the greatest distance from the pin to the axis of the head being small enough to prevent the steel wire, which forms the index of the head, from passing the pin. The torsion head can then be turned only a little more than π in either direction from zero.

Care must be taken not to bend the wires near the soldered joints. A bend at B or D will alter the effective value of a_2. If the wire AB is bent near A, the effect, when the torsion head is turned, will be the same as if the point A describes a small horizontal circle. This causes changes in a_1 as the head is turned, and, what is more serious, causes the bar EF to turn through angles which are by no means negligible, in addition to the angles directly due to the torsion of the wires. For this reason, annealed wires are more suitable for the experiment than hard drawn wires, as they are more easily straightened.

The torsion heads are read on circles divided at intervals of 45°, the dividing lines being scribed on the board XY.

§ 4. *Theory of the method.* If each torsion head is turned from its zero through ϕ radians in either direction, the bar EF will turn in the same direction until the bifilar and torsional couples are equal. If EF turns through θ, the whole twist of each wire is $\phi - \theta$.

Let the radius and the length of each wire be r cm. and l cm.,

and the rigidity of the metal n dyne cm.$^{-2}$. Since the wires are *nearly* vertical, the couple, due to torsion, exerted by the *pair* upon the bar is $\pi nr^4(\phi - \theta)/l^*$, to a close approximation.

The small couple due to the *bending* of the wires assists the bifilar couple; Kohlrausch† takes account of this small couple by writing in place of (1),

$$G = \frac{a_1 a_2 \sin\theta}{l'}.Mg, \qquad \ldots\ldots(3)$$

where

$$l' = l - r^2\{2\pi E/Mg\}^{\frac{1}{2}}, \qquad \ldots\ldots(4)$$

and E is Young's modulus.

Equating the torsional to the (corrected) bifilar couple, we have

$$\sin\theta = C(\phi - \theta), \qquad \ldots\ldots(5)$$

where

$$C = \frac{\pi nr^4 l'}{Mga_1 a_2 l}. \qquad \ldots\ldots(6)$$

Then

$$n = \frac{ga_1 a_2 l}{\pi r^4 l'}.MC. \qquad \ldots\ldots(7)$$

§ 5. *Experimental details.* The distances $AC = 2a_1$, $BD = 2a_2$ are measured. The diameters of the wires are taken at a number of points and the mean radius is found.

The total mass, M grm., of the system carried by the wires is found. The masses of the screws are found before they are soldered to the wires.

The torsion heads are first set to zero, and the scales on which the pointers K, L are read are adjusted to be perpendicular to KL. If a goniometer is used, it is set so that its arm is in the central position when the goniometer wire coincides with its own image.

To eliminate errors due to slight bends in the wires, the readings must be taken over the range $-\pi$ to π for ϕ; the theory assumes absence of hysteresis. But in experimental work in elasticity we must realise that hysteresis effects are unavoidable, when the strains are more than infinitesimal. To ensure that the effects of hysteresis shall be orderly and not irregular, the torsion heads are taken through a *complete cycle* from π to $-\pi$ and back to π. To make the two readings for $\phi = \pi$ agree as closely as possible, a preliminary half cycle from $-\pi$ to π is done. To make the conditions uniform throughout the cycle and a half, the readings for the preliminary settings are taken and recorded; this will secure approximately constant time intervals between successive readings. Thus the heads are set in succession at the following multiples of $\pi/4$:

$$-4, -3, -2, -1, 0, 1, 2, 3,$$
$$4, 3, 2, 1, 0, -1, -2, -3, -4, -3, -2, -1, 0, 1, 2, 3, 4.$$

* G. F. C. Searle, *Experimental Elasticity*, § 39.
† Kohlrausch, *Wied. Ann.*, XVII, p. 737, 1882.

The first 8 are the preliminary readings, and only the last 17 are used.

If ϕ goes through a complete cycle, and θ is plotted against ϕ, a narrow hysteresis loop will be obtained. When readings are taken as above, there will be *two* values of θ for each value of ϕ except $\phi = -\pi$. With careful work, the two values of θ for $\phi = \pi$ will be exactly or very nearly identical; for the wires used in § 7, I have seldom found a difference between these two values as great as one minute. As a rough method of eliminating the effects of hysteresis, the mean of the two values of θ for each value of ϕ is taken as *the* value of θ for that value of ϕ.

The effect of bends in the wires near their upper ends, A, C (Fig. 2), will be the same as if these points described small horizontal circles about the centres A_0, C_0, as in Fig. 5. Let ϕ be measured

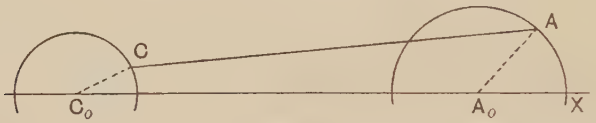

Fig. 5.

from C_0A_0X, and let $AA_0X = \phi + \alpha$, $CC_0X = \phi + \gamma$, while $A_0A = r$, $C_0C = s$, $A_0C_0 = 2a_1$. Then, if ϵ is the small angle between AC and A_0C_0,

$$\tan\epsilon = \frac{r\sin(\phi+\alpha) - s\sin(\phi+\gamma)}{2a_1 + r\cos(\phi+\alpha) - s\cos(\phi+\gamma)}.$$

When r and s are small compared with $2a_1$, $\tan\epsilon$ may be replaced by ϵ and the variable terms in the denominator may be neglected. Then, putting

$$(r\sin\alpha - s\sin\gamma)/2a_1 = P, \qquad (r\cos\alpha - s\cos\gamma)/2a_1 = Q,$$

we have

$$\epsilon = P\cos\phi + Q\sin\phi. \qquad \text{.....(8)}$$

Here P and Q are the values of ϵ when $\phi = 0$ and $\phi = \frac{1}{2}\pi$.

If the line BD makes an angle θ with A_0C_0 when the heads read ϕ, the angle between BD and AC is $\theta - \epsilon$. The wires will not be quite free from torsion when the heads read zero; let η be the mean twist of the wires when $\phi = 0$. We must thus write $\sin(\theta - \epsilon)$ for $\sin\theta$ and $\phi + \eta - \theta$ for $\phi - \theta$ in the equilibrium equation (5), which thus becomes

$$\sin(\theta - \epsilon) = C(\phi + \eta - \theta). \qquad \text{.....(9)}$$

To evade difficulties, θ is kept small. Then, since ϵ is also small, we may replace the sine by the angle in (9), and thus obtain

$$\theta = \frac{C}{1+C}(\phi + \eta) + \epsilon = D(\phi + \eta) + \epsilon. \quad \text{.....(10)}$$

If θ_0, ϵ_0 correspond to $\phi = 0$, we have, since $\epsilon_0 = P$,

$$\theta_0 = D\eta + \epsilon_0 = D\eta + P. \quad \text{.....(11)}$$

Thus

$$\theta - \theta_0 = D\phi + \epsilon - P = D\phi + P\cos\phi + Q\sin\phi - P. \quad \text{.....(12)}$$

Since this equation is *linear* in θ, we may take θ_0 as corresponding to *any* initial position of the bar which is *near* its ideal zero position.

Thus, if β is the angle at any time between the bar and some nearly ideal zero position,

$$\beta = \theta - \theta_0. \quad \text{.....(13)}$$

Since β, though small—say less than 0·2 radian—is not infinitesimal, some correction should be made. An exact solution cannot be given, but accuracy is gained by writing $\sin\beta$ for β, and then the final formula becomes

$$\sin\beta = D\phi + P\cos\phi + Q\sin\phi - P. \quad \text{.....(14)}$$

To eliminate P and Q, we combine the observations. Let β_m correspond to $\phi = m\pi/4$. Then, putting $\phi = \pi$ and $\phi = -\pi$, so that $m = 4$ and $m = -4$, we have

$$\pi D = \tfrac{1}{2}(\sin\beta_4 - \sin\beta_{-4}). \quad \text{.....(15)}$$

A second value for πD is found by giving m the values 3, -3, 1, -1. Then

$$\pi D = \sin\beta_3 - \sin\beta_{-3} - (\sin\beta_1 - \sin\beta_{-1}). \quad \text{.....(16)}$$

The two values of D are usually in good agreement, although, when β is plotted against ϕ, the curve differs considerably from a straight line. The mean value of πD is used to find C. Thus

$$C = \frac{\pi D}{\pi - \pi D}. \quad \text{.....(17)}$$

Then n is found by (7).

The actual values of P and Q are easily found. Thus

$$P = -\tfrac{1}{4}(\sin\beta_4 + \sin\beta_{-4}), \quad \text{.....(18)}$$

$$Q = \tfrac{1}{2}(\sin\beta_2 - \sin\beta_{-2} - \pi D). \quad \text{.....(19)}$$

§ 6. *Conversion table.* A goniometer, such as those used at the Cavendish Laboratory, gives the *tangent* of the angle ψ through which the arm is turned from its zero. To find $\sin\psi$ we subtract from $\tan\psi$ the small quantity s given in the table.

$\tan\psi$	s	$\tan\psi$	s	$\tan\psi$	s
·01	·00000	·10	·00050	·19	·00334
·02	·00000	·11	·00066	·20	·00388
·03	·00001	·12	·00085	·21	·00448
·04	·00003	·13	·00108	·22	·00514
·05	·00006	·14	·00135	·23	·00585
·06	·00011	·15	·00166	·24	·00663
·07	·00017	·16	·00201	·25	·00746
·08	·00025	·17	·00240		
·09	·00036	·18	·00285		

Simple interpolation, by "proportional parts," will give s with an error not exceeding unity in the fifth place of decimals. Thus, if $\tan\psi = \cdot124$, we find $s = \cdot00095$, and then

$$\sin\psi = \tan\psi - s = \cdot12305.$$

§ 7. *Practical example.* The following results were obtained for a pair of soft brass wires.

The distances AC, BD were each 6·00 cm. Hence $a_1 = a_2 = 3$ cm.
Mean radius of wires $= r = 0{\cdot}0352$ cm.
Length of each wire $= l = 47{\cdot}30$ cm.
Mass of suspended system, *excluding* the weight W (Fig. 2) = 417·6 gm.
The small correction for the buoyancy of the damper was neglected.

The deflexions were observed by a goniometer. The distance from the centre of the pivot to the scale was 40·00 cm. The central, or zero, reading is 10·00 cm. The following goniometer readings were obtained for the last 17 of the values of ϕ specified in § 5.

ϕ radians	Reading cm.	Reading cm.	Mean reading	x cm.	$\tan\beta = x/40$	$\sin\beta$ obsd.	$\sin\beta$ calcd.
π	↓ 14·10	14·10	14·100	4·115	·1029	·1023	·1033
$\frac{3}{4}\pi$	12·96	13·04	13·000	3·015	·0754	·0752	·0749
$\frac{1}{2}\pi$	↓ 11·85	11·98	11·915	1·930	·0482	·0482	·0471
$\frac{1}{4}\pi$	10·80	10·96	10·880	0·895	·0224	·0224	·0220
0	9·90	10·07	**9·985**	0·000	·0000	·0000	·0000
$-\frac{1}{4}\pi$	9·16	↑ 9·33	9·245	−0·740	−·0185	−·0185	−·0204
$-\frac{1}{2}\pi$	8·30	↑ 8·43	8·365	−1·620	−·0405	−·0405	−·0415
$-\frac{3}{4}\pi$	7·37	↑ 7·44	7·405	−2·580	−·0645	−·0644	−·0653
$-\pi$	6·33		6·330	−3·655	−·0914	−·0910	−·0921

The value of x was found by subtracting from the mean reading, as given in column 4, the mean zero reading 9·985 cm. corresponding to $\phi = 0$. The differences between the readings in columns 2 and 3 are due to hysteresis. The seventh column shows that $\sin\beta$ is not proportional to ϕ.

By (15), $\pi D = \frac{1}{2}(\cdot1023 + \cdot0910) = \cdot09665$,

and by (16), $\pi D = \cdot0752 + \cdot0644 - (\cdot0224 + \cdot0185) = \cdot09870$.

Mean value of $\pi D = 0{\cdot}0977$.

Then, by (17) $$C = \frac{\pi D}{\pi - \pi D} = 0{\cdot}03210.$$

By (18), $$P = -\frac{1}{4}(\cdot1023 - \cdot0910) = -0{\cdot}0028,$$

and by (19), $$Q = \frac{1}{2}(\cdot0482 + \cdot0405 - \cdot0977) = -0{\cdot}0045.$$

In the table, the column "sin β calcd." gives sin β as calculated by (14), using the values of πD, P and Q just found; there is fair agreement between the calculated and observed values of sin β.

The total load M was 417·6 + 4999 = 5416·6 grm.

Taking $E = 10^{12}$ dyne cm.$^{-2}$, we have $r^2(2\pi E/Mg)^{\frac{1}{2}} = 1{\cdot}35$ cm., and hence, by (4), $l' = 47{\cdot}30 - 1{\cdot}35 = 45{\cdot}95$ cm.

Then, by (7),

$$n = \frac{ga_1a_2l}{\pi r^4 l'}.MC = \frac{981 \times 3^2 \times 47{\cdot}30}{\pi \times 0{\cdot}0352^4 \times 45{\cdot}95} \times 5416{\cdot}6 \times 0{\cdot}03210$$

$$= 3{\cdot}277 \times 10^{11} \text{ dyne cm.}^{-2}.$$

A similar set of observations, in which M was 3417·1 grm., gave the following values of sin β:

·1510, ·1123, ·0743, ·0361, ·0000, −·0345, −·0735, −·1122, −·1540.

Mean value of $\pi D = 0{\cdot}1532$. Hence $C = 0{\cdot}05127$.

Also $l' = 47{\cdot}30 - 1{\cdot}70 = 45{\cdot}60$ cm.

Then

$$n = \frac{981 \times 3^2 \times 47{\cdot}30}{\pi \times 0{\cdot}0352^4 \times 45{\cdot}60} \times 3417{\cdot}1 \times 0{\cdot}05127 = 3{\cdot}326 \times 10^{11} \text{ dyne cm.}^{-2}.$$

An independent determination of n was made by attaching a bar, of moment of inertia $K = 4{\cdot}766 \times 10^4$ grm. cm.^{2}, to each of the two wires in turn; the mean periodic time of the torsional vibrations was $T = 10{\cdot}55$ sec. Hence

$$n = \frac{8\pi Kl}{T^2 r^4} = \frac{8\pi \times 4{\cdot}766 \times 10^4 \times 47{\cdot}30}{10{\cdot}55^2 \times 0{\cdot}0352^4} = 3{\cdot}316 \times 10^{11} \text{ dyne cm.}^{-2}.$$

The Rotation of the Non-Spinning Gyrostat. By G. T. Bennett, M.A., F.R.S., Emmanuel College, Cambridge.

[*Read* 8 March 1920.]

§ 1. The following extract is taken from an old examination paper*:

> "A symmetrical wheel free to rotate about its axle is moved from rest in any position by means of the axle and is finally restored to a position in which the axle again points in the same direction as formerly. Shew that the wheel, again at rest, will have rotated through a plane angle equal to the solid angle of the cone described by the varying directions of the axle."

The proof of this result may be put briefly in a geometrical form. Translational and rotational movements being independent, the centroid of the wheel may be treated as stationary. As the gyrostat has no component rotation about its axis, the axis of rotation is at any moment some diameter of the wheel. This line has the central plane of the wheel as locus for the body-axode, and has a closed cone of arbitrary form as locus for the space-axode. The angular movement is therefore representable by the rolling of the plane on the cone. The angle of ultimate rotation of the wheel is thus (for cones of ordinary type) the excess of the four right angles of the plane surface above the total surface-angle of the cone. This difference is equal to the solid angle of the reciprocal cone described by the axis of the wheel. And hence follows the result quoted; namely, that the solid angle described by the axis of the wheel is equal to the circular measure of the plane angle of the resultant displacement of the wheel about its axis. Further, the sense of the displacement accords with the sense of circulation associated with the solid angle.

§ 2. The result may be extended to the case in which the initial and final directions of the axis are different, say a and b. For the axis may be restored to its original direction a by a subsequent movement in the plane ba; and this latter movement, which is a rotation about the normal to a and b, leaves unaltered the angle that any diameter of the wheel makes with the plane ab. Hence the original movement, shifting the axis of the wheel from a to b

* Emmanuel and other Colleges, Second Year Problems, Wed. June 8, 1898 Question 11.

by any conical movement, alters the angle between the plane ab and any diameter of the wheel by an angle equal to the solid angle enclosed by the cone formed by the conical surface ab together with the plane ba.

§ 3. A geometrical integration of Euler's equation leads to the same result as § 1. The axis, with its direction given by spherical polar coordinates θ and ϕ (radial and azimuthal), generates a solid angle

$$\sigma = \int(1 - \cos\theta).d\phi. \qquad \ldots\ldots(1)$$

The equation of motion, being

$$\dot{\phi}\cos\theta + \dot{\psi} = 0, \qquad \ldots\ldots(2)$$

with zero initial values for ϕ and ψ, has as its integral

$$\phi + \psi = \sigma. \qquad \ldots\ldots(3)$$

If the axis of reference $\theta = 0$ is supposed (conveniently) external to the cone then ϕ is zero finally as well as initially, and ψ is the angle of resultant displacement of the wheel and is equal to the solid angle σ.

If, more generally, the gyrostat has a constant spin Ω about its axis, the Euler equation becomes

$$\dot{\phi}\cos\theta + \dot{\psi} = \Omega \qquad \ldots\ldots(4)$$

with

$$\phi + \psi = \sigma + \Omega t \qquad \ldots\ldots(5)$$

as its integral. And the final rotation of the gyrostat is then given by the solid angle of the cone described by the axis plus the time-integral of the spin. It may be noticed that the angle $\phi + \psi$, with a value independent of the choice of coordinates, gives in itself a natural measure of the total rotation of the wheel, as followed and estimated by projection on the plane $\theta = \pi/2$. For on that plane the circular disc shows as an ellipse, with ϕ as the azimuth of the direction of the minor axis, and ψ as the eccentric angle, measured from the minor axis, of the projection of the revolving diameter of the wheel. A distant observer on the axis $\theta = 0$, able to distinguish the two faces of the wheel, would in this way precisely reckon the amount of rotation, whole turns and fractional. He does not give merely the ultimate position, by naming a plane angle to a modulus of four right angles, but assigns the multiple of the modulus necessary for a correct account of the movement intervening between the initial and final positions.

A kinematic representation of the angle $\phi + \psi$ may be obtained by supposing the circular rim of the disc to have rolling contact with the rim of another equal disc whose plane keeps parallel to the plane $\theta = \pi/2$. The angle of rotation of this latter disc about its axis (which keeps the invariable direction $\theta = 0$) is then $\phi + \psi$.

§ 4. For the special case in which θ is constant, so that the axis of the gyrostat describes a circular cone, the rotation is stated by Sir George Greenhill* to be $2\pi -$ (conical angle described by the axle), as against the solid angle itself found above. The difference of sign of the latter can be accounted for by a reverse sign-convention: but the term 2π is unnecessary if 2π is implied as a modulus, and it appears to be wrong if the precise angle of turning is intended. If, specially, the axis of the gyrostat described only a small cone, then the angle of consequent rotation is certainly a small angle, and not an angle nearly equal to four right angles.

He adds the remark that the movement "can be shown experimentally with a penholder held between the fingers and moved round in a cone by the tip of a finger applied at the end." But the illustration is inapt; for the creep of the penholder occurs in the sense opposite to that of the conical movement. The body-axode is a circular cone and not a plane, and it rolls inside a slightly larger circular cone as space-axode; and hence the reverse movement.

§ 5. The movement of the non-spinning gyroscope here considered is not yet among those that are familiarly recognised, though it has important practical applications and deserves to rank as a dynamical commonplace. Bodies suspended from a point on an axis of symmetry behave in the same way and for the same reason. No matter how the point of suspension may be moved about, and no matter what complicated conical movement is consequently executed by the axis, the applied forces have no moment about the axis, and the spin remains zero if originally zero. The resultant rotation is then given, as above, by the solid angle of the cone described by the axis.

Aeroplane compasses, in particular, are found to keep their cards practically parallel to the floor, under the combined action of gravity and lateral acceleration, during a banked turn of the aeroplane. Hence, from inertia alone, and apart from all other sources of control or disturbance, the compass-card would be rotated, as a consequence of the turn, through an angle equal to the solid angle described by the normal to the card. For an angle of banking α and a change of course β the solid angle is not much less than $(1 - \cos\alpha)\beta$ if the banking is taken and left quickly; and for very steep banking this angle is nearly equal to the change of course itself, and the card would almost appear to "stick." As compared with considerations of magnetic disturbance due to the vertical component of the earth's field, and of mechanical disturbance due to rotation of the bowl and liquid, the pure inertia effect

* Advisory Committee for Aeronautics. Reports and Memoranda, No. 146, *Report on Gyroscopic Theory*, p. 13, § 14.

of the conical movement seems to need more emphasis than it has hitherto been awarded. It is here explicitly isolated.

The gyroscopic compass, like the magnetic compass, may at times suffer disturbance from this same source, if the compass-position in the ship and the run of the sea are such as to produce a circular or elliptical movement of the binnacle.

§ 6. It would be hard to trace to its primitive source the knowledge of the small piece of mechanics here discussed. It is really implicit in all treatises on Rigid Dynamics, but fails to emerge clearly amid the pressure of more important movements. Among empiricists it must be well-nigh prehistoric. The sailor in coiling a rope makes a winding motion of the feeding hand to remove the kinks from the overtwist of the piece which is to form the next turn of the coil. The circus clown, with the vertex of his conical cap resting on his finger-tip, or the end of a stick, easily makes it turn round and round; and the postman collecting his mail knows how to twist up the neck of his bag with a circular movement of the hand he holds it by. Later among empiricists are those who, accustomed to handle magnetic compasses, are very familiar with the rotation of the card produced so readily by giving the bowl a horizontal circular translational movement (without rotation). More lately still Mr S. G. Brown has noticed the conical motion and its effect. In the abstract of his lecture to the British Association* it is described as a "new phenomenon" and is stated as being "explainable mathematically." More fully in his lecture to the Royal Institution† he states that in virtue of the "wobbling" (*videlicet* conical) motion, "the needles and card would then have a force applied trying to carry the moving system round in the direction of the wobble." This mode of expression is of course entirely illegitimate. The rotational movement observed needs no "force" to explain it; the very essence of the inertia effect is that it occurs with no spin about the axis of rotation and no couple about that line either. Mr Brown announces also (but without demonstration) that if his compass-disc "is carried round in a horizontal circular path without any wobble the plate still goes round or tries to go round with the circular movement" and that this "should be of interest to mathematicians." It seems likely that the sheer paradox in angular momentum thus propounded will readily dissolve when all the relevant physical data are revealed: and meanwhile the interest is but that of a heresy resting on hearsay.

* British Association, Bournemouth, 1919. Evening Lecture, Fr. Sept. 12, "The Gyroscopic Compass." *Abstract*, ll. 9–14.

† *Nature*, March 11, 1920, p. 45, col. 2.

Proof of the equivalence of different mean values. By ALFRED KIENAST. (Communicated by Professor G. H. Hardy.)

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If $a_1, a_2, \ldots a_n, \ldots$ denote the terms of a sequence of complex numbers, and

$$S_n^{(0)} = a_1 + \ldots + a_n,$$
$$S_n^{(1)} = S_1^{(0)} + \ldots + S_n^{(0)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n^{(\kappa)} = S_1^{(\kappa-1)} + \ldots + S_n^{(\kappa-1)},$$

then $\lim_{n\to\infty} S_n^{(\kappa)} \Big/ \binom{n+\kappa-1}{\kappa}$ is called Cesàro's κth mean* of the sequence $S_n^{(0)}$.

Putting
$$h_n^{(0)} = a_1 + \ldots + a_n,$$
$$h_n^{(1)} = \frac{1}{n}\{h_1^{(0)} + \ldots + h_n^{(0)}\};$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h_n^{(\kappa)} = \frac{1}{n}\{h_1^{(\kappa-1)} + \ldots + h_n^{(\kappa-1)}\},$$

then $\lim_{n\to\infty} h_n^{(\kappa)}$ is called Hölder's κth mean† of the sequence $h_n^{(0)}$.

In a paper "Extensions of Abel's Theorem and its converses‡" I found it convenient to introduce the expressions

$$s_n^{(0)} = a_1 + \ldots + a_n \qquad (n = 1, 2, \ldots),$$
$$s_n^{(1)} = \frac{1}{n}\{s_1^{(0)} + \ldots + s_{n-1}^{(0)}\} \qquad (n = 2, 3, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$s_n^{(\kappa)} = \frac{1}{n}\{s_\kappa^{(\kappa-1)} + \ldots + s_{n-1}^{(\kappa-1)}\} \qquad (n = \kappa+1, \kappa+2, \ldots),$$

and proved various theorems concerning the limits $\lim_{n\to\infty} s_n^{(\kappa)}$.

Several writers have proved

Theorem 1. *Whenever Cesàro's (Hölder's) κth mean exists and is finite, then Hölder's (Cesàro's) κth mean exists too, and both have the same value.*

* Bromwich, *Infinite series*, p. 310. † *Ibid.*, p. 313.
‡ *Proc. Cambridge Phil. Soc.*, vol. XIX, 1918, pp. 129–147.

I propose to complete the researches of my above quoted paper by proving the theorem:

Theorem 2. *Whenever Hölder's (and therefore Cesàro's) κth mean exists and is finite, then* $\lim_{n\to\infty} s_n^{(\kappa)}$ *exists too, and both have the same value, and vice versa.*

The demonstration of both theorems is based upon relations between the mean values which it is possible to calculate completely, as I have found, in a most simple manner.

In §§ I to VI I determine the expression of $s_n^{(\kappa)}$ by

$$h_\lambda^{(\kappa)} \qquad (\lambda = 1, 2, \ldots (n-\kappa)),$$

in § VII the expression of $h_n^{(\kappa)}$ by

$$s_{\lambda-\kappa}^{(\kappa)} \qquad (\lambda = 1, 2, \ldots n),$$

in § VIII the expression of $S_n^{(\kappa)}$ by

$$h_\lambda^{(\kappa)} \qquad (\lambda = 1, 2, \ldots n),$$

and finally in § IX I consider two more general mean values.

I. From the definitions follow

$$s_n^{(0)} = h_n^{(0)}, \quad h_n^{(0)} = n h_n^{(1)} - (n-1) h_{n-1}^{(1)},$$

$$s_n^{(1)} = \frac{1}{n} \sum_{\lambda=1}^{n-1} h_\lambda^{(0)} = \frac{1}{n} \sum_{\lambda=1}^{n-1} \{\lambda h_\lambda^{(1)} - (\lambda-1) h_{\lambda-1}^{(1)}\} = \frac{n-1}{n} h_{n-1}^{(1)},$$

$$h_n^{(1)} = n h_n^{(2)} - (n-1) h_{n-1}^{(2)},$$

$$s_n^{(2)} = \frac{1}{n} \sum_{\lambda=2}^{n-1} \frac{\lambda-1}{\lambda} h_{\lambda-1}^{(1)} = \frac{1}{n} \left[\frac{1}{2} h_1^{(2)} + \frac{2}{3} (2h_2^{(2)} - h_1^{(2)}) + \ldots \right.$$

$$+ \frac{n-2}{n-1} \{(n-2) h_{n-2}^{(2)} - (n-3) h_{n-3}^{(2)}\}$$

$$\left. + \frac{n-1}{n} \{(n-2) h_{n-2}^{(2)} - (n-2) h_{n-2}^{(2)}\} \right].$$

Adding a term which is zero,

$$s_n^{(2)} = \frac{(n-1)(n-2)}{n^2} h_{n-2}^{(2)} - \frac{1}{n} \left\{ \frac{1}{6} h_1^{(2)} + \frac{1}{6} h_2^{(2)} + \ldots + \frac{n-2}{n(n-1)} h_{n-2}^{(2)} \right\},$$

etc. Now I suppose that, by continuing in this manner, I have arrived at the formula

$$s_n^{(\kappa)} = c_{n,\kappa} h_{n-\kappa}^{(\kappa)} - \frac{1}{n} \sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa} h_\lambda^{(\kappa)} \quad \ldots\ldots\ldots\ldots\ldots (1),$$

where $c_{n,\kappa}$ is a function of the indices n and κ, and the coefficients $d_{\lambda,\kappa} (\lambda = 1, 2, \ldots)$ are, for each κ, definite numbers which are the values of a function of λ for $\lambda = 1, 2, \ldots$.

Proceeding to build up the expression for $s_n^{(\kappa+1)}$, applying the same transformations as above, we find

$$s_n^{(\kappa+1)} = \frac{1}{n}\sum_{\lambda=\kappa+1}^{n-1}\Big[c_{\lambda,\kappa}\{(\lambda-\kappa)\,h_{\lambda-\kappa}^{(\kappa+1)} - (\lambda-\kappa-1)\,h_{\lambda-\kappa-1}^{(\kappa+1)}\} - \frac{1}{\lambda}\sum_{\nu=1}^{\lambda-\kappa} d_{\nu,\kappa}\{\nu h_\nu^{(\kappa+1)} - (\nu-1)\,h_{\nu-1}^{(\kappa+1)}\}\Big] + \frac{1}{n}c_{n,\kappa}\{(n-\kappa-1)\,h_{n-\kappa-1}^{(\kappa+1)} - (n-\kappa-1)\,h_{n-\kappa-1}^{(\kappa+1)}\},$$

$$s_n^{(\kappa+1)} = c_{n,\kappa}\frac{n-\kappa-1}{n}h_{n-\kappa-1}^{(\kappa+1)} - \frac{1}{n}\sum_{\lambda=1}^{n-\kappa-1} d_{\lambda,\kappa+1}h_\lambda^{(\kappa+1)} \quad \ldots\ldots\ldots\ldots(2);$$

from which we conclude

$$c_{n,\kappa+1} = c_{n,\kappa}\frac{n-\kappa-1}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and a series of relations involving the numbers $d_{\lambda,\kappa}$ and $d_{\lambda,\kappa+1}$.

Equation (2) is of the same formation as (1), and therefore (1) gives the required expression of $s_n^{(\kappa)}$ by the numbers $h_\lambda^{(\kappa)}$.

II. Since $c_{n,1} = \dfrac{n-1}{n}$, (3) leads to

$$c_{n,\kappa} = n^{-\kappa}(n-\kappa)(n-\kappa+1)\ldots(n-1),$$

from which follows, for $\kappa = 2$, $c_{n,2} = \dfrac{(n-1)(n-2)}{n^2}$, which is in accordance with the expression for $s_n^{(2)}$ above.

III. $d_{\lambda,\kappa}$ may be determined in the following way. Putting

$$a_1 = a_2 = \ldots = a_{\lambda-1} = 0, \quad a_\lambda = 1, \quad a_{\lambda+1} = \ldots = 0 \quad \ldots\ldots(4),$$

we find

$$s_1^{(0)} = \ldots = s_{\lambda-1}^{(0)} = 0, \qquad s_\lambda^{(0)} = 1, \qquad s_{\lambda+1}^{(0)} = 1, \qquad \ldots,$$

$$s_2^{(1)} = \ldots = s_\lambda^{(1)} = 0, \qquad s_{\lambda+1}^{(1)} = \frac{1}{\lambda+1}, \quad s_{\lambda+2}^{(1)} = \frac{2}{\lambda+2}, \quad \ldots,$$

$$s_3^{(2)} = \ldots = s_{\lambda+1}^{(2)} = 0, \qquad s_{\lambda+2}^{(2)} = \frac{1}{(\lambda+1)(\lambda+2)}, \quad \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$s_{\kappa+1}^{(\kappa)} = \ldots = s_{\lambda+\kappa-1}^{(\kappa)} = 0, \quad s_{\lambda+\kappa}^{(\kappa)} = \frac{1}{(\lambda+1)(\lambda+2)\ldots(\lambda+\kappa)}, \quad \ldots;$$

and $$h_1^{(0)} = \ldots = h_{\lambda-1}^{(0)} = 0, \quad h_\lambda^{(0)} = 1, \quad h_{\lambda+1}^{(0)} = 1, \quad \ldots,$$

$$h_1^{(1)} = \ldots = h_{\lambda-1}^{(1)} = 0, \quad h_\lambda^{(1)} = \frac{1}{\lambda}, \quad h_{\lambda+1}^{(1)} = \frac{2}{\lambda+1}, \quad \ldots,$$

$$h_1^{(2)} = \ldots = h_{\lambda-1}^{(2)} = 0, \quad h_\lambda^{(2)} = \frac{1}{\lambda^2}, \quad \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$h_1^{(\kappa)} = \ldots = h_{\lambda-1}^{(\kappa)} = 0, \quad h_\lambda^{(\kappa)} = \lambda^{-\kappa}, \quad \ldots.$$

Writing equation (1) for $n = \lambda + \kappa$, and substituting these special values, we obtain

$$d_{\lambda,\kappa} = (\lambda+\kappa)\left\{\frac{\lambda(\lambda+1)\ldots(\lambda+\kappa-1)}{(\lambda+\kappa)^\kappa} - \frac{\lambda^\kappa}{(\lambda+1)(\lambda+2)\ldots(\lambda+\kappa)}\right\},$$

which is, for $\kappa = 2$, in accordance with the above expression for $s_n^{(2)}$.

IV. *Lemma α. The coefficient $d_{\lambda,\kappa}$ is a positive number for* $\lambda = 1, 2, \ldots, \kappa = 2, 3, \ldots$.

It is easy to verify the inequalities

$$\frac{\lambda+\mu}{\lambda+\kappa} \geqq \frac{\lambda}{\lambda+\kappa-\mu} \qquad (\mu = 0, 1, 2, \ldots, \kappa-1),$$

from which results, by multiplication of all the left-hand and all the right-hand sides,

$$\frac{\lambda(\lambda+1)\ldots(\lambda+\kappa-1)}{(\lambda+\kappa)^\kappa} \geqq \frac{\lambda^\kappa}{(\lambda+1)(\lambda+2)\ldots(\lambda+\kappa)},$$

which demonstrates the assertion.

Lemma β. The coefficient $d_{\lambda,\kappa}$ satisfies the equation

$$\lim_{n\to\infty}\left\{d_{n,\kappa} - \frac{1}{6}(\kappa-1)\,\kappa\,(\kappa+1)\,\frac{1}{n}\right\} = 0 \quad \ldots\ldots\ldots(5).$$

To show this we expand $d_{\lambda,\kappa}$ in the form

$$d_{\lambda,\kappa} = \frac{\frac{1}{6}(\kappa-1)\,\kappa\,(\kappa+1)\,\lambda^{2\kappa-3} + \cdots}{\lambda^{2\kappa-2}(1+\alpha\lambda^{-1}+\ldots)} = \frac{1}{6}(\kappa-1)\,\kappa\,(\kappa+1)\,\frac{1}{\lambda} + \frac{c}{\lambda^2} + \ldots,$$

and the proposition follows.

A consequence of (5) is

$$\lim_{n\to\infty} \sum_{\lambda=1}^{n} d_{\lambda,\kappa} = \infty,$$

and therefore the conditions of Stolz's theorem are satisfied. Thus we can state:

Lemma γ. If $\lim_{n\to\infty} h_n^{(\kappa)} = H$ *exists and is finite, then*

$$\lim_{n\to\infty} \sum_{\lambda=1}^{n} d_{\lambda,\kappa} h_\lambda^{(\kappa)} \Big/ \sum_{\lambda=1}^{n} d_{\lambda,\kappa} = H.$$

V. Let $a_1 = 1, \quad a_2 = a_3 = \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6);$

then $s_n^{(0)} = 1 \qquad (n = 1, 2, \ldots),$

$$s_n^{(1)} = \frac{n-1}{n} \qquad (n = 2, 3, \ldots):$$

therefore $\lim_{n\to\infty} s_n^{(1)} = 1$, and consequently

$$\lim_{n\to\infty} s_n^{(2)} = \lim_{n\to\infty} s_n^{(3)} = \ldots = 1.$$

Furthermore $h_n^{(0)} = h_n^{(1)} = h_n^{(2)} = \ldots = 1 \quad (n = 1, 2, \ldots).$

From (1) and (6) we obtain

$$\lim_{n\to\infty} s_n^{(\kappa)} = 1 = \lim_{n\to\infty} c_{n,\kappa} - \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n-\kappa} d_{\lambda,\kappa},$$

or

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa} = 0.$$

VI. Now passing in equation (1) to the limit $n \to \infty$, we find

$$\lim_{n\to\infty} s_n^{(\kappa)} = \lim_{n\to\infty} h_{n-\kappa}^{(\kappa)} - \lim_{n\to\infty} \left(\frac{1}{n} \sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa} \right) \lim_{n\to\infty} \left\{ \sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa} h_\lambda^{(\kappa)} \Big/ \sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa} \right\},$$

and this equation leads to the theorems:

Theorem 3. *Whenever* $\lim_{n\to\infty} h_{n-\kappa}^{(\kappa)}$ *exists and is finite, then* $\lim_{n\to\infty} s_n^{(\kappa)}$ *exists too and has the same value.*

More generally

Theorem 4. *When the function* $h_{n-\kappa}^{(\kappa)}$ *oscillates between finite limits, then* $s_n^{(\kappa)}$ *oscillates between the same limits.*

VII. The reverse propositions can be established in the same way. From the definitions follow

$$h_n^{(1)} = \frac{n+1}{n} s_{n+1}^{(1)}, \qquad s_\lambda^{(1)} = (\lambda+1) s_{\lambda+1}^{(2)} - \lambda s_\lambda^{(2)},$$

$$h_n^{(2)} = \frac{1}{n} \left[\sum_{\lambda=1}^{n} \frac{\lambda+1}{\lambda} \{(\lambda+2) s_{\lambda+2}^{(2)} - (\lambda+1) s_{\lambda+1}^{(2)}\} + \frac{n+2}{n+1} \{(n+2) s_{n+2}^{(2)} - (n+2) s_{n+2}^{(2)}\} \right]$$

$$= \frac{(n+2)^2}{n(n+1)} s_{n+2}^{(2)} + \frac{1}{n} \sum_{\lambda=1}^{n} \frac{\lambda+2}{\lambda(\lambda+1)} s_{\lambda+2}^{(2)}.$$

Continuing this process we find

$$h_n^{(\kappa)} = e_{n,\kappa} s_{n+\kappa}^{(\kappa)} + \frac{1}{n} \sum_{\lambda=1}^{n} f_{\lambda,\kappa} s_{\lambda+\kappa}^{(\kappa)} \quad \ldots\ldots\ldots\ldots\ldots(7).$$

In the same way as we determined $c_{n,\kappa}$ by (2) and (3), we obtain here

$$e_{n,\kappa+1} = e_{n+1,\kappa} \frac{n+\kappa+1}{n}.$$

Thus, since $e_{n,1} = \frac{n+1}{n}$,

$$e_{n,\kappa} = \frac{(n+\kappa)^{\kappa}}{n(n+1)\ldots(n+\kappa-1)}.$$

Taking the values (4) for the numbers a_λ, we find from (7)

$$f_{\lambda,\kappa} = \lambda \left\{ \frac{(\lambda+1)(\lambda+2)\ldots(\lambda+\kappa)}{\lambda^{\kappa}} - \frac{(\lambda+\kappa)^{\kappa}}{\lambda(\lambda+1)\ldots(\lambda+\kappa-1)} \right\}.$$

The considerations in § IV show that

$$f_{\lambda,\kappa} \geqq 0, \qquad (\lambda = 1, 2, \ldots;\ \kappa = 2, 3, \ldots).$$

Expansion of $f_{\lambda,\kappa}$ in descending powers of λ gives

$$f_{\lambda,\kappa} = \frac{1}{6}(\kappa-1)\,\kappa\,(\kappa+1)\frac{1}{\lambda} + \frac{c}{\lambda^2} + \ldots;$$

thus

$$\lim_{n\to\infty} \sum_{\lambda=1}^{n} f_{\lambda,\kappa} = \infty.$$

Introducing in equation (7) the values (6) for the numbers a_λ, we find

$$1 = \lim_{n\to\infty} e_{n,\kappa} + \lim_{n\to\infty} \left(\frac{1}{n} \sum_{\lambda=1}^{n} f_{\lambda,\kappa} \right) \lim_{n\to\infty} \sum_{\lambda=1}^{n} f_{\lambda,\kappa} s_{\lambda+\kappa}^{(\kappa)} \Big/ \sum_{\lambda=1}^{n} f_{\lambda,\kappa};$$

which gives, on account of Stolz's theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\lambda=1}^{n} f_{\lambda,\kappa} = 0.$$

Thus equation (7) is completely determined and leads to

Theorem 5. *Whenever* $\lim_{n\to\infty} s_{n+\kappa}^{(\kappa)}$ *exists and is finite, then* $\lim_{n\to\infty} h_n^{(\kappa)}$ *exists too and has the same value.*

Theorem 6. *Whenever the function* $s_{n+\kappa}^{(\kappa)}$ *oscillates finitely, then* $h_n^{(\kappa)}$ *oscillates between the same limits.*

Theorems 3 and 5 together constitute theorem 2.

VIII. The relation connecting Cesàro's and Hölder's means can be deduced in the same way. We have

$$S_n^{(0)} = h_n^{(0)}, \qquad h_n^{(0)} = n h_n^{(1)} - (n-1)\, h_{n-1}^{(1)},$$

$$\frac{1}{n} S_n^{(1)} = \frac{1}{n} \sum_{\lambda=1}^{n} \{\lambda h_\lambda^{(1)} - (\lambda-1)\, h_{\lambda-1}^{(1)}\} = h_n^{(1)},$$

$$\frac{2 S_n^{(2)}}{n(n+1)} = \frac{2}{n(n+1)} \left[\sum_{\lambda=1}^{n} \lambda \{\lambda h_\lambda^{(2)} - (\lambda-1) h_{\lambda-1}^{(2)}\} + (n+1)\{n h_n^{(2)} - n h_n^{(2)}\} \right]$$

$$= 2 h_n^{(2)} - \frac{2}{n(n+1)} \sum_{\lambda=1}^{n} \lambda h_\lambda^{(2)}.$$

Assuming therefore

$$\frac{S_n^{(\kappa)}}{\binom{n+\kappa-1}{\kappa}} = c_{n,\kappa} h_n^{(\kappa)} - \frac{1}{\binom{n+\kappa-1}{\kappa}} \sum_{\lambda=1}^{n} d_{\lambda,\kappa} h_\lambda^{(\kappa)} \quad \ldots\ldots(8),$$

we find $$S_n^{(\kappa+1)} = \binom{n+\kappa}{\kappa} n c_{n+1,\kappa} h_n^{(\kappa+1)} - \sum_{\lambda=1}^{n} d_{\lambda,\kappa+1} h_\lambda^{(\kappa+1)}.$$

Hence $$c_{n,\kappa+1} = (\kappa+1)\, c_{n+1,\kappa},$$

or $$c_{n,\kappa} = \kappa!.$$

Starting from the numbers (4), we have

$$S_1^{(0)} = \ldots = S_{\lambda-1}^{(0)} = 0, \; S_\lambda^{(0)} = S_{\lambda+1}^{(0)} = \ldots = 1,$$

$$S_1^{(1)} = \ldots = S_{\lambda-1}^{(1)} = 0, \; S_\lambda^{(1)} = 1, \; S_{\lambda+1}^{(1)} = 2, \; \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_1^{(\kappa)} = \ldots = S_{\lambda-1}^{(\kappa)} = 0, \; S_\lambda^{(\kappa)} = 1, \; \ldots, \; S_n^{(\kappa)} = \binom{n-\lambda+\kappa}{\kappa}, \; \ldots;$$

as is easily verified by the formula

$$S_n^{(\kappa+1)} = \sum_{\nu=1}^{n} S_\nu^{(\kappa)} = \sum_{m=0}^{n-\lambda} \binom{m+\kappa}{\kappa} = \binom{n-\lambda+\kappa+1}{\kappa+1}.$$

Writing (8) for $n = \lambda$, we find

$$d_{\lambda,\kappa} = \lambda(\lambda+1)\ldots(\lambda+\kappa-1) - \lambda^\kappa \geqq 0.$$

Starting with the numbers (6), that is with the numbers (4) for $\lambda = 1$, we have

$$S_n^{(\kappa)} = \binom{n+\kappa-1}{\kappa}, \qquad h_n^{(\kappa)} = 1,$$

and from formula (8) follows

$$\sum_{\lambda=1}^{n} d_{\lambda,\kappa} = (\kappa! - 1) \binom{n+\kappa-1}{\kappa},$$

so that finally

$$\frac{S_n^{(\kappa)}}{\binom{n+\kappa-1}{\kappa}} = \kappa! h_n^{(\kappa)} - (\kappa!-1)\frac{\sum_{\lambda=1}^{n} d_{\lambda,\kappa} h_\lambda^{(\kappa)}}{\sum_{\lambda=1}^{n} d_{\lambda,\kappa}} \quad \ldots\ldots\ldots(9).$$

Analogous considerations lead to

$$h_n^{(\kappa)} = \frac{1}{\kappa!}\frac{S_n^{(\kappa)}}{\binom{n+\kappa-1}{\kappa}} + \left(1-\frac{1}{\kappa!}\right)\frac{\sum_{\lambda=1}^{n} f_{\lambda,\kappa} S_\lambda^{(\kappa)}}{\sum_{\lambda=1}^{n} f_{\lambda,\kappa}\binom{\lambda+\kappa-1}{\kappa}} \quad \ldots(10),$$

$$f_{\lambda,\kappa} = \lambda\left\{\frac{1}{\lambda^\kappa} - \frac{1}{\lambda(\lambda+1)\ldots(\lambda+\kappa-1)}\right\} \geqq 0,$$

$$\sum_{\lambda=1}^{n} f_{\lambda,\kappa}\binom{\lambda+\kappa-1}{\kappa} = n\left(1-\frac{1}{\kappa!}\right).$$

Formulae (9) and (10) prove theorem 1.

IX. By similar considerations it is possible to arrive at a statement about the equivalence of two means of the kind examined in Part II of my above quoted paper.

Let b_κ, c_κ denote the terms of two infinite sequences of positive real numbers, which have, when we write

$$\sum_1^n b_\kappa = B_\kappa, \qquad \sum_1^n c_\kappa = C_\kappa,$$

the properties (i) $\lim_{n\to\infty} B_n = \infty$, $\lim_{n\to\infty} C_n = \infty$,

(ii) $\frac{1}{n}\sum_1^n \frac{\kappa b_\kappa}{B_\kappa}$, $\frac{1}{n}\sum_1^n \frac{\kappa c_\kappa}{C_\kappa}$

tend to limits or oscillate between finite limits. Then putting as before

$$s_n^{(0)} = \sum_1^n a_n,$$

the means

$$s_n^{(1)} = \frac{1}{B_n}\sum_2^n b_\lambda s_{\lambda-1}^{(0)} \quad \ldots\ldots\ldots(11),$$

$$t_n^{(1)} = \frac{1}{C_n}\sum_2^n c_\lambda s_{\lambda-1}^{(0)} \quad \ldots\ldots\ldots(12),$$

are connected by two analogous relations. From (11) and (12) follow

$$b_n s_{n-1}^{(0)} = B_n s_n^{(1)} - B_{n-1} s_{n-1}^{(1)} \quad \ldots\ldots\ldots(13),$$

$$c_n s_{n-1}^{(0)} = C_n t_n^{(1)} - C_{n-1} t_{n-1}^{(1)} \ldots\ldots\ldots(14).$$

Substituting (14) in (11) we find, on adding a term which is zero,

$$s_n^{(1)} = \frac{1}{B_n}\left[\frac{b_2}{c_2}C_2 t_2^{(1)} + \frac{b_3}{c_3}\{C_3 t_3^{(1)} - C_2 t_2^{(1)}\} + \cdots \right.$$
$$\left. \cdots + \frac{b_n}{c_n}\{C_n t_n^{(1)} - C_{n-1} t_{n-1}^{(1)}\} + \frac{b_{n+1}}{c_{n+1}}\{C_n t_n^{(1)} - C_n t_n^{(1)}\}\right]$$
$$= \frac{1}{B_n}\sum_{\lambda=2}^{n}\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda t_\lambda^{(1)} + \frac{b_{n+1}}{B_n}\frac{C_n}{c_{n+1}} t_n^{(1)}.$$

Since $$\sum_{1}^{n}\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda = \sum_{2}^{n}\frac{b_\lambda}{c_\lambda}(C_\lambda - C_{\lambda-1}) + \frac{b_1 C_1}{c_1} - \frac{b_{n+1}}{c_{n+1}} C_n$$
$$= B_n\left[1 - \frac{b_{n+1}}{B_n}\frac{C_n}{c_n}\right],$$

we can write

$$s_n^{(1)} = t_n^{(1)} + \left(1 - \frac{b_{n+1}}{B_n}\frac{C_n}{c_{n+1}}\right)\left\{-t_n^{(1)} + \frac{\sum_{2}^{n}\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda t_\lambda^{(1)}}{\sum_{2}^{n}\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda}\right\} \quad \ldots(15).$$

Now there may be distinguished two possibilities:

Theorem 7. *If*

$$\left|\frac{\sum_{2}^{n}\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda t_\lambda^{(1)}}{\sum_{2}^{n}\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda}\right| < K \text{ (\textit{fixed}),}$$
$$\lim_{n\to\infty}\frac{C_n}{c_{n+1}}\frac{b_{n+1}}{B_n} = 1,$$

and if $t_n^{(1)}$ approaches a finite limit (or oscillates between finite limits), then $s_n^{(1)}$ approaches the same limit (or oscillates between the same limits).

Theorem 8. *If*

$$\lim_{n\to\infty}\frac{\sum_{2}^{n}\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda t_\lambda^{(1)}}{\sum_{2}^{n}\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda} = \lim_{n\to\infty} t_n^{(1)},$$
$$\frac{b_{n+1}}{B_n}\frac{C_n}{c_{n+1}} < K \text{ (\textit{fixed}),}$$

and if $t_n^{(1)}$ approaches a limit, then $s_n^{(1)}$ approaches the same limit.

This is a known theorem*.

The second relation results by substituting from (13) in (12) and proceeding in the same way. The same formula is arrived at by interchanging in (15) b_κ and c_κ, B_n and C_n, $t_n^{(1)}$ and $s_n^{(1)}$. From it we infer two theorems analogous to (7) and (8).

* Bromwich, *Infinite Series*, p. 386, Theorem V.

Notes on the Theory of Vibrations. (1) *Vibrations of Finite Amplitude.* (2) *A Theorem due to Routh.* By W. J. Harrison, M.A., Fellow of Clare College.

[*Read* 3 May 1920.]

(I) Lord Rayleigh in his *Theory of Sound*, Vol. I, has considered the effect of introducing terms depending on x^2 and x^3 into the simple equation of vibratory motion $\frac{d^2x}{dt^2} + n^2x = 0$. He treats the added terms as small and employs the method of successive approximation. The object of this note is to point out that exact integrals can be obtained in the form of the series of which Rayleigh determined the first two or three terms. The solutions now obtained are valid for any relative magnitude of the added terms subject to the motion remaining vibratory.

(*a*) *The Symmetric System.*

The equation of motion is

$$\frac{d^2x}{dt^2} + n^2x \mp 2\beta x^3 = 0,$$

where β is positive, and the upper sign is taken in the first instance.

A first integral is

$$\left(\frac{dx}{dt}\right)^2 = n^2a^2 - \beta a^4 - n^2x^2 + \beta x^4,$$

where a is the amplitude of the vibration.

We have

$$\left(\frac{dx}{dt}\right)^2 = (a^2 - x^2)(n^2 - \beta a^2 - \beta x^2),$$

or

$$\left(\frac{dx_1}{dt}\right)^2 = (n^2 - \beta a^2)(1 - x_1^2)(1 - k^2x_1^2),$$

where

$$ax_1 = x, \quad k^2 = \beta a^2/(n^2 - \beta a^2).$$

Hence*

$$\begin{aligned} x &= ax_1 \\ &= a \operatorname{sn}\{(n^2 - \beta a^2)^{\frac{1}{2}}t, k\}, \quad (x = 0 \text{ when } t = 0) \\ &= \frac{2\pi a}{Kk} \sum_0^\infty \frac{q^{m+\frac{1}{2}}}{1 - q^{2m+1}} \sin \frac{(2m+1)\pi (n^2 - \beta a^2)^{\frac{1}{2}}t}{2K}. \end{aligned}$$

* For the expansions of elliptic functions quoted in this paper see Whittaker and Watson, *Modern Analysis*, 1915, p. 504, and Example (5), p. 513; or Hancock, *Theory of Elliptic Functions*, Vol. I, 1910, pp. 486, 494, 495.

Let the units of length and time be chosen so that $a = 1$, $n = 1$. It is necessary that $\beta < \frac{1}{2}$, otherwise $\frac{dx}{dt}$ vanishes first for $0 < x < 1$.

The effect of the term $2\beta x^3$ on the vibrations can be exhibited by the results of numerical calculation given in the following table:

β	x	p
·01	1·0006 sin pt + ·0006 sin $3pt$	9924
·1	1·0074 sin pt + ·0074 sin $3pt$	·9214
·3	1·0335 sin pt + ·0348 sin $3pt$ + ·0012 sin $5pt$	·7309
·4	1·0632 sin pt + ·0676 sin $3pt$ + ·0046 sin $5pt$ + ·0003 sin $7pt$	·5997
·45	1·0928 sin pt + ·1028 sin $3pt$ + ·0108 sin $5pt$ + ·0011 sin $7pt$ + ·0001 sin $9pt$	·5063
·5	tanh $(t/\sqrt{2})$	

We proceed to consider the equation

$$\frac{d^2x}{dt^2} + n^2x + 2\beta x^3 = 0.$$

If a is the amplitude of the motion as before, we have

$$\left(\frac{dx}{dt}\right)^2 = (a^2 - x^2)\,(n^2 + \beta a^2 + \beta x^2),$$

or
$$\left(\frac{dx_1}{dt}\right)^2 = (n^2 + \beta a^2)\,(1 - x_1^2)\,(1 + \mu^2 x_1^2),$$

where
$$ax_1 = x, \text{ and } \mu^2 = \beta a^2/(n^2 + \beta a^2).$$

Write $1 - x_1^2 = z^2$, so that

$$\left(\frac{dz}{dt}\right)^2 = (n^2 + 2\beta a^2)\,(1 - z^2)\,(1 - k^2 z^2),$$

where
$$k^2 = \mu^2/(1 + \mu^2) = \beta a^2/(n^2 + 2\beta a^2).$$

Thus $z = \operatorname{sn}\{(n^2 + 2\beta a^2)^{\frac{1}{2}}t, k\}$, $(z = 0 \text{ when } t = 0)$.

Therefore

$$\begin{aligned} x &= ax_1 \\ &= a \operatorname{cn}\{(n^2 + 2\beta a^2)^{\frac{1}{2}}t, k\} \\ &= \frac{2\pi a}{Kk} \sum_0^\infty \frac{q^{m+\frac{1}{2}}}{1 + q^{2m+1}} \cos \frac{(2m+1)\,\pi\,(n^2 + 2\beta a^2)^{\frac{1}{2}}t}{2K}. \end{aligned}$$

In this case there is no limit to the value of β, the motion remains vibratory, but the period of the gravest mode decreases

as β increases. The results of calculation, with $n = 1$, $a = 1$, are as follows:

β	x	p
·1	·995 cos pt + ·005 cos 3pt	1·072
·5	·9818 cos pt + ·0179 cos 3pt + ·0003 cos 5pt	1·318
1·0	·9742 cos pt + ·0253 cos 3pt + ·0006 cos 5pt	1·569
10·0	·9582 cos pt + ·0402 cos 3pt + ·0016 cos 5pt + ·0001 cos 7pt	3·975
100·0	·9555 cos pt + ·0427 cos 3pt + ·0018 cos 5pt + ·0001 cos 7pt	12·03

(*b*) *The Asymmetric System.*

The equation of motion is

$$\frac{d^2x}{dt^2} + n^2x + \tfrac{3}{2}\alpha x^2 = 0,$$

where α may be assumed to be positive, as changing the sign of α is equivalent to reversing the direction of the axis of x.

Let the scale of time be such that $n = 1$, and the scale of length chosen so that the amplitude of the motion measured from $x = 0$ in the direction of x positive is unity. Then

$$\left(\frac{dx}{dt}\right)^2 = (1 - x)(1 + \alpha + x + \alpha x^2)$$
$$= (1 - x)(b + x)(c + x),$$

where
$$b = \tfrac{1}{2}\{1 + \alpha - (1 - 2\alpha - 3\alpha^2)^{\frac{1}{2}}\}/\alpha,$$
$$c = \tfrac{1}{2}\{1 + \alpha + (1 - 2\alpha - 3\alpha^2)^{\frac{1}{2}}\}/\alpha.$$

The limits of the vibration are $x = 1$ and $x = -b$. It is necessary that α should be less than $\frac{1}{3}$, so that the greatest value of b is 2.

Writing $1 - x = (b + 1)y^2$, we have

$$\left(\frac{dy}{dt}\right)^2 = \tfrac{1}{4}\alpha(c + 1)(1 - y^2)(1 - k^2y^2),$$

where $k^2 = (b + 1)/(c + 1)$.

Hence

$$y = \operatorname{sn}\{\tfrac{1}{2}\alpha^{\frac{1}{2}}(c + 1)^{\frac{1}{2}}t, k\},$$

and
$$x = 1 - (b + 1)\operatorname{sn}^2\{\tfrac{1}{2}\alpha^{\frac{1}{2}}(c + 1)^{\frac{1}{2}}t, k\}$$
$$= 1 - \frac{b + 1}{k^2}\left\{1 - \frac{E}{K} - \frac{2\pi^2}{K^2}\sum_1^\infty \frac{mq^m}{1 - q^{2m}}\cos\frac{m\pi\alpha^{\frac{1}{2}}(c + 1)^{\frac{1}{2}}t}{2K}\right\}.$$

The results of calculation are as follows:

a	b	x	p
·1	1·1125	$-$ ·0838 + 1·0557 cos pt + ·0284 cos $2pt$ + ·0006 cos $3pt$	·9855
·2	1·2680	$-$ ·2059 + 1·1306 cos pt + ·0712 cos $2pt$ + ·0033 cos $3pt$	·9477
·3	1·5657	$-$ ·4634 + 1·2634 cos pt + ·1783 cos $2pt$ + ·0190 cos $3pt$ + ·0018 cos $4pt$ + ·0002 cos $5pt$	·8152

The calculations have been performed for illustrative purposes only, and no special care has been taken to ensure the accuracy of the digits in the final decimal places.

(*c*) The solution of the equation

$$\frac{d^2x}{dt^2} + n^2x + \tfrac{3}{2}\alpha x^2 + 2\beta x^3 = 0$$

in the form of a Fourier Series requires rather more elaboration of the algebra.

The motion presents one novel feature which does not appear in the previous solutions. If β be positive, however small, the motion remains vibratory for any finite value of α, and if α and α/β be great, the amplitude of the motion on one side is approximately α/β times its amplitude on the other.

(II) Routh has shown (*vide Advanced Rigid Dynamics*, 1905, p. 56) that an increase in the inertia of any part of a vibrating system will increase all the periods in such a way that the modified periods are separated by the periods of the original system. This is true in general if the inertia of only one part of the system be increased, the definition of a single part being that the effect of increasing its inertia can be represented by a single term

$$\tfrac{1}{2}(\mu_1\dot{q}_1 + \mu_2\dot{q}_2 + \ldots)^2$$

in the expression for the kinetic energy, where q_1, q_2, ... are the normal coordinates of the original system. For example, the theorem is applicable to the case of an additional mass attached at a single point of a stretched string, but not to the case of an increase of mass spread over a portion of the string, or to the case of two or more masses attached at different points.

The theorem may be simply proved as follows. Let the modified kinetic energy be

$$\tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \ldots) + \tfrac{1}{2}(\mu_1\dot{q}_1 + \mu_2\dot{q}_2 + \ldots)^2,$$

and the potential energy be

$$\tfrac{1}{2}(\lambda_1^2 q_1^2 + \lambda_2^2 q_2^2 + \ldots).$$

The equations of motion are typified by

$$\ddot{q}_r + \lambda_r^2 q_r + \mu_r(\mu_1\ddot{q}_1 + \mu_2\ddot{q}_2 + \ldots) = 0.$$

The determinantal equation for the periods is

$$\begin{vmatrix} \lambda^2(1+\mu_1^2)-\lambda_1^2, & \mu_1\mu_2\lambda^2, & \mu_1\mu_3\lambda^2, \ldots \\ \mu_1\mu_2\lambda^2, & \lambda^2(1+\mu_2^2)-\lambda_2^2, & \mu_2\mu_3\lambda^2, \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} = 0. \quad \ldots(1)$$

Let $\lambda_1^2, \lambda_2^2, \ldots$ be arranged in ascending order of magnitude. If $\lambda^2 = 0$, the left-hand side of (1) is $(-1)^n$ as regards sign. If $\lambda^2 = \lambda_1^2$, the left-hand side of (1) is equal to

$$\mu_1^2 \begin{vmatrix} \lambda_1^2, & \mu_2\lambda_1^2, & \mu_3\lambda_1^2, \ldots \\ \mu_2\lambda_1^2, & (\mu_2^2+1)\lambda_1^2-\lambda_2^2, & \mu_2\mu_3\lambda_1^2, \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix}$$

$$= \mu_1^2 \begin{vmatrix} \lambda_1^2 & \mu_2\lambda_1^2 & \mu_3\lambda_1^2 \ldots \\ 0 & \lambda_1^2-\lambda_2^2 & 0 \ldots \\ 0 & 0 & \lambda_1^2-\lambda_3^2 \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix},$$

and this is $(-1)^{n-1}$ as regards sign.

Hence all the roots in λ^2 are decreased and they are separated by $\lambda_1^2, \lambda_2^2, \ldots$.

The validity of this proof depends on (1) the non-equality of any of the values of $\lambda_1^2, \lambda_2^2, \ldots$, (2) the non-evanescence of any of the constants $\mu_1, \mu_2, \ldots$. In case of (1) one period at least of the modified system is equal to a period of the original, but the theorem may be held to cover this case.

In case of (2) the theorem does not remain true. Suppose the μ's are all zero except $\mu_r, \mu_s, \mu_t, \ldots$. Then only the periods corresponding to $q_r, q_s, q_t, \ldots$ are changed. The periods belonging to these coordinates will be increased and their new values will be separated by their old values. But these new periods bear no relation to the periods belonging to the remaining coordinates and can occupy any position in regard to them except as specified above. Hence the theorem does not seem to indicate where the modified periods must lie in regard to the complete system of periods of the undisturbed system.

An example is afforded by the modification introduced into the periods of a stretched string by a load attached at a point dividing the string into two lengths which are commensurable. Rayleigh's argument (*vide Theory of Sound*, Vol. I, p. 122), which serves to maintain the validity of the theorem in this case, is acceptable owing to the strictly defined relations which exist between the periods in both states. But in an ordinary dynamical problem the theorem must be held to break down in the exceptional cases under consideration since it fails completely to indicate the position of the modified periods in relation to the original periods.

Experiments with a plane diffraction grating. By G. F. C. Searle, Sc.D., F.R.S., University Lecturer in Experimental Physics.

[*Read* 3 May 1920.]

Part I. Parallel Light.

§ 1. *Introduction**. When a plane grating is employed in accurate measurements of wave length, the rulings are set perpendicular to the direction of the incident beam of parallel light. When these two directions are not at right angles, the diffracted beam is no longer parallel to a plane containing the directions of (*a*) the incident beam and (*b*) a line intersecting the rulings at right angles. The formulae applicable to this general case are obtained in §§ 4, 5, 7; they are tested by the experiment of §§ 8, 9 for the restricted case in which the directions (*a*) and (*b*) are at right angles.

§ 2. *The grating axes.* It is necessary to specify the three axes of a plane grating and the origin from which they start.

For a transmission grating, the origin O is a point on the centre line of one of the openings. In a reflecting grating, O would lie on the centre line of one of the reflecting portions.

The axes are

(1) The normal ON to the plane of the grating.

(2) The transverse axis OT, a line through O cutting the rulings at right angles.

(3) The longitudinal axis OL, a line parallel to the rulings.

The grating interval, i.e. the common interval measured along OT from centre to centre of the openings, will be denoted by d.

§ 3. *Diffracted wave front and ray.* At a distance of thousands of wave lengths from the grating, the wavelets due to the separate openings will merge into practically a single wave. For the mathematical purposes of this paper we shall speak of this wave as the diffracted wave front and of a normal to it as the diffracted ray. We may speak of the diffracted wave front passing through the origin O, if we understand it to be a surface through O cutting at right angles the normals to the distant wave fronts. The normal through O may be called the diffracted ray through O.

In the case of reflexion or refraction at a polished surface, the time of passage from an incident wave front to a reflected or

* I have to thank Dr J. A. Wilcken of Christ's College, and Mr C. L. Wiseman, M.A. of Peterhouse. Dr Wilcken took the observations of § 12, Part I, and assisted in other ways. Mr Wiseman gave valuable help and criticism in the mathematical parts of the paper.

refracted front is independent of the particular ray. But, in the case of a grating, the time of passage from an incident to a diffracted front increases or diminishes by $i\tau$ as the point of incidence of the "ray" is moved from the centre of one opening to the centre of the next. Here τ is the periodic time of the vibration and i is a positive integer.

§ 4. *Diffraction of a plane wave; general case.* Take the axes of x, y, z to coincide with the axes ON, OT, OL of the grating, as in Fig. 1. Let R be a point on the centre line of the qth opening and let the coordinates of R be $0, qd, h$.

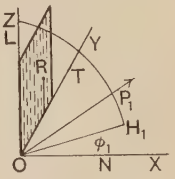


Fig. 1.

Let the direction cosines of the *forward* direction OP_1 of the incident beam be l_1, m_1, n_1, and let those of the *forward* direction OP_2 of the diffracted beam of order i be l_2, m_2, n_2.

Through O draw planes perpendicular to these two directions. The distance of R from the first plane, counted positive when the incident wave front reaches O before it reaches R, is $m_1qd + n_1h$. The distance of R from the second plane, counted positive when the diffracted front leaves O before it leaves R, is $m_2qd + n_2h$. If v_0 is the velocity of light and λ_0 the wave length in a vacuum, and if μ_1, μ_2 are the refractive indices of the media on the two sides of the grating, the times corresponding to the two distances are

$$\mu_1 (m_1qd + n_1h)/v_0 \text{ and } \mu_2 (m_2qd + n_2h)/v_0,$$

and these differ by $qi\tau$. Thus, since $\tau v_0 = \lambda_0$, we have

$$\mu_2 (m_2qd + n_2h) - \mu_1 (m_1qd + n_1h) = \mp qi\lambda_0.$$

This result must hold good for *all* positions of R on the grating, for which q is integral. We thus obtain

$$\mu_2 m_2 = \mu_1 m_1 \mp i\lambda_0/d, \quad \ldots\ldots(1)$$

$$\mu_2 n_2 = \mu_1 n_1. \quad \ldots\ldots(2)$$

These equations completely determine the directions of the diffracted beams of order i.

Let the incident and diffracted beams make angles ϵ_1, ϵ_2 with OT and angles η_1, η_2 with OL. Then

$$\cos \epsilon_1 = m_1, \quad \cos \epsilon_2 = m_2, \quad \ldots\ldots(3)$$

$$\cos \eta_1 = n_1, \quad \cos \eta_2 = n_2. \quad \ldots\ldots(4)$$

Hence (1) and (2) may be written

$$\mu_2 \cos \epsilon_2 = \mu_1 \cos \epsilon_1 \mp i\lambda_0/d, \quad \ldots\ldots(5)$$

$$\mu_2 \cos \eta_2 = \mu_1 \cos \eta_1. \quad \ldots\ldots(6)$$

Since $m_2^2 + n_2^2$ cannot exceed unity when the direction cosines are real, the condition that a diffracted beam may exist is $m_2^2 + n_2^2 \leqq 1$, or $\cos^2 \epsilon_2 \leqq \sin^2 \eta_2$. If ϵ_2 and η_2 lie between 0 and $\frac{1}{2}\pi$, this requires that $\eta_2 + \epsilon_2 \geqq \frac{1}{2}\pi$.

It is noteworthy that ϵ_2 depends only upon ϵ_1 and $i\lambda_0/d$, and that η_2 depends only upon η_1.

We shall not further consider the case in which μ_1 and μ_2 are unequal, but shall confine the work to the special case of $\mu_1 = \mu_2$. The reader will find no difficulty in making the necessary modifications.

§ 5. *Diffraction of a plane wave; single medium.* In practice each medium is air, of refractive index μ relative to a vacuum. If λ is the wave length in air, $\lambda_0 = \mu\lambda$. We then obtain the simple equations

$$m_2 = m_1 \mp i\lambda/d, \quad \text{or} \quad \cos \epsilon_2 = \cos \epsilon_1 \mp i\lambda/d, \quad \ldots\ldots(7)$$

$$n_2 = n_1, \quad \text{or} \quad \cos \eta_2 = \cos \eta_1. \quad \ldots\ldots(8)$$

Since η may be restricted to lie between 0 and π, we have

$$\eta_2 = \eta_1 = \eta. \quad \ldots\ldots(9)$$

The direction of the diffracted ray is easily constructed on a spherical diagram. Let the axes of the grating intersect a sphere about O as centre in N, T, L (Fig. 2), and let NON' be a diameter. Let the continuation of the incident ray through O meet the sphere in P_1. The great circle arc TP_1 measures ϵ_1. Calculate ϵ_2 by (7) and take $TQ = \epsilon_2$ on TP_1. About T and L as poles draw small circles through Q and P_1. Then $LP_1 = \eta$. If the small circles do not intersect, there will be no diffracted beam either by transmission or by reflexion. If the small circles intersect in the points P_2, P_2', then OP_2, OP_2' will be the directions of the two diffracted beams. Of the arcs NP_2, NP_2' of the great circle $NP_2P_2'N'$, one is greater and one less than $\frac{1}{2}\pi$. If NP_2 is *less* than $\frac{1}{2}\pi$, it corresponds to the transmitted beam, and then NP_2' corresponds to the reflected beam.

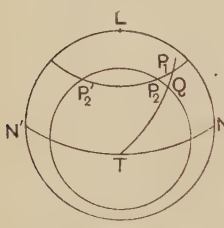


Fig. 2.

When ϵ_1 and i are given, there are *two* values of ϵ_2, and hence there are *two* points Q_- and Q_+ on TP_1. Thus there will be *two* directions $(OP_2)_-$ and $(OP_2)_+$ for the transmitted diffracted beam, and similarly for the reflected beam.

It may happen that only *one* of the two beams $(OP_2)_-$ and $(OP_2)_+$ exists. Unless P_1 is on the great circle LN, there will be two distinct values of m_2^2, and the condition $m_2^2 + n_2^2 \leqq 1$ may be satisfied by the smaller value of m_2^2 but not by the larger.

Since η_2 has the constant value η, while m_2 or $\cos \epsilon_2$ depends upon λ, it follows that, if white light is used, to each λ there will correspond a position of P_2 on the small circle through P_1 with L for pole.

§ 6. *The deviation.* If D is the angle ($< \pi$) between the forward directions of the incident and the transmitted diffracted beams, $\cos D = l_1 l_2 + m_1 m_2 + n_1 n_2$. If the plane ZOP_1 (Fig. 1) cuts OXY in OH_1, where $XOH_1 = \phi_1$, then, since $P_1OZ = \eta$,

$$l_1 = \sin \eta \cos \phi_1, \quad m_1 = \sin \eta \sin \phi_1, \quad n_1 = \cos \eta,$$

and similarly for P_2. Thus

$$1 - 2 \sin^2 \tfrac{1}{2}D = \cos D = \sin^2 \eta \cos (\phi_1 - \phi_2) + \cos^2 \eta,$$

and hence

$$\sin \tfrac{1}{2}D = \sin \eta \sin \tfrac{1}{2} (\phi_1 \sim \phi_2), \quad \text{.....(10)}$$

as can also be shown from the isosceles spherical triangle P_1LP_2 in Fig. 2.

In the case of the transmitted diffracted beam, $l_1 l_2$ is *positive*. Noting that $n_2 = n_1 = \cos \eta$, putting $m_1 = a + b$, $m_2 = a - b$, and substituting for l_1, l_2, we find

$$2 (\sin^2 \tfrac{1}{2}D - b^2) = \sin^2 \eta - a^2 - b^2 - [(\sin^2 \eta - a^2 - b^2)^2 - 4a^2 b^2]^{\frac{1}{2}}.$$

Thus $\sin^2 \frac{1}{2}D$ is greater than b^2 except when $a = 0$, and then the two are equal. When $a = 0$, $m_2 = - m_1$. If we take m_1 positive, we see, by (7), that, since i is positive, $m_1 = - m_2 = i\lambda/2d$. Hence $b = i\lambda/2d$. Thus, if D_0 is the minimum deviation,

$$\sin \tfrac{1}{2}D_0 = i\lambda/2d. \quad \text{.....(11)}$$

Since η does not occur in (11), $m_1 = - m_2$ gives a minimum of D for any given value of η—a minimum having the same value for *all* values of η.

§ 7. *The sloped grating.* For the experiment of §§ 8, 9 it is convenient to use axes differing from those of § 4. Now let OY (Fig. 3) coincide with OT and let OL make an angle θ with OZ. Then the direction cosines of OL are $\sin \theta$, 0, $\cos \theta$, those of OT are 0, 1, 0, and those of ON are $\cos \theta$, 0, $- \sin \theta$.

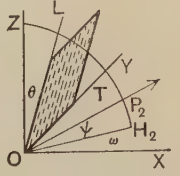


Fig. 3.

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of the forward directions OP_1, OP_2 of the incident and the transmitted diffracted beams. Let $n_2 = \sin \psi$, so that the diffracted ray OP_2 makes an angle ψ with the plane OXY, counted positive when $P_2OZ < \frac{1}{2}\pi$. Let the plane ZOP_2 cut OXY in OH_2 and let $XOH_2 = \omega$. Then, if η is the common angle between OP_1 or OP_2 and OL,

$$l_2 \sin\theta + n_2 \cos\theta = \cos\eta = l_1 \sin\theta + n_1 \cos\theta. \quad \ldots(12)$$

We also have, by (7), if $P_1OT = \epsilon_1$, $P_2OT = \epsilon_2$,

$$\cos\epsilon_2 = m_2 = m_1 \mp i\lambda/d = \cos\epsilon_1 \mp i\lambda/d. \quad \ldots\ldots(13)$$

Hence m_2 is known at once. Using $l_2^2 = 1 - m_2^2 - n_2^2$, we have, by (12),

$$\sin^2\theta\,(1 - m_2^2 - n_2^2) = (\cos\eta - n_2\cos\theta)^2.$$

Solving for n_2 and taking the *negative* sign in the ambiguity, we have

$$\sin\psi = n_2 = \cos\theta\cos\eta - \sin\theta\,(\sin^2\eta - m_2^2)^{\frac{1}{2}}. \quad \ldots(14)$$

Using this value of n_2 in (12), we find

$$l_2 = \sin\theta\cos\eta + \cos\theta\,(\sin^2\eta - m_2^2)^{\frac{1}{2}}. \quad \ldots\ldots\ (15)$$

Since $\cos NOP_2 = l_2\cos\theta - n_2\sin\theta$, we find from (14) and (15) that

$$\cos NOP_2 = (\sin^2\eta - m_2^2)^{\frac{1}{2}}. \quad \ldots\ldots\ldots\ (16)$$

Thus the *negative* sign has been correctly chosen in (14) for the *transmitted* beam, since for this $\cos NOP_2$ must be positive. If the positive sign is used in (14), $\cos NOP_2$ is negative, corresponding to the reflected diffracted beam.

In terms of ψ and ω, the direction cosines of OP_2 are $\cos\psi\cos\omega$, $\cos\psi\sin\omega$, $\sin\psi$. Hence $m_2 = \cos\psi\sin\omega$, and thus

$$\sin\omega = m_2/\cos\psi = \cos\epsilon_2/\cos\psi. \quad \ldots\ldots\ldots(17)$$

The two angles ψ and ω completely determine the direction of the diffracted ray.

In the experiment of §§ 8, 9 the incident rays are parallel to the axis OX of Fig. 3. Hence $l_1 = 1$, $m_1 = 0$, $n_1 = 0$. We then have

$$\cos\epsilon_2 = m_2 = \mp i\lambda/d, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

$$\cos\eta = \sin\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

and thus, since $\sin^2\eta - \cos^2\epsilon_2 = \sin^2\epsilon_2 - \cos^2\eta$,

$$\sin\psi = n_2 = \sin\theta\cos\theta - \sin\theta\,(\sin^2\epsilon_2 - \sin^2\theta)^{\frac{1}{2}}, \ldots(20)$$

$$l_2 = \sin^2\theta + \cos\theta\,(\sin^2\epsilon_2 - \sin^2\theta)^{\frac{1}{2}}, \ldots\ldots\ldots\ldots(21)$$

$$\sin\omega = m_2/\cos\psi = \cos\epsilon_2/\cos\psi. \quad \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

Since ϵ_2 and θ may be restricted to be less than $\frac{1}{2}\pi$, we see that no diffracted beam will be formed if θ exceeds its critical value ϵ_2.

§ 8. *Apparatus.* The general arrangement is shown in Fig. 4. The grating G is attached to a horizontal shaft A, with its plane parallel to A and its rulings perpendicular to A. A horizontal collimator L has horizontal and vertical cross-wires intersecting at C in its focal plane; these are illuminated by a sodium flame S. The straight line joining C to the appropriate nodal point of the

lens is the line of collimation, or axis, of the collimator. The parallel beam defined by C is parallel to this line. After the light has passed through the grating, it is received by a goniometer K, and an image of the collimator wires is formed in its focal plane. To fix the line of collimation, cross-wires are placed in the focal plane; they intersect in D. The goniometer is carried on a horizontal revolving shaft B, and its line of collimation is perpendicular to the shaft. One cross-wire is parallel and the other perpendicular to the shaft; the latter is also perpendicular to the shortest distance from D to the axis of the shaft. The shafts are provided with divided circles E, F, which are read by aid of the pointers U, U', V, V'. A balance weight W is attached to the circle F. The point of intersection of the line of collimation of K with the axis of B should lie approximately on the centre of the grating, and the line of collimation of L should pass through the same point.

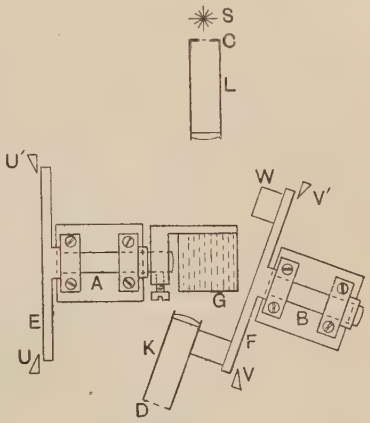


Fig. 4.

The angles θ and ψ are measured by the circles E and F.

§ 9. *Experimental details.* The shaft A is set horizontal by aid of a level. The collimator is adjusted optically by an auto-collimating method. The plane of the grating is set horizontal by a level, and the shaft is then turned through 90°, as measured by the circle E, so that the plane of the grating is vertical. Light from a flame is then reflected by a plate of glass held at 45° past the cross-wires and through the lens on to the grating, and the collimator, previously set for "infinity," is adjusted so that the image of C, the intersection of the wires, coincides with C itself. If the coincidence is recovered when the grating shaft is turned through 180°, the plane of the grating is parallel to the shaft. The line of collimation is then both horizontal and also perpendicular to the grating shaft.

The line of collimation of the goniometer is set perpendicular to the goniometer shaft B by an optical method. An auxiliary collimator, set for "infinity," is placed so that it is approximately perpendicular to the shaft. A plate of plane parallel glass is attached to the circle F near its centre so that its faces are approximately parallel to the shaft. It is convenient to make the shaft vertical; the glass plate can then be supported on a small levelling table resting on the circle. By adjusting both the collimator and the

plate, the faces of the plate are made parallel to the shaft and the axis of the collimator is made perpendicular to the shaft. In this case the image of the collimator wires can, by turning the circle, be made to coincide with those wires, when *either* side of the plate faces the collimator. If the plate has been suitably placed, it will be possible, by turning the goniometer on its shaft, to receive the image of the collimator wires on the focal plane of the goniometer. The "vertical" cross-wire of the goniometer, i.e. the wire perpendicular to the shaft, is now adjusted so that it coincides with the image of the corresponding wire of the collimator. The line of collimation is then perpendicular to the shaft. The goniometer is then put into position and its shaft is levelled.

The axes of the collimator, of the grating shaft and of the goniometer shaft are adjusted to be approximately in the same horizontal plane. The plane of the grating is made vertical, and the goniometer stand is adjusted in azimuth so that one of the diffracted images of the collimator wires can be made to coincide with the goniometer wires by turning the goniometer on its shaft.

When the adjustments already described have been effected, and when the plane of the grating G is vertical, the diffracted beams are parallel to the plane OTN. If OT is inclined at an angle δ to the grating shaft, the direction of OT will be changed by 2δ if G is turned through 180° about the axis of the shaft from Position 1 to Position 2, when the plane is again vertical. The goniometer is turned to receive a diffracted beam when G is in Position 1. If, when G is turned into Position 2, the inclination of the beam to a horizontal plane is changed, the grating must be turned in its own plane until the inclination is the same for both positions.

Since ψ is always small, $\cos\psi$ is nearly unity and hence, by (22), $\sin\omega$ has a nearly constant value, for m_2 is independent of θ. Hence, if the image of C, the intersection of the collimator wires, lies on D, the intersection of the goniometer wires, when the plane of G is vertical, the image of C will always lie very near the "vertical" cross-wire, and one setting of the goniometer stand will suffice for all values of θ.

The plane of the grating is made horizontal and the index reading is taken. It is then turned through 90°; it is now vertical and in its zero position. The goniometer is next adjusted so that the image of C lies on the horizontal wire of K. The goniometer is then in its zero position.

The grating is now turned through 10°, 20°, ... on either side of the zero, and the goniometer is turned on its shaft to bring the image of C on to the horizontal wire of K in each case. If the grating circle has *two* indices, the grating is turned through 10°, 20°, ... as indicated by one index. In reducing the observations the mean of the angles furnished by *both* indices is used.

Since $\epsilon_1 = \frac{1}{2}\pi$, $\cos\epsilon_2 = m_2 = i\lambda/d$. From the values of the interval d and the wave length λ, $\cos\epsilon_2$ is found and then the values of ψ corresponding to the mean values of θ are calculated by (20). These values are compared with the mean values of ψ given by the goniometer readings.

§ 10. *Distortion of the image.* As θ, and consequently ψ, increases, the observer sees that the angle between the images of the collimator wires undergoes great changes. When $\theta = 0$, the images are at right angles, but the angle diminishes rapidly as θ reaches its critical value. The theory shows that they are actually tangential one to the other when θ has its critical value, but, as no light is transmitted in the critical position, the phenomenon cannot be observed. If the collimator wires are stretched across a small circular opening, the image of the edge is distorted into an oval, which is practically an ellipse having the images of the wires as conjugate diameters. When, however, θ approaches its critical value, the oval begins to deviate from an ellipse.

In Fig. 5 let OX, OY or OT, OZ, OL, ON meet a sphere described about O as centre in X, T, Z, L, N. Let OJ be the diffracted ray corresponding to the incident ray OX; the ray OX corresponds to the line of collimation of the collimator and OJ to that of the goniometer, when the image of C is brought to the intersection of the goniometer wires. Let OP_1 be a ray nearly parallel to OX and let OP_2 be the corresponding diffracted ray. Let the great circles through Z and P_1, J, P_2 cut the great circle TXS in H, K, M. Let S be the pole of ZJK and let the great circle SJ meet ZP_2M in Q. Then $ZJQ = \frac{1}{2}\pi$.

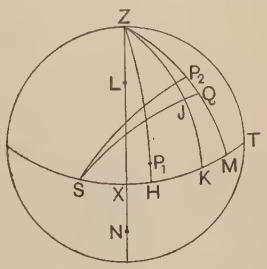


Fig. 5.

If the goniometer is mounted as described in § 8, and if its line of collimation coincides with OJ, its horizontal cross-wire will correspond to SJQ and its "vertical" wire to ZJK. The rays parallel to OP_2 will come to a focus in the focal plane of the goniometer at D', whose coordinates referred to the horizontal and vertical wires through D (Fig. 4) are $f \times$ angle QOJ and $f \times$ angle P_2OQ, where f is the focal length of the lens.

If points on a curve CC' in the focal plane of the collimator give rise to diffracted rays whose directions are shown by points on the curve JP_2 on the sphere, and if the image of CC' in the focal plane of the goniometer is DD', the angle between the "vertical" cross-wire and the tangent to DD' at D is equal to I, the angle between the great circle JZ and the tangent at J to the curve JP_2.

If $JK = \psi$, $P_2M = \psi'$, $XK = \omega$, $XM = \omega'$, then

$$\tan I = \text{limit of } \frac{JQ}{P_2Q} = \cos\psi \left(\frac{d\omega'}{d\psi'}\right)_\psi . \quad \ldots\ldots\ldots\ldots(23)$$

If $XH = \alpha$, $P_1H = \gamma$, the direction cosines of OP_1, OP_2 are

$$l_1 = \cos\gamma\cos\alpha, \qquad m_1 = \cos\gamma\sin\alpha, \qquad n_1 = \sin\gamma,$$
$$l_2 = \cos\psi'\cos\omega', \qquad m_2 = \cos\psi'\sin\omega', \qquad n_2 = \sin\psi'.$$

Since $ZOL = \theta$, the direction cosines of OL are $\sin\theta$, 0, $\cos\theta$. But $P_1OL = P_2OL = \eta$, $P_1OT = \epsilon_1$, $P_2OT = \epsilon_2$, and thus the fundamental equations (13) and (12) become

$$\cos\psi'\sin\omega' = \cos\gamma\sin\alpha \mp i\lambda/d, \quad \ldots\ldots\ldots\ldots(24)$$

$$\cos\psi'\cos\omega'\sin\theta + \sin\psi'\cos\theta = \cos\gamma\cos\alpha\sin\theta + \sin\gamma\cos\theta. \quad \ldots\ldots(25)$$

The vertical collimator wire corresponds to the great circle ZX, and for this $\alpha = 0$, but γ varies. If I_V is the inclination to JZ of the corresponding path described by P_2,

$$\tan I_V = \cos\psi \left(\frac{\partial\omega'}{\partial\gamma}\right)_0 \Big/ \left(\frac{\partial\psi'}{\partial\gamma}\right)_0 .$$

Differentiating (24) with respect to γ and then putting $\gamma = 0$, so that ψ', ω' become ψ, ω, we have

$$-\sin\psi\sin\omega\,(\partial\psi'/\partial\gamma)_0 + \cos\psi\cos\omega\,(\partial\omega'/\partial\gamma)_0 = 0.$$

Hence $$\tan I_V = \sin\psi\tan\omega.$$

The horizontal collimator wire corresponds to the great circle XT and for this $\gamma = 0$, but α varies. Differentiating (25) with regard to α and then putting $\alpha = 0$, we have

$$-\sin\psi\cos\omega\sin\theta\,(\partial\psi'/\partial\alpha)_0 - \cos\psi\sin\omega\sin\theta\,(\partial\omega'/\partial\alpha)_0 + \cos\psi\cos\theta\,(\partial\psi'/\partial\alpha)_0 = 0.$$

Hence, if I_H is the inclination to JZ of the corresponding path of P_2,

$$\tan I_H = \cos\psi \left(\frac{\partial\omega'}{\partial\alpha}\right)_0 \Big/ \left(\frac{\partial\psi'}{\partial\alpha}\right)_0 = \frac{\cos\psi\cos\theta - \sin\psi\cos\omega\sin\theta}{\sin\omega\sin\theta}. \quad \ldots\ldots(26)$$

Multiplying numerator and denominator by $\cos\omega$, and replacing $\cos^2\omega$ by $1 - \sin^2\omega$, we find

$$\tan I_H = \tan I_V + \frac{l_2\cos\theta - n_2\sin\theta}{\sin\omega\cos\omega\sin\theta}. \quad \ldots\ldots\ldots (27)$$

Since the direction cosines of ON are $\cos\theta$, 0, $-\sin\theta$, and l_2, n_2 in (27) refer to OJ, $l_2\cos\theta - n_2\sin\theta = \cos NOJ$. In the critical position, J lies on the great circle LT, which corresponds to the

plane of the grating. Then $\cos NOJ = 0$, and the difference between the tangents vanishes, i.e. the two curves touch at J.

Since the distance of ZLX from T is constant, the curve corresponding to the vertical cross-wire is a small circle passing through J with T for pole. At J the small circle is perpendicular to the great circle TJ and the value of $\tan I_V$ can be verified by spherical trigonometry.

The horizontal cross-wire is represented by the great circle XT, but now both ϵ_1 and η vary and no simple construction is available for the *whole* curve through J corresponding to this wire. The curve touches at J the small circle passing through J and X with L as pole, and cuts at right angles the great circle LJ. Hence $\tan I_H = \cot LJZ$, and then (26) can be verified by spherical trigonometry. If we find $\cos LJZ$ and $\sin LJZ$ and divide, we obtain the alternative form

$$\tan I_H = \frac{\cos\theta - \sin\psi \sin\theta}{\cos\psi \sin\omega \sin\theta}.$$

If the angle between the two small circles which intersect in J is Δ, then Δ is the supplement of LJT. But $LJ = \frac{1}{2}\pi - \theta$, $JT = \epsilon_2$, $LT = \frac{1}{2}\pi$, and hence

$$\cos\Delta = \tan\theta \cot\epsilon_2.$$

In the experiment $\cos\epsilon_2$ has the constant value $\mp i\lambda/d$, and thus $\cos\Delta$ depends only upon θ. The angle Δ will vanish when $\cos\Delta = 1$, and this occurs when $\theta = \epsilon_2$, i.e. when θ has its critical value.

We can make visible a finite arc of the small circle with L for pole. If we illuminate with white light the *small* opening across which the wires are stretched, the position of J on this small circle will be different for the different colours. The short length of cross-wire will correspond for any colour to a small arc of a curve touching the small circle at practically its middle point. The envelope of these small arcs will be the small circle itself. The image of the horizontal wire will thus be a dark curved line running across the spectrum.

§ 11. *The critical values.* The critical position of the grating is reached when $\theta = \epsilon_2$, and we have, by (20), (22), the critical values

$$\sin\psi_c = \sin\epsilon_2 \cos\epsilon_2, \qquad \sin\omega_c = \frac{\cos\epsilon_2}{(1 - \sin^2\epsilon_2 \cos^2\epsilon_2)^{\frac{1}{2}}},$$

$$\tan\omega_c = \frac{\cos\epsilon_2}{\sin^2\epsilon_2}, \qquad (\tan I_V)_c = \frac{\cos^2\epsilon_2}{\sin\epsilon_2}.$$

For the grating used in § 12 and with $i = 1$,

$\cos\epsilon_2 = 0{\cdot}33568$, $\sin\epsilon_2 = 0{\cdot}94198$, $\epsilon_2 = \frac{1}{2}\pi - 19^\circ\,36'\,49''$.

Then $\psi_c = 18^\circ\,26'$, $\omega_c = 20^\circ\,43'\,18''$ and $(I_V)_c = 6^\circ\,49'\,17''$.

When $\theta = 0$, and therefore $\psi = 0$,

$$\sin \omega = \cos \epsilon_2, \qquad \omega = 19^\circ\ 36'\ 49''.$$

Thus the maximum change in ω is only $1^\circ\ 6'\ 29''$.

§ 12. *Practical example.* The following results were obtained by Dr J. A. Wilcken, using a grating having 14,468 lines per inch, and, hence, an interval $d = 1{\cdot}7556 \times 10^{-4}$ cm.

The adjustments described in § 9 were either effected or tested. The plane of the grating was not *quite* parallel to the grating shaft, but as both images of the first order were observed, the mean results will be hardly affected. The calculated values of ψ were found on the assumption that the axis of the collimator is perpendicular to the transverse axis of the grating. Sodium light of mean wave length $\lambda = 5{\cdot}893 \times 10^{-5}$ cm. was used. Then, since the images were of the *first* order throughout,

$$m_2 = \frac{\lambda}{d} = \frac{5{\cdot}893 \times 10^{-5}}{1{\cdot}7556 \times 10^{-4}} = 0{\cdot}33568 = \sin 19^\circ\ 36'\ 49''.$$

Thus $\epsilon_2 = \cos^{-1} m_2 = 70^\circ\ 23'\ 11''$, and $m_2^2 = 0{\cdot}11268$.

Each of the observed values of ψ given in the table is the mean of four. Each of the first order images was observed, and for each image two values of θ, one on either side of zero, were taken. The grating circle, which was printed on card, was a little eccentric relative to the shaft (it was a "home-made" affair), and, consequently, although the settings were made to integral degrees by one index, the other index did not always read integral degrees. Some of the mean values of θ are, therefore, not integral degrees. The calculated values of ψ are those found from equation (20), which for convenience is written

$$\sin \psi = n_2 = \tfrac{1}{2} \sin 2\theta - \sin \theta\, [(\cos \theta + m_2)(\cos \theta - m_2)]^{\frac{1}{2}}.$$

For the sake of interest, the calculated values of ω, I_V and I_H have been added. The last line in the table gives the critical values as found by calculation.

θ mean obsd.	ψ mean obsd.	ψ calcd.	ω calcd.	I_V	I_H
° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0 0 0	0 0 0	0 0 0	19 36 49	0 0 0	90 0 0
9 54 0	0 36 0	0 34 51	19 36 52	0 12 25	86 38 28
19 54 0	1 12 45	1 12 33	19 37 7	0 25 51	83 1 5
29 50 0	1 55 30	1 55 36	19 37 34	0 41 13	78 53 41
39 51 0	2 51 45	2 50 18	19 38 19	1 0 46	73 42 30
49 48 0	4 7 30	4 7 25	19 39 58	1 28 20	66 31 55
59 49 0	6 26 15	6 22 31	19 44 27	2 16 55	54 29 53
64 44 0	8 32 15	8 29 8	19 50 24	3 2 53	44 1 14
67 45 0	10 54 0	10 51 13	19 59 10	3 55 4	33 20 15
69 44 0	14 18 0	14 10 3	20 15 19	5 9 40	20 20 49
70 23 11		18 26 0	20 43 18	6 49 17	6 49 17

The agreement between the observed and calculated values of ψ is satisfactory.

Part II. Non-parallel Light.

§ 1. *Introduction.* In the following experiments the incident light does not form a parallel beam. The diffraction now not merely changes the direction of the axial ray of the beam, but also, in general, introduces astigmatism into, or changes the astigmatism of, the incident beam. The exception is when the incident and diffracted axial rays are perpendicular to the rulings and the deviation is a minimum. The diffracted rays will, in general, pass through two focal lines when the aperture is small. If the aperture is increased, aberration will appear and all the rays will not pass accurately through the two lines. Aberration can be minimised by keeping the aperture small, but astigmatic effects are inseparable from the diffraction in the general case.

The formulae for the general case are easily obtained, but are complicated. We shall, therefore, consider only the case in which the axial ray of the incident beam is perpendicular to the rulings.

§ 2. *Diffraction of an astigmatic beam.* In Fig. 1 let OX, OY, OZ coincide with ON, OT, OL, the axes of the grating, as defined in Part I, § 2. For convenience, OZ will be taken as vertical.

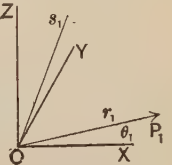


Fig. 1.

Let a beam, which started from a luminous point and therefore has a wave front, fall upon the grating near O. Let OP_1 be the continuation of the ray through O, *which has been restricted to lie in the plane* OXY, and let OP_1 be taken as the axial ray of the beam. Let $P_1OX = \theta_1$. Take OP_1 as the axis of r_1 in a new set of axes Or_1, Os_1, Ot_1, of which Os_1 is in the plane OXY and Ot_1 coincides with OZ. Let the equation to the incident wave front passing through O be

$$r_1 = \tfrac{1}{2}S_1 s_1^2 + W_1 s_1 t_1 + \tfrac{1}{2}T_1 t_1^2. \quad \ldots\ldots\ldots\ldots(1)$$

Let R be a point on the grating and let its coordinates referred to the grating axes be 0, qd, z, where d is the grating interval and q is an integer. Then the coordinates of R referred to the axes of the incident beam are

$$r_1 = qd \sin\theta_1, \quad s_1 = qd\cos\theta_1, \quad t_1 = z. \quad \ldots\ldots\ldots\ldots(2)$$

If a line through R parallel to OP_1 cuts the wave front OF_1 in F_1, the second and third coordinates of F_1 are $qd\cos\theta_1$ and z. By (1) and (2), the distance of F_1 from the plane $r_1 = 0$, which touches the wave front at O, is $\frac{1}{2}S_1 q^2 d^2 \cos^2\theta_1 + W_1 qdz\cos\theta_1 + \frac{1}{2}T_1 z^2$, and the distance of R from the same plane is $qd\sin\theta_1$. Hence

$$F_1R = qd\sin\theta_1 - \tfrac{1}{2}S_1 q^2 d^2\cos^2\theta_1 - W_1 qdz\cos\theta_1 - \tfrac{1}{2}T_1 z^2 \ldots.(3)$$

When R and F_1 approach O, F_1R becomes more and more nearly the normal at F_1, and, for a small aperture, may be treated as the normal in the estimation of *distances*. Thus, ultimately, F_1R is the ray distance from the wave front OF_1 to R.

Let OP_2 be a diffracted ray of order i. By symmetry, OP_2 is in the plane OXY, since OP_1 is in that plane. Let $P_2OX = \theta_2$. Take the axial ray OP_2 as the axis of r_2 in a set of axes Or_2, Os_2, Ot_2, of which Os_2 is in the plane OXY and Ot_2 coincides with OZ. Let the equation to the diffracted wave front passing through O be

$$r_2 = \tfrac{1}{2}S_2s_2^2 + W_2s_2t_2 + \tfrac{1}{2}T_2t_2^2. \quad \text{.....(4)}$$

Then, if F_2R, parallel to OP_2, cuts the diffracted wave front OF_2 in F_2, the distance F_2R is ultimately the ray distance from F_2 to R. We then have

$$F_2R = qd \sin\theta_2 - \tfrac{1}{2}S_2q^2d^2\cos^2\theta_2 - W_2qdz\cos\theta_2 - \tfrac{1}{2}T_2z^2 \text{....(5)}$$

The optical condition is that F_2R differs from F_1R by $qi\lambda$, where i is a positive integer. We thus obtain

$$F_2R = F_1R \pm qi\lambda.$$

Since this holds for all values of z and all integral values of q, we have, by (3) and (5),

$$\sin\theta_2 = \sin\theta_1 \pm i\lambda/d, \quad \text{.....(6)}$$

$$S_2 = k^2S_1, \quad W_2 = kW_1, \quad T_2 = T_1, \quad \text{.....(7)}$$

where $k = \cos\theta_1/\cos\theta_2$. Since θ_1 and θ_2 both lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ for a transmitted beam, k is positive.

The direction of the axial ray of the diffracted beam is given by (6) and is independent of the constants S_1, W_1, T_1. Equations (7) give the form of the diffracted wave front which passes through O.

If the deviation of the axial ray is a minimum, it follows from Part I, § 6, or otherwise, that $\sin\theta_2 = -\sin\theta_1$. Since θ_1 and θ_2 both lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, $\cos\theta_2 = \cos\theta_1$, and thus $k = 1$. Hence, in the case of minimum deviation, the form of the wave front is unchanged and the diffraction merely turns it through 2θ about OZ. The restriction stated in § 2 must be noted.

If the planes of the principal sections of the incident wave front are OXY and ZOP_1, or, what is the same thing, the planes Or_1s_1, Or_1t_1, then $W_1 = 0$. It follows, from (7), that $W_2 = 0$, and thus the principal planes of the diffracted wave front are OXY and ZOP_2.

When $W_1 = 0$, the section of the incident front by the horizontal plane $t_1 = 0$ is $r_1 = \frac{1}{2}S_1s_1^2$, and the distance of the centre of curvature of this section from O is S_1^{-1}. The vertical focal line of the beam passes through this centre of curvature. Similarly, the horizontal focal line is at a distance T_1^{-1} from O. The distances

from O of the vertical and horizontal focal lines of the diffracted beam are S_2^{-1} and T_2^{-1}.

If the incident beam is stigmatic, $T_1 = S_1$ and $W_1 = 0$. Then $S_2 = k^2 S_1$, $T_2 = S_1$. Hence $S_2 = k^2 T_2$, and so the diffracted beam is astigmatic, unless $k = 1$, i.e. unless the deviation is a minimum.

§ 3. *The principal curvatures.* The principal curvatures of the diffracted front can be found in terms of those of the incident front.

Let the principal planes of the incident front intersect the tangent plane at O in $O\eta_1$, $O\zeta_1$ (Fig. 2). Take these, with $O\xi_1$ along OP_1, as axes for the front. Let the radii of curvature of the sections by $O\xi_1\eta_1$ and $O\xi_1\zeta_1$ be B_1^{-1} and C_1^{-1}. The equation to the incident front is then

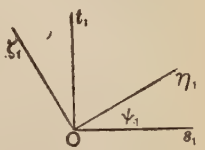


Fig. 2.

$$\xi_1 = \tfrac{1}{2}B_1\eta_1^2 + \tfrac{1}{2}C_1\zeta_1^2. \quad \text{.....(8)}$$

Let $O\eta_1$ make an angle ψ_1 with Os_1, as in Fig. 2. Then

$$\xi_1 = r_1, \quad \eta_1 = s_1\cos\psi_1 + t_1\sin\psi_1, \quad \zeta_1 = -s_1\sin\psi_1 + t_1\cos\psi_1,$$

and hence (8) is equivalent to

$$r_1 = \tfrac{1}{2}B_1(s_1\cos\psi_1 + t_1\sin\psi_1)^2 + \tfrac{1}{2}C_1(-s_1\sin\psi_1 + t_1\cos\psi_1)^2. \quad \text{...(9)}$$

Comparing (9) with (1), we find

$$\left.\begin{aligned} S_1 &= \tfrac{1}{2}(B_1 + C_1) + \tfrac{1}{2}(B_1 - C_1)\cos 2\psi_1, \\ W_1 &= \tfrac{1}{2}(B_1 - C_1)\sin 2\psi_1, \\ T_1 &= \tfrac{1}{2}(B_1 + C_1) - \tfrac{1}{2}(B_1 - C_1)\cos 2\psi_1. \end{aligned}\right\} \quad \text{.....(10)}$$

Then S_2, W_2, T_2 can be found by (7).

If the equation to the diffracted front referred to its principal axes is

$$\xi_2 = \tfrac{1}{2}B_2\eta_2^2 + \tfrac{1}{2}C_2\zeta_2^2, \quad \text{.....(11)}$$

and if $\eta_2 O s_2 = \psi_2$, then B_2, C_2, ψ_2 are related to S_2, T_2, W_2 by equations similar to (10). Solving for B_2, C_2, ψ_2, we have

$$\left.\begin{matrix} B_2 \\ C_2 \end{matrix}\right\} = \tfrac{1}{2}(S_2 + T_2) \pm [\tfrac{1}{4}(S_2 - T_2)^2 + W_2^2]^{\frac{1}{2}}, \quad \text{.....(12)}$$

$$\tan 2\psi_2 = 2W_2/(S_2 - T_2). \quad \text{.....(13)}$$

The ambiguity in (12) has been settled so that, when $\psi_1 = 0$, $B_2 = k^2 B_1$, $C_2 = C_1$. Apart from mere reversals of direction, (13) gives two values of ψ_2 differing by $\frac{1}{2}\pi$, and corresponding to the axes $O\eta_2$, $O\zeta_2$. The arrangement of signs in (12) implies that when $\psi_1 = 0$, $\psi_2 = 0$. Since, by (7), W_1 and W_2 vanish together, ψ_1 and ψ_2 must reach $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, ... together, and it follows that, for intermediate values, ψ_2 must lie in the same quadrant as ψ_1.

Equations (12), (13) with (7) give B_2, C_2, ψ_2 in terms of S_1, T_1, W_1, which are given in terms of B_1, C_1, ψ_1 by (10).

§ 4. *A simple case.* If we take $C_1 = 0$, the incident wave front is cylindrical. We then obtain

$$B_2 = \tfrac{1}{2}B_1\{k^2 + 1 + (k^2 - 1)\cos 2\psi_1\}, \quad C_2 = 0, \quad \ldots(14)$$

$$\tan 2\psi_2 = \frac{2k \sin\psi_1 \cos\psi_1}{k^2\cos^2\psi_1 - \sin^2\psi_1} = \tan 2 \tan^{-1}\left(\frac{\tan\psi_1}{k}\right)$$

or

$$\tan\psi_2 = k^{-1} . \tan\psi_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

The maximum difference between ψ_1 and ψ_2 occurs when $\tan\psi_1 = k^{\frac{1}{2}}$, and then $\sin(\psi_1 - \psi_2) = (k-1)/(k+1)$.

Since $C_2 = 0$, the diffracted wave front is cylindrical. There is therefore only one focal line at a finite distance from O and this distance is B_2^{-1}. If the principal planes of the incident front are turned round, ψ_1 will change, so causing B_2 to vary, and the distance of this focal line from O will vary.

§ 5. *Measurement of wave length.* The results of § 2 can be applied in the determination of the wave length of sodium light by measurements made on an optical bench. On the bench slide three carriages D, H, K, as shown in plan in Fig. 3; D carries a

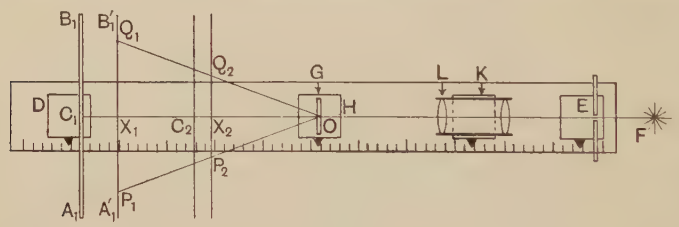


Fig. 3.

horizontal glass scale divided in mm., H carries the grating G (with vertical rulings), whose centre is O, and K carries the converging lens system L, of focal length f. At the end of the bench is a vertical slit illuminated by a sodium flame F; to identify a point E on the slit, a wire may be stretched across the slit. The divided face of the scale faces the slit and this face and the plane of the grating are perpendicular to the bench. The line through the nodal points of L is parallel to the bench and passes through E.

The scale is first placed in the position A_1B_1, at a distance from E exceeding $4f$ by about 30 cm., and the lens is adjusted to form a real undiffracted image on the scale at C_1. The axial ray of the incident beam is normal to the grating, and thus, if ϕ is the angle between the normal and the axial ray of either diffracted beam of order i, we have $k^2 = \sec^2\phi$. Since the incident beam

corresponding to the point E of the slit is stigmatic, the vertical focal lines of the diffracted beams of order i will be at a distance OC_1/k^2 from O, along lines OP_1, OQ_1, each making an angle ϕ with OC_1. Sharp images of the slit will pass through P_1, Q_1 and can be focussed on the scale if it is moved to A'_1B_1'. With the grating used in § 9, the two sodium lines can easily be separated. Then $\sin\phi = \frac{1}{2}P_1Q_1/OP_1$. Since OP_1 is difficult to measure, we suppose OX_1 known, where X_1 is the mid-point of P_1Q_1. If $OX_1 = x_1$, $P_1Q_1 = 2y_1$, $\tan\phi = \frac{1}{2}P_1Q_1/OX_1 = y_1/x_1$.

The glass plate protecting the grating prevents an accurate measurement of OX_1. We therefore move the lens carriage along the bench so that the undiffracted image is focussed on the scale at C_2. If the scale is moved further towards O, the diffracted images can be focussed at P_2, Q_2. If $OX_2 = x_2$, $P_2Q_2 = 2y_2$, $\tan\phi = y_2/x_2$. Hence

$$\tan\phi = (y_1 - y_2)/(x_1 - x_2). \quad \text{.....(16)}$$

Putting $\theta_1 = 0$, $\theta_2 = \phi$ in (6), we find

$$\lambda = d\sin\phi/i, \quad \text{.....(17)}$$

where i is the order of the image and d is the grating interval. From (16) and (17), λ is determined.

Since it is an *angle* we measure, small errors of focussing will be of little account, for, in spite of them, the point in which the *axial* ray cuts the scale in each case will be correctly estimated, and this is all that is necessary.

§ 6. *Test of law of obliquity.* Let $OC_1 = u_1$, $OC_2 = u_2$, $OP_1 = v_1$, $OP_2 = v_2$. Then $v_1 - v_2 = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$. But, since $k^2 = \sec^2\phi$, we have, by § 5, $v_1 = u_1\cos^2\phi$, $v_2 = u_2\cos^2\phi$, and thus $v_1 - v_2 = (u_1 - u_2)\cos^2\phi$. Since $u_1 - u_2$ is known from the bench readings, we can test the law for the vertical focal lines by comparing the two values of $v_1 - v_2$. As we are now concerned with the *positions* of images the focussing must be accurate.

If the slit is not too narrow, the diffracted images of order i of the horizontal wire stretched across it may be focussed on the scale. If these are at p_1, q_1 when $OC = \xi_1$ and at p_2, q_2 when $OC = \xi_2$, and if $p_1q_1 = 2\eta_1$, $p_2q_2 = 2\eta_2$, then

$$p_1p_2 = [(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2]^{\frac{1}{2}}.$$

Since, by § 2, $T_2 = T_1$, we have $p_1p_2 = u_1 - u_2$. The two values of p_1p_2 are compared.

It is difficult to obtain satisfactory readings for x and ξ. This is largely due to the fact that the diffracted rays in the horizontal plane through O do not meet in a point but touch a caustic of large

radius. If Q_1R_1, drawn perpendicular to Q_1O in Fig. 3, cuts OC_1 in R_1, the radius of curvature of the caustic at Q_1 is $3Q_1R_1$. The length of the caustic between the points of contact of the tangents from M_1, N_1, where M_1N_1 is the width of grating actually used, is $3\,(Q_1M_1 \sim Q_1N_1)$ approximately.

§ 7. *Adjustment of the lens.* The lens, a converging system, is adjusted optically. Let its focal length be f and the distance between its nodal points be t, where t is positive when the distance between the principal foci exceeds $2f$; for a projection lens as shown in Fig. 3, t will be negative. When the distance of the luminous point E from the scale ACB exceeds $4f + t$, there are two positions of the lens for which an image of E is formed on the scale. Let M, N (Fig. 4) be the nodal points corresponding to the principal

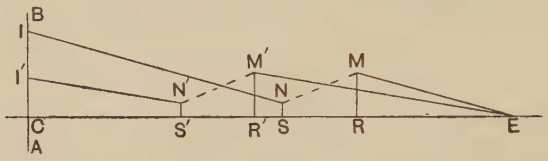


Fig. 4.

foci to the right and left of L. In Fig. 4 let EC be the horizontal line through E parallel to the bench and let the other lines be projections upon the horizontal plane through EC. Let MR, NS be the perpendiculars from M, N on EC. Let $RE = p$, $SC = q$. Then in the second position of the lens, $R'E = q$, $S'C = p$. When the angles are small, $RS = R'S' = t$. Let I, I' be the images of E in the two cases.

Take CE as axis of x and horizontal and vertical lines through C as axes of y and z. Let the second and third coordinates of M, N, I, I' be y, z, η, ζ, Y, Z, Y', Z'. Since IN is parallel to ME, and $I'N'$ to $M'E$,

$$Y = \eta + yq/p,\ \ Z = \zeta + zq/p,\ \ Y' = \eta + yp/q,\ \ Z' = \zeta + zp/q.$$

Since $(p + q)/pq = 1/f$,

$$Y - Y' = y\,(q - p)/f,\ \ Z - Z' = z\,(q - p)/f.$$

Hence, if the image has the same position for *both* cases, then $y = 0$, and $z = 0$, and thus M lies on EC. The emergent ray through N will then be *parallel* to EC for all positions of the lens carriage.

The lens is best mounted so that it can turn about a vertical pivot whose axis passes through M. If the support to which the pivot is fixed is moved through the distance MR at right angles to the bench, M will be brought on to EC. Then by turning the lens about M, N can be brought on to EC.

To identify C, a pin is mounted on a carriage so that its tip coincides with E. The carriage is moved along the bench so that the tip touches the scale AB. The point of contact is C. If L is adjusted on its carriage so that I coincides with C for both cases, then M, N lie on EC.

§ 8. *Other experimental details.* The scale AB is set perpendicular to the bench. A set square XYZ, with the right angle at X, is held with XY in contact with AB. A pin is held close to XZ. If, when the carriage D is moved along the bench, the distance from the pin to XZ is constant, AB is correctly placed on its carriage. The scale must be horizontal and the slit vertical. The plane of the grating can be set perpendicular to EC optically. The lens L is removed and a small triangle of white paper is fixed to AB so that a vertex coincides with C. The grating G is placed midway between C and E and is adjusted on its carriage so that the image of C by reflexion at G coincides with E. To allow a close test of parallax, a few grains of lycopodium may be placed on AB when the image of the slit does not fall on a dividing line.

§ 9. *Practical example.* Using a grating with $d = 1{\cdot}7526 \times 10^{-4}$ cm., the following results were obtained:

The image of first order was used; thus $i = 1$.

Bench reading 118·50 cm., glass scale readings 97·52, 73·84 cm.
" " 141·21 cm., " " " 89·64, 82·18 cm.

Hence

$y_1 = \frac{1}{2}(97{\cdot}52 - 73{\cdot}84) = 11{\cdot}84$ cm., $y_2 = \frac{1}{2}(89{\cdot}64 - 82{\cdot}18) = 3{\cdot}73$ cm.

Also $x_1 - x_2 = 141{\cdot}21 - 118{\cdot}50 = 22{\cdot}71$ cm.

Hence $\tan\phi = (y_1 - y_2)/(x_1 - x_2) = 8{\cdot}11/22{\cdot}71 = \tan 19° 39' 7''$.

Then $\lambda = d \sin\phi/i = 5{\cdot}894 \times 10^{-5}$ cm.

§ 10. *Experiment with an astigmatic incident beam.* The experimental test of the results of § 4 is a good exercise in optical manipulation. Fig. 5 is a plan of the apparatus. Two cross-wires, intersecting in E, are fitted into a tube turning about a horizontal axis

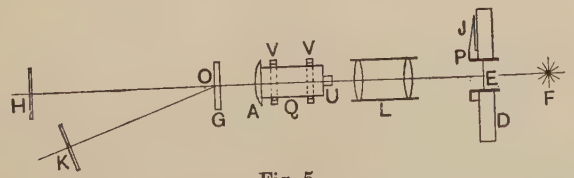


Fig. 5.

in a hole in the board D. A circular scale is attached to D and P has a pointer J which indicates its angular position. Only *one* wire is used in the measurements, but the second wire is useful as identifying E. The wires are illuminated by the sodium flame F. A lantern projection lens L is placed so that E is at its principal focus; for the best results, that end of L should face E which faces the lantern slide. Beyond L is a cylindrical tube Q, resting in two V's, V, V, and against a stop U, and thus having only one degree of freedom. A plano-cylindrical lens A is attached by its plane face to one end of Q. A lens of about + 2·5 dioptres, such as is used in spectacles, is suitable. The grating, centre O, is placed at G. A ground glass screen H can slide on the main optical bench, which also carries G, Q, L; if possible, D should be carried on the bench. On a short auxiliary bench slides a second screen K; the angle between the benches is ϕ, where $\sin\phi = i\lambda/d$. The ground sides of the screens face O.

Suppose, for a moment, that E is a luminous point. Then E, at the focus of L, gives rise to a parallel beam falling on the cylindrical lens A. This lens converts the plane wave front into a cylindrical front. If the "power" of A is + F dioptres, a "real" focal line, parallel to the generators of A's surface, will be formed $100/F$ cm. from A. This focal line can be received on the screen H. By § 4, the diffracted front is cylindrical and there is only one focal line at a finite distance. This focal line can be received on the screen K. If A is turned by turning Q on its axis, the focal line of the diffracted beam will turn about the axial ray and the distance of the focal line from O will change. The experiment tests the relation between the linear displacement of K and the angular displacement of A.

When a wire is used instead of a luminous point, images of the wire will be formed on H and K when the generators of A are parallel to the wire. If the pointer J is set in any position, a sharp image can be obtained by turning Q.

When the adjustments of § 11 have been made, H is set to receive the image of the wire, and HO is measured. As a correction we may add t/μ, where t is the thickness and μ the index of the plate covering the grating. A line ruled on H is made vertical by aid of a set square and a level, and P and Q are adjusted so that the image is vertical and $\psi_1 = 0$. Then K is set so that the diffracted image is in focus on it, and the reading of K on its bench is taken. Then P is turned by steps of 10° or 15°, Q is turned in response, and K is adjusted in each case so that the image is focussed. When P has been turned through 90°, so that $\psi_1 = \frac{1}{2}\pi$, the image is horizontal, and, by (14), since $\psi_1 = \frac{1}{2}\pi$, its distance from O is equal to the measured distance OH. When the image is vertical, $\psi_1 = 0$ or π. Since $B_1^{-1} = OH$, $B_2^{-1} = OK$, we have, by (14),

$$OK = \frac{2 \cdot OH}{k^2 + 1 + (k^2 - 1)\cos 2\psi_1}, \quad \ldots\ldots\ldots\ldots(18)$$

where $k = \sec \phi$ and $\sin \phi = i\lambda/d$.

To compare theory with experiment, we may plot the value of OK given by (18) against the bench reading of K. If the zero of this bench is at the end nearest O, the points will lie about a straight line equally inclined to both axes. An alternative method is used in § 12.

§ 11. *Experimental details.* The cross-wires should be mounted so that E is as nearly as possible on the axis of P. The lines joining the nodal points of L to E are made coincident and parallel to the bench by the method of § 7. The axis of Q is set approximately parallel to the bench; optical methods are available. The cylindrical lens A is adjusted optically. For a given direction of the wire at E, there are two positions of Q, 180° apart, in which A forms a sharp image of the wire on H. If the positions of these images are not identical, the error can be corrected by moving A at right angles to its generators across the end of Q.

To set the lens L so that E is at its focus, a plane mirror is substituted for H, Q and G are removed, the cross-wires are illuminated and L and the mirror are adjusted so that E coincides with its own image. The plane of G is made perpendicular to the bench by the same method, the plate covering the grating serving as the plane mirror. The bench on which K slides is adjusted optically. First set P and Q so that a vertical image of the wire is formed on K. Then slide L along the main bench and readjust K. If the position of the image relative to K is unchanged, the auxiliary bench is correctly placed. If a micrometer eyepiece is used in place of the screen K, *two* images will be seen except when the wire is horizontal, since sodium light has a double spectrum line. Unless the wire is very fine, the images will overlap. The doubling of the images causes no inconvenience.

§ 12. *Practical example.* The following results were obtained with a grating of 14,493 lines per inch.

For this grating, $d = 1{\cdot}7526 \times 10^{-4}$ cm. The wave length was $5{\cdot}893 \times 10^{-5}$ cm. The image of first order was used; thus $i = 1$. Hence

$$\sin \phi = 0{\cdot}33625, \quad k^2 = \sec^2 \phi = 1{\cdot}1275, \quad \phi = 19° 38' 55''.$$

A cylindrical lens of + 2·5 dioptre was used. The corrected value of OH was 38·91 cm. The angle ψ_1 was varied from 0° to 180° by steps of 15°. The bench readings in columns 2 and 4 are theoretically identical, and their mean is given in column 5. The values of OK calculated by (18) are given in column 6. To facilitate comparison, the mean difference between columns 5 and 6 has

been added to column 5, as suggested by Dr Wilcken, and the results are entered in column 7.

ψ_1	Bench reading cm.	ψ_1	Bench reading cm.	Mean reading cm.	*OK* calcd. cm.	*OK* obsd. cm.	ψ_2 calcd.
°		°					° ′
0	18·52	180	18·51	18·52	34·51	34·36	0 0
15	18·62	165	19·10	18·86	34·77	34·70	14 10
30	19·54	150	19·90	19·72	35·51	35·56	28 32
45	20·68	135	21·21	20·94	36·58	36·78	43 17
60	21·87	120	21·96	21·92	37·71	37·76	58 29
75	22·74	105	22·89	22·82	38·58	38·66	74 7
90	22·91			22·91	38·91	38·75	90 0

The observed value of *OK* is a little low at 0° and 90° and a little high at 45°. Probably the incident wave front was not accurately cylindrical.

The last column gives ψ_2 as calculated from $\tan \psi_2 = k^{-1} \tan \psi_1$. The difference between ψ_2 and ψ_1 is too small to admit of measurement with simple apparatus.

The Shadow Electroscope. By R. Whiddington, M.A., St John's College.

[*Received* 15 June 1920.]

A simple form of Electrostatic Voltmeter of low capacity is frequently useful in the laboratory. The instrument under description is of the gold leaf type designed primarily for class instruction and while not capable of the highest precision is yet sufficiently accurate for many purposes*.

All leaf electroscopes with which I am familiar require some sort of optical system such as a microscope to view the leaf. Attempts have been made to use a scale placed near the leaf for measuring purposes, but when too near, disturbing electrostatic effects are encountered, placed too far away parallax errors become obtrusive.

It occurred to me that the difficulties might be overcome by simply throwing a shadow of the leaf on a semitransparent scale some centimetres away, using a small 2-volt lamp as a source of light.

The first instrument made on these lines consisted of a tin cigarette box with the lamp at one end, a transparent scale at the other end and the gold leaf system with its insulation in the middle. It was found as expected that quite a sharp shadow could be obtained when the lamp filament was nearly parallel to the leaf.

The final design of electroscope is shown in section in the figure, the photographically reproduced scale, graduated in volts, being shown below. It will be seen that the scale is practically even from 100 to 500 above which the leaf becomes unstable.

The quadrant shape of metal box was chosen as being most likely to give an even scale and a constant capacity over its working range.

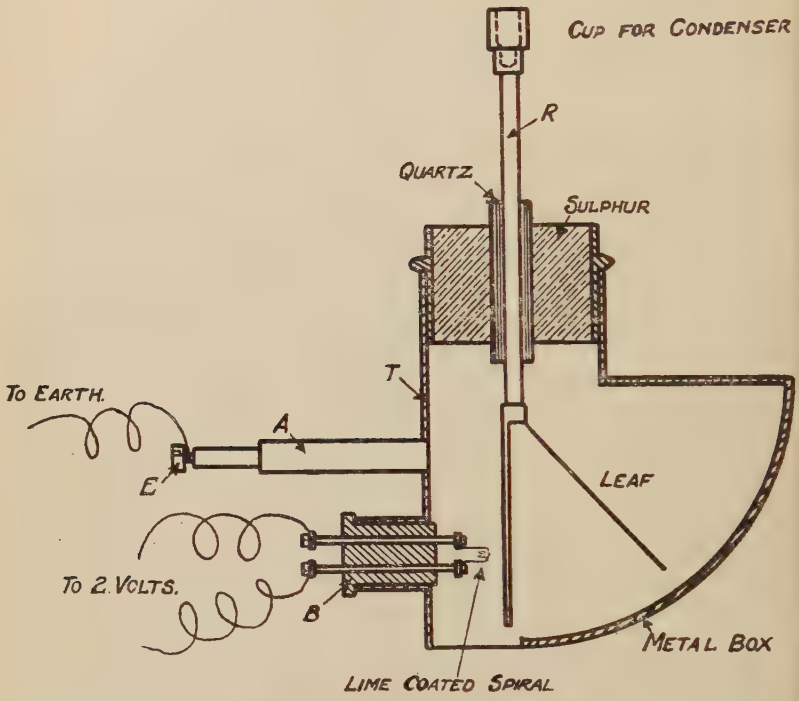
The tube (T) carries a well fitting sulphur plug fitted centrally with a quartz tube down which passes the rod (R) which carries the leaf within the case and a small cup at the top.

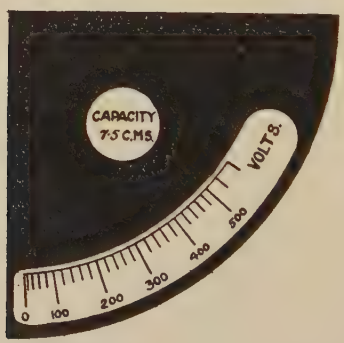
The metal arm (A) is for clamping and tilting purposes and carries an earthing terminal (E).

Just below (A), a short side tube is arranged carrying an ebonite block (B) in which a small lime coated spiral is fitted. When B is pushed home the spiral finds a place behind R. Its object is, when heated from a 2-volt cell, to provide a source of ions for

* The original instrument, of which this one is the final form, was designed in 1919 for the Naval officers under instruction in Physics at the Cavendish Laboratory.

experiments on ionization, its position behind the leaf precluding the possibility of disturbing convection currents.





The voltage range of the instrument is from 100 to 500 volts and with a good leaf it is possible to estimate to 1 volt, an accuracy sufficient for most purposes.

The scale was graduated by applying known voltages from a small direct current generator*, measuring them by a standard Weston Voltmeter.

I have found that with this instrument and the scale reproduced above, it is sufficient, when no more than approximate results are required, to register the shadow of the leaf for two positions only—zero and one other, say 200 volts. To effect this it will generally be necessary to alter the sensitiveness somewhat by adjusting the height of the sulphur block in T. This is no doubt due to the non-uniform aluminium leaf† available.

Charging the Electroscope.

After connecting E to earth, the leaf may be charged positively by induction from a rubbed ebonite rod. If a negative charge is required care should be taken not to overcharge the leaf. If an appreciable leak is observed a small piece of smooth silk rubbed lightly over Q will almost certainly cure it.

Insulation troubles are nearly always traceable to hairs and dust particles attracted under the comparatively high voltages used. It is therefore best to conduct the experiments in a dust free room.

The following are a few of the experiments which can be carried out with this instrument.

Experiment 1. *To determine the capacity (C_e) of the electroscope by comparison with that of a sphere of radius r cm.*

Method. Charge the leaf to a voltage V_1 as indicated by the scale reading (with the case earthed), and then share the charge on the‡ leaf with the insulated sphere thereby causing a drop in potential to V_2.

Then since $q = C_e V_1 = (C_e + r) V_2$, q being the original charge,

$$C_e = \frac{r}{V_1/V_2 - 1} \text{ cm.}$$

The following table shows a series of measurements taken on

* Kindly lent by the Electric Construction Company, Wolverhampton.

† Cut with scissors from leaf approximately ·0004 cms. thick.

‡ It is here assumed that the capacity of the sphere is equal to its radius. This is only true when the sphere is far removed from other conductors, a condition which can be approximately realised in practice if a long thin stiff vertical wire be inserted in the cup of the electroscope (or stalk of the condenser as the case may be) and the sphere touched to the top of the wire. If this precaution be neglected the results obtained will be too small.

Further, it must be remembered that when bringing up the sphere to the electroscope for charge sharing, any charge on the insulating handle will affect the leaf by induction and spoil the results. This effect may be got rid of by passing the handle through a flame occasionally, merely touching the ebonite is often sufficient to produce a charge.

these lines using an insulated brass sphere of radius 3·25 cm.*; V_1 and V_2 are the scale readings in volts.

V_1	V_2	V_1/V_2
490	330	1·485
330	235	1·405
235	160	1·470
160	110	1·452

From the above readings the mean value of $V_1/V_2 = 1{\cdot}450$, whence $C_e = 7{\cdot}2$ cm.

Experiment 2. *To determine the capacity of a parallel plate air condenser by the method of Experiment* 1.

The readings tabulated below were obtained with a specially designed circular plate air condenser†. The diameter of the central plate being 4·25 cm., and its distance from two outer earthed plates being 0·15 cm., the capacity C_a can be calculated from the formula for a parallel plate air condenser, viz. $2\frac{A}{4\pi d}$ cm.

Inserting the proper values for the present case leads to the value 60·2 cm.

The experimentally determined value may be expected, if anything, to be rather greater than this calculated value owing to the extra capacity of the edges of the central plate.

The method is essentially the same as in Exp. 1 but in this case the formula is

$$C_a + C_e = \frac{r}{V_1/V_2 - 1}.$$

Using the same sphere as in Exp. 1 the following results were obtained, the insulated central stalk of the condenser fitting in the electroscope cup (see figure) and the outer plates being connected to earth.

* An ordinary bedstead knob mounted on an insulating ebonite rod.

† The main point in the design is the protection of the central insulated plate from dust, small hairs etc., which under the comparatively high potentials employed would be attracted to it with resulting insulation troubles. The central plate is therefore sandwiched between two outside parallel plates, one of which is provided with a peripheral spacing ring which in butting up against the other outer plate completely encloses the inner insulated one. Insulating grooved buttons of ebonite form the insulation.

V_1	V_2	V_1/V_2
350	332	1·054
320	306	1·046
500	480	1·042
460	441	1·043
420	402	1·044

From these readings the mean value of $V_1/V_2 = 1{\cdot}045$, whence $C_a + C_e = 72{\cdot}2$ cm., and since $C_e = 7{\cdot}2$, $C_a = 65{\cdot}0$ cm.

Experiment 3. *To determine the Specific Inductive Capacity of Ebonite.*

This can be readily carried out by using a second condenser exactly similar to the one used in the previous experiment but with circular plates of ebonite separating the plates instead of air.

Then if this condenser (capacity C_b) is placed on the electroscope in the manner of the previous experiment, and charged to a potential (V_1), and the sphere is used in the manner previously described, the resulting collapse of the leaf will be so small as to be hardly readable owing to the large capacity of the ebonite condenser. It is therefore more convenient to use the air condenser of measured capacity C_a in place of the sphere. It is sufficient to hold C_a by its outer case for earthing purposes, touching its central plate momentarily to the corresponding plate of the condenser C_b on the electroscope. The potential resulting from this sharing of charge (V_2) is noted.

We then have that

$$\frac{C_a + C_b + C_e}{C_b + C_e} = \frac{V_1}{V_2},$$

whence

$$C_b + C_e = \frac{C_a}{V_1/V_2 - 1},$$

in which both C_a and C_e have been previously determined by

V_1	V_2	V_1/V_2
330	250	1·320
340	262	1·296
258	192	1·341
480	362	1·336
350	265	1·322
260	195	1·318

experiment. The above table gives the results of an experiment. From which the mean value of V_1/V_2 comes out to be 1·322.

By calculation from this value $C_b = 194{\cdot}7$ cm.

Assuming the identical dimensions of the two condensers* the Specific Inductive Capacity of Ebonite is just the ratio

$$C_b/C_a = \frac{194{\cdot}7}{65{\cdot}0}$$
$$= 2{\cdot}98.$$

A value not far removed from the accepted value which according to the table of Kaye and Laby will usually lie between 2·7 and 2·9.

Experiment 4. *The comparison of two capacities by the ionization leak method.*

It is convenient to illustrate this method by giving as an example the results of an experiment using the same two condensers as the preceding experiment.

Method. If when the lime coated spiral is glowing steadily the slow leak of the electroscope be observed firstly with C_a in position and then with C_b in position, the capacities can at once be compared, for if T_a and T_b be these times it can be shown that

$$\frac{C_a + C_e}{C_b + C_e} = \frac{T_a}{T_b}.$$

The following table gives some results obtained with this method. In order to eliminate as far as possible any variations in the amount of ionization (which depends very greatly on the temperature of the filament and therefore on the E.M.F. of the power supplying cell) the readings for T_a and T_b were taken alternately and as quickly as possible. It will be seen that under

Times in seconds	
T_a	T_b
9·2	25·0
9·2	24·6
9·6	25·8
9·4	25·4
Mean 9·35	25·2

* This can easily be tested experimentally.

the conditions of this experiment, in which a well charged 2-volt lead accumulator was used, there is very fair concordance between the various readings.

Leak observed from 400 volts to 200 volts.

If now in the above-mentioned expression we assume the previously determined values of C_a and C_e, viz. 65·0 cm. and 7·2 cm. respectively, the value of C_b comes out to 188·0 cm. leading to a value for the specific inductive capacity of ebonite of 2·90. This value is in as good agreement as is to be expected with the determination of Exp. 3.

On the Hart circle of a spherical triangle. By Professor H. F. Baker.

[*Read* 9 February 1920].

This note is concerned with the problem, given three arbitrary plane sections of any quadric, of finding a fourth section which shall be tangent to four of the tangent planes of the three given sections. If the three given sections are concurrent on the quadric they have only four tangent sections, and the fourth section is unique, the projection of the figure on to a plane (from the point of concurrence) giving rise to Feuerbach's theorem of the nine-point circle. In general the three given sections have eight common tangent planes; in fact any two of these sections lie on two quadric cones, and the six vertices of the cones so obtainable lie by threes on four coplanar lines; the three cones whose vertices are on any one of these lines have a pair of common tangent planes, which thus touch the three sections. The eight tangent planes of these are thus accounted for. There are now fourteen ways of selecting, from these eight tangent planes, four which all touch another section; six of these ways, in which the four tangent planes selected are tangent to a fourth section passing through the point of concurrence of the three given sections, are easy to recognise, and do not need further consideration. There are however eight ways of choosing four from the tangent planes which shall all touch another section lying in a plane ϖ forming with the planes of the three given sections a finite tetrahedron.

§ 1. We are thus lead to the problem of the condition necessary and sufficient in order that the sections of a quadric by the four faces of a tetrahedron should have four common tangent planes; and the main object of this note is to state this condition in a form which in fact leads to great simplification of what is generally presented as a somewhat intricate theory, and to point out several results, apparently new, which follow from this. Let the tetrahedron be O, X, Y, Z; denote the intersections of the quadric with OX by A, A', those with OY by B, B' and those with OZ by C, C'; similarly denote the intersections with YZ by U, U', those with ZX by V, V' and those with XY by W, W'. In general, if each edge of the tetrahedron be joined by planes to the two points in which the quadric meets the opposite edge, the twelve planes so obtained touch another quadric. But it may happen that this new quadric degenerates into two points, say S and S'; then, with a proper choice of notation, the four lines AU, BV, CW are concurrent in a point S, and the four lines $A'U'$, $B'V'$, $C'W'$ con-

current in another point S'. That this should be so is a necessary and sufficient condition that the four sections of the quadric by the faces of the tetrahedron should have four common tangent planes. The condition may be stated in another form; take on the edge OX, the point A_1 separated harmonically from A by O and X, and the point A_1' separated harmonically from A' by O and X; in the same way take on each edge of the tetrahedron the harmonic conjugates, with regard to the vertices of the tetrahedron lying on that edge, respectively of the intersections of the quadric with that edge. The twelve new points so obtained lie on another quadric, which we may describe as the harmonic conjugate of the original in regard to the tetrahedron. The condition in question then is that the harmonically conjugate quadric should break up into two planes, say σ and σ'; these will be the polar planes of S and S' in regard to the original quadric.

We may illustrate this condition by applying it to the (Feuerbach) case of three sections of the quadric which are concurrent on the quadric, say in O. The fourth section of the quadric touched by the four common tangent planes of the three given sections OYZ, OZX, OXY is then constructed as follows: on the plane YOZ take the line p through O, harmonically conjugate with respect to OY, OZ, to the line in which the plane YOZ is met by the tangent plane of the quadric at O; let this line p meet the quadric again in P; obtain the points Q, R of the sections ZOX, XOY in a similar way. The plane PQR is the fourth plane required. In this case one of the planes σ, σ' is the tangent plane at O.

§ 2. We may obtain a direct verification of the sufficiency of the condition in general by using it to obtain any one of the eight (Hart) sections ϖ which can be associated with three given sections YOZ, ZOX, XOY, so as to form four sections with four common tangent planes. Let the quadric, referred to YOZ, ZOX, XOY and the polar plane of O, have the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = t_1^2;$$

with an arbitrary choice of the signs of $\sqrt{a}, \sqrt{b}, \sqrt{c}$, take

$$u = \tfrac{1}{2}(f + \sqrt{b}\sqrt{c}), \quad v = \tfrac{1}{2}(g + \sqrt{c}\sqrt{a}), \quad w = \tfrac{1}{2}(h + \sqrt{a}\sqrt{b}),$$

and then l, m, n so that

$$mn = u, \quad nl = v, \quad lm = w;$$

the eight planes required are then expressed by

$$lx + my + nz - t_1 = 0.$$

It is at once seen that this follows from the condition stated above. If we introduce λ, μ, ν so that

$$f = \sqrt{b}\sqrt{c}\cos\lambda, \quad g = \sqrt{c}\sqrt{a}\cos\mu, \quad h = \sqrt{a}\sqrt{b}\cos\nu,$$

a plane of this latter form is

$$x\sqrt{a}\frac{\cos\frac{1}{2}\mu\cos\frac{1}{2}\nu}{\cos\frac{1}{2}\lambda}+y\sqrt{b}\frac{\cos\frac{1}{2}\nu\cos\frac{1}{2}\lambda}{\cos\frac{1}{2}\mu}$$
$$+z\sqrt{c}\frac{\cos\frac{1}{2}\lambda\cos\frac{1}{2}\mu}{\cos\frac{1}{2}\nu}-t_1=0, \qquad \text{(H)};$$

on the other hand a common tangent plane of the three given sections in $x=0$, $y=0$, $z=0$ is at once found to be

$$x\sqrt{a}\cos(s-\lambda)+y\sqrt{b}\cos(s-\mu)+z\sqrt{c}\cos(s-\nu)-t_1=0, \quad \text{(I)}$$

where $s=\frac{1}{2}(\lambda+\mu+\nu)$; and it is easy to see that the section (I) touches the section (H) at the point of the plane (H) which lies on

$$x\sqrt{a}:y\sqrt{b}:z\sqrt{c}=p(q-r)^2:q(r-p)^2:r(p-q)^2,$$

where, for brevity, p, q, r stand respectively for

$$\sin(s-\lambda),\ \sin(s-\mu),\ \sin(s-\nu).$$

The four planes (I) which touch the section (H), as well as the original sections in $x=0$, $y=0$, $z=0$, are obtained from the above equation by replacing λ, μ, ν by $\pm\lambda$, $\pm\mu$, $\pm\nu$, respectively.

The eight sections (H) are obtainable from that above by replacing $\sqrt{a}, \sqrt{b}, \sqrt{c}, \lambda, \mu, \nu$ respectively by

$$(\sqrt{a},\sqrt{b},\sqrt{c},\lambda,\mu,\nu),\ (-\sqrt{a},\sqrt{b},\sqrt{c},\lambda,\pi+\mu,\pi+\nu),$$
$$(\sqrt{a},-\sqrt{b},\sqrt{c},\lambda+\pi,\mu,\nu+\pi),\ (\sqrt{a},\sqrt{b},-\sqrt{c},\lambda+\pi,\mu+\pi,\nu)$$

together with those obtainable from these by changing the sign of t_1.

§ 3. The following result gives a construction for the position, upon the section, i, of the quadric by the plane (I), of the point in which this section is touched by the plane (H). Upon i we have three points, its contacts with the sections in $x=0$, $y=0$, $z=0$; we also have two points, namely those in which i is met by the plane from O to the intersection of the planes ABC, $A'B'C'$, which plane is at once found to have the equation

$$x\sqrt{a}+y\sqrt{b}+z\sqrt{c}=0.$$

The point to be constructed is the apolar complement of the two latter points in regard to the three former points. This result may be made clearer perhaps by stating it for a sphere in Euclidian geometry: If D, E, F be the mid-points respectively of the sides BC, CA, AB of a spherical triangle, the planes of the great circle arcs EF, BC give a diameter, and the three diameters so obtained are coplanar; let I, J denote the intersections of their plane with the inscribed circle of the triangle ABC; let P, Q, R be the points of contact of this inscribed circle with the sides BC, CA, AB. Then,

upon this inscribed circle, the point of contact with the Hart circle, which touches this and certain other three tangent circles of the sides of the triangle, is the apolar complement of I, J in regard to P, Q, R. For the particular case of the nine point circle of a plane triangle the result has been remarked by Prof. F. Morley, as was pointed out to the writer by Mr J. H. Grace, *Bulletin of the American Math. Soc.*, 1, 1895, 116–124 ("Apolar triangles on a conic").

§ 4. Another result may also be stated here. To introduce it and render its meaning clearer we state it first for the Hart circle of a spherical triangle in Euclidian geometry. If this circle meet the sides of the spherical triangle ABC respectively in U, U' on BC, V, V' on CA, W, W' on AB, then, with proper choice of notation, the arcs AU, BV, CW are concurrent, say in S, and the arcs $A'U'$, $B'V'$, $C'W'$ are concurrent, say in S'. The result in question is that S, S' are the centres of similitude of the circumscribed circle of the triangle ABC and the Hart circle. It is a direct generalisation of the corresponding familiar fact for the nine point circle of a plane triangle.

Stated in the more general way here adopted, which is also the more precise way, the theorem is that the lines OS, OS' are each the intersection of two planes through O which touch both the section ϖ and the section by the plane ABC. If PQR, $P'Q'R'$ be two sets of three points lying respectively on two plane sections of a quadric, such that PP', QQ', RR' are concurrent, the sections lie on a quadric cone having this point of concurrence for vertex; thus a plane through O touching the section μ by the plane ABC equally touches the section μ' by the plane $A'B'C'$. Now S, the point of concurrence of AU, BV, CW, is the vertex of one cone containing the sections μ, ϖ; and S' is similarly the vertex of one cone containing the sections μ', ϖ. The line OS', joining the vertex of one cone containing the sections μ', ϖ to the vertex of one cone containing the sections μ, μ', passes through one of the vertices of the two cones containing the sections μ, ϖ; as OSS' are not collinear, this line OS' passes through the vertex other than S of a cone containing μ and ϖ. The two cones containing μ and ϖ thus have their vertices on OS and OS'. Now to each of these cones there can be drawn from O two tangent planes, which intersect in the line joining O to the vertex of the cone; the four planes so obtained touch the sections μ and ϖ, and thus are the four common tangent planes of the cones with vertex O standing on the sections μ, ϖ. Two of these planes therefore intersect in OS and two in OS'; which is the result we desired to obtain.

There are as we have said eight sections ϖ each touched by four of the common tangent planes of the sections in YOZ, ZOX, XOY. These fall into four pairs, the planes of a pair intersecting on the

polar plane of O, being harmonic conjugates in regard to this plane and O; for the pair associated as above with the two planes ABC, $A'B'C'$ the lines OS, OS' are the same. There is another pair associated similarly with the planes $AB'C'$, $A'BC$, a third pair with the planes $BC'A'$, $B'CA$ and a fourth pair associated with the planes $CA'B'$, $C'AB$. And it may be remarked that the sections by the planes ABC, $AB'C'$, $BC'A'$, $CA'B'$ are all touched by four planes, as follows at once from the fact that AA', BB', CC' are concurrent; so also the sections by the planes $A'B'C'$, $A'BC$, $B'CA$, $C'AB$ are all touched by four planes.

§ 5. Another remark may be made, relating to a property which appears in Euclidian geometry as Salmon's theorem that the tangent of the radius of the circumcircle of a spherical triangle is twice the tangent of the radius of the Hart circle.

Let P be the pole of any plane section of a quadric, upon which any point A is taken, and O be any other point; denote by ρ the Cayley separation of the lines OP, OA in regard to the quadric, and by δ the Cayley separation of P from O. It can then be shown that ρ is independent of the position of A upon the section, and is indeed symmetrical in regard to P and O, being connected with δ by an equation $\sin\delta \sin\rho = \pm 1$. Calling ρ the radius of the section in regard to the point O, it can be shown that if ρ, ρ' be the radii of any two sections α, α' whose planes intersect in a line l, and the planes joining l to O and to the vertex of one of the two cones containing α and α' be respectively denoted by ω and γ, then $\tan\rho/\tan\rho'$ is equal to the homography $(\gamma, \omega; \alpha, \alpha')$ or to the negative of this. In particular when the planes γ, α are harmonically separated by ω and α', this leads to $\tan\rho = 2\tan\rho'$. In our figure the plane σ, which is the polar of S in regard to the quadric, passes through the line of intersection of the planes ABC and ϖ, since S is the vertex of one of the cones containing the section by ABC and the Hart section ϖ, and this plane σ also contains the vertex of the other cone containing these sections; *it can easily be proved that the plane ω which joins O to the line of intersection of the planes ABC and ϖ is harmonically separated from ϖ by the planes ABC and σ*; thus the planes σ, ABC, ω, ϖ have the relation of the respective planes γ, α, ω, α' in the general description just given. It follows that if ρ, R be the radii of the sections ϖ, ABC, we have $\tan R = 2\tan\rho$; which is what we wished to prove.

§ 6. A last remark may be added bringing into relief the connexion between the present point of view and that of the Euclidian geometry. As hitherto, let $OXYZ$ be a tetrahedron whose faces meet a quadric in sections having four common tangent planes. Denote by $i\alpha$, $i\beta$, $i\gamma$ the Cayley separations OX, OY, OZ in regard

to the quadric; by $i\alpha', i\beta', i\gamma'$ the Cayley separations YZ, ZX, XY; by (A), (B), (C) the Cayley separations of the pairs of planes meeting respectively in OX, OY, OZ; and by (A'), (B'), (C') the Cayley separation of the plane XYZ respectively from the planes YOZ, ZOX, XOY. Each of these separations is ambiguous in sign and by additive multiples of π, unless we enter into further detail. There are however equations by which all of them are deducible from α, β, γ; and these equations may be represented, when proper regard is paid to the ambiguities, by

$$\alpha' = i\pi + \beta - \gamma, \quad \beta' = i\pi + \gamma - \alpha, \quad \gamma' = i\pi + \alpha - \beta,$$

$$\tan(A) = \frac{\sinh \alpha}{\cosh(\epsilon + \alpha)}, \quad \tan(A') = \frac{\sinh(\beta - \gamma)}{\cosh(\epsilon + \beta + \gamma)}$$

where ϵ is such that

$$2\tanh \epsilon = \tanh \alpha \tanh \beta \tanh \gamma - \tanh \alpha - \tanh \beta - \tanh \gamma.$$

And these lead to

$$(A') = (B) - (C), \quad (B') = (C) - (A), \quad (C') = (A) - (B),$$

which may be used to define the Hart section.

§ 7. In what has preceded we have stated a *sufficient* condition for the Hart section, namely that AU, BV, CW are concurrent. It can however be proved that this is also a *necessary* consequence of the existence of the four sections of the quadric all touched by four other planes, provided we exclude certain particular possibilities which are easily stated. Precisely, given three arbitrary plane sections of a quadric, no one of which degenerates into two straight lines, so that the equation of the quadric referred to these and the polar plane of their point of intersection is of the form $(abcfgh\mathfrak{Y}xyz)^2 = t_1^2$, in order that these with a fourth section (also not two straight lines) should form a set of four sections all touched by four planes, if no relations are assumed to hold among the coefficients a, b, c, f, g, h, it is necessary that the condition in question (that AU, BV, CW are concurrent) should hold.

In order that the sections by $x = 0$, $y = 0$, $z = 0$, $t = 0$ of the quadric $(abcdfghuvw\mathfrak{Y}xyzt)^2 = 0$ should have four common tangent planes, the cones enveloping the quadric along these sections must be concurrent; if Δ be the four-rowed determinantal discriminant, and A, B, ... the minors therein, it follows that the necessary and sufficient condition for this is that the equation

$$(abcdfghuvw\mathfrak{Y}\sqrt{A}, \sqrt{B}, \sqrt{C}, \sqrt{D})^2 = \Delta$$

should be satisfied for four choices of the signs of $\sqrt{A}, \sqrt{B}, \sqrt{C}, \sqrt{D}$. It proves to be possible to examine all the ways in which this can happen, and the result is as stated.

On a property of focal conics and of bicircular quartics. By Professor H. F. Baker.

[*Read* 9 February 1920.]

The property of focal conics referred to in the title is the well-known one that if P, R be any two points of the focal hyperbola of a system of confocal quadrics, and Q, S be any two points of the focal ellipse, then the distances PQ, PS have the same difference as the distances RQ, RS. The theorem remains true if every one of the distances be replaced by the Cayley separation of its end points in regard to an arbitrary quadric of the confocal system, and the original theorem is then obtainable by making the parameter of this arbitrary quadric increase without limit. It is shown that the generalised theorem is equivalent to the geometrical theorem that two enveloping cones of the arbitrary confocal exist, each of which touches the four lines PQ, QR, RS, SP. The theorem that the sum of the two focal distances of a point of an ellipse is constant may similarly be replaced by the theorem that the sum of the Cayley separations of a point of the ellipse from the foci is constant, in regard to an arbitrary confocal conic; a theorem is obtained which includes both this last result and the former. It is unnecessary to point out that this last result is equivalent with Chasles's theorem that a variable tangent plane of a quadric cone makes angles with the planes of circular section whose sum is constant (Chasles, *Géom. Supér.*, 1880, § 812, p. 517).

The property of bicircular quartics referred to is that the angles which a variable bitangent circle of one mode of generation makes with two fixed bitangent circles of another mode of generation, have a constant sum (Jessop, *Quart. Journ.*, XXIII, 1889, 375). This is shown to be equivalent to the former theorem.

There exist much more general theorems in regard to the generation of a quadric with the help of a thread of constant length, whose systematic investigation is in connexion with the theory of hyperelliptic functions (Chasles, *Liouville*, XI, 1846, 15; Darboux, *Théorie des surfaces*, Livre IV, Ch. XIV, 296–312; Staude, *Math. Ann.*, XX and XXII, 1883; Finsterwalder, *Math Ann.*, XXVI, 1886; Maxwell, *Works*, II, 156 or *Quart. Journ.*, 1867). I have added some lines in regard to this general point of view.

§ 1. If P, Q, R, S be four coplanar points of a quadric, and through the lines SP, PQ, QR, RS be drawn four arbitrary planes, respectively, α, β, γ, δ, the lines $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ meeting the quadric

again respectively in P', Q', R', S', then (1) the points P', Q', R', S' are equally on a plane, (2) if by the angle between the sections of the quadric by the planes α, β be understood the Cayley separation of these planes, measured by the homography of these planes in regard to the two tangent planes to the quadric drawn from their line of intersection, then the sum of the angles at P, R, determined respectively by the sections α, β and γ, δ, is equal to the sum of the angles at Q, S, determined respectively by the sections β, γ and δ, α.

That P', Q', R', S' lie in a plane follows from the fact that the four quadrics consisting of (i) the original quadric, (ii) the planes α, γ, (iii) the planes β, δ, (iv) the planes PQR, $P'Q'R'$, have seven, and therefore eight points in common. For the relation between the angles, denote by θ the section by the plane $PQRS$, and in general by (α, β) the angle between the sections (α, β). Then we have

$$\pi = (\theta, \alpha) + (\alpha, \beta) + (\beta, \theta) = (\theta, \gamma) + (\gamma, \delta) + (\delta, \theta),$$

and therefore

$$(\alpha, \beta) + (\gamma, \delta) = 2\pi - (\theta, \alpha) - (\theta, \beta) - (\theta, \gamma) - (\theta, \delta),$$

which is also the value of $(\beta, \gamma) + (\delta, \alpha)$, the ambiguities of interpretation being properly settled in each case.

In a plane we have the theorem that if P, Q, R, S be concyclic points through which pass pairs of four circles α, β, γ, δ, namely α, β through P, β, γ through Q, γ, δ through R and δ, α through S, then the two angles (α, β), (γ, δ) have the same sum as the two angles (β, γ), (δ, α); and this, not depending on the Axiom of parallels, may well be regarded as a fundamental theorem. Further if P' be the other intersection of α and β, etc., the points P', Q', R', S' are concyclic. The connexion of this result with the theorem of the angles is incidentally remarked by Prof. W. McF. Orr, *Trans. Camb. Phil. Soc.*, XVI, 1897, 95.

§ 2. Regard the bicircular quartic in question as the projection on to an arbitrary plane of the section of a quadric by a quadric cone of general position, the centre of projection being an arbitrary point of the quadric. An arbitrary tangent plane of the cone cuts the quadric in a section projecting into a conic having two points of contact with the bicircular quartic, and this conic, passing through the nodes of the quartic, is for us a bitangent *circle*, of one mode of generation. The other three modes are obtained by considering the other three quadric cones through the intersection of the quadric and the first cone. Take then two bitangent circles of the bicircular quartic of the first mode of generation, say α and γ; their points of contact will be on another circle, say ρ, as appears from the three dimensional figure. Take also two

bitangent circles of a second mode of generation, say β and δ, with points of contact on a circle, σ. We shall prove that the eight points of intersection of the pairs of circles (α, β), (β, γ), (γ, δ), (δ, α) lie on two circles θ, θ', four on each. These circles θ, θ' pass through the two intersections of the circles ρ, σ, and separate these circles harmonically; the circle ρ is orthogonal to the principal circle to which the bitangent circles of the first mode are all orthogonal, with a similar statement for σ. For the proof, let $\alpha = 0$, $\gamma = 0$ be the equations of any two tangent planes of a quadric cone, whose generators of contact lie on a plane $\rho = 0$, so that the cone has the equation $\alpha\gamma - \rho^2 = 0$. Let $\beta\delta - \sigma^2 = 0$ be another quadric cone, whereof $\beta = 0$, $\delta = 0$ are tangent planes touching the cone on $\sigma = 0$. Then a quadric $E = 0$ through the curve of intersection of the two cones has an equation of the form

$$E = \alpha\gamma - \rho^2 - m^2(\beta\delta - \sigma^2) = 0,$$

so that the four lines $\alpha = 0$, $\beta = 0$; $\beta = 0$, $\gamma = 0$; $\gamma = 0$, $\delta = 0$; $\delta = 0$, $\alpha = 0$, in which the two first planes α, γ meet the two latter planes β, δ, intersect the quadric $E = 0$ in eight points lying in the two planes $\rho + m\sigma = 0$, $\rho - m\sigma = 0$.

We have then a proof of Jessop's theorem in regard to the bicircular quartic curve*.

§ 3. Reciprocally let any two conics be taken in space, not intersecting one another. Consider a quadric touched by the common tangent planes of these two conics. Then if A, C be any two points of the first conic, and B, D any two points of the second conic, it follows from § 2 that the pairs of tangent planes to this quadric from the lines AB, BC, CD, DA touch two enveloping cones of the quadric, say F and G. Or, as a line lying in a tangent plane of a cone is a tangent line of the cone, there are two enveloping cones of the quadric which touch the lines AB, BC, CD, DA. And, comparing the equations of § 2, the vertices of these cones lie on the line joining the points R, S, in the planes of the conics, which are the poles respectively of AC, BD in regard to these conics, and separate R, S harmonically; the positions of the vertices depend on the quadric taken to touch the common tangent planes of the conics. Moreover, as the reciprocal of the

* The direct analytical proof is, of course, simple. Let the fundamental quadric be $x^2 + y^2 + z^2 + t^2 = 0$, and bitangent circles of two modes be obtained by projection of the polar sections respectively of the two points

$$[(a-d)\,x,\ (b-d)\,y,\ (c-d)\,z,\ 0],\ [(a-c)\,\xi,\ (b-c)\,\eta,\ 0,\ (d-c)\,\tau].$$

Then the angle between these circles, being the Cayley separation of these points, is the angle, in rectangular Cartesian coordinates, between the two lines

$$X/px = Y/qy,\ X/p\xi = Y/q\eta, \text{ where } p^2 = (a-d)\,(a-c),\ q^2 = (b-d)\,(b-c).$$

This generalises at once to the Cyclide; cf. Jessop, *Quartic surfaces*, 1916, p. 106.

theorem in regard to the angles, if we consider the homography of A, B in regard to the quadric, say a, and take the corresponding homographies for the pairs B, C; C, D; D, A respectively, say b, c, d, we have $ac = bd$, or $a/d = b/c$. In words, the difference of the Cayley separations of A from B and D, in regard to the quadric, is the same, for unaltered positions of B, D on the second conic, when A is replaced by any other point C of the first conic.

This result includes the particular case of the focal conics of a confocal system, for which we may also consider the further particular case of actual Euclidian distances between the points. (Cf. § 10 below, where the relation between the separation and the distance is given.)

§ 4. If we assume that the sides of the skew quadrilateral $ABCD$ in § 3 touch an enveloping cone of the quadric, we can deduce the relation between the Cayley separations in another way. In fact if the sides of a skew quadrilateral touch any quadric having ring contact with a given quadric, the sum of the Cayley separations belonging to the sides of the quadrilateral, each taken in proper sense, is zero, the separations being measured by the latter quadric. For if AT be a tangent to a quadric V, which has ring contact with a quadric U, drawn from a point A, the Cayley separation AT in regard to U is independent of T. If A be (ξ, η, ζ, τ), T be (x, y, z, t), so that, with usual notation, $V_x = 0$, $V_{x\xi} = 0$, and U be $V + P^2 = 0$, then $U_x = P_x^2$, $U_{x\xi} = P_x P_\xi$, and hence

$$U_{x\xi}/(U_x U_\xi)^{\frac{1}{2}} = P_x P_\xi / P_x (U_\xi)^{\frac{1}{2}} = P_\xi / (U_\xi)^{\frac{1}{2}},$$

which is independent of x, y, z, t; and $U_{x\xi}/(U_x U_\xi)^{\frac{1}{2}}$ is the cosine of the separation in question. Therefore, if the sides AB, BC, CD, DA of the skew quadrilateral touch V respectively at L, M, X, Y, we have the following relations among the separations

$$(AB) = (AL) - (BL), \quad (BC) = (BM) - (CM),$$
$$(CD) = (CX) - (DX), \quad (DA) = (DY) - (AY),$$
$$(AY) = (AL), \quad (BL) = (BM), \quad (CM) = (CX), \quad (DX) = (DY),$$

leading to

$$(AB) + (BC) + (CD) + (DA) = 0,$$

or

$$(AB) - (AD) = (CB) - (CD).$$

In the application of this result above, V was a cone.

§ 5. We may however make an application in which U is a cone, and V not a cone, U being an enveloping cone of V. Namely, if the sides of a skew quadrilateral touch a quadric, the sum of the four Cayley separations of the vertices, each in proper sense, in regard to any enveloping cone of the quadric, is zero. The reciprocal theorem, is that if two plane sections α, γ of a quadric

be both touched by each of two other sections β, δ—and if, taking a fifth arbitrary section, ω, of the quadric, we measure the angle between the planes of two sections α, β, which touch one another, in the usual way, by considering the homography of these planes in regard to the tangent planes drawn from their line of intersection to the section ω—then, with proper sense of measurement, $[\alpha, \beta]$ denoting the angle between these planes, we have

$$[\alpha, \beta] + [\beta, \gamma] + [\gamma, \delta] + [\delta, \alpha] = 0.$$

Now take one of the two quadric cones containing the sections α, γ, and regard this cone, and the section ω, as fundamental; speak of α, γ as circular sections of this cone, of opposite systems because each has two points common with the other and with ω. Then we have Chasles's theorem that a variable tangent plane of a quadric cone makes angles of constant sum with two planes of circular section of the cone, of opposite systems.

§ 6. The reciprocal theorem is that a generator of a quadric cone makes angles of constant sum with two conjugate focal lines of the cone, that is, considering the conic in which the plane of ω cuts the cone, and the quadrilateral formed by the common tangents of this conic and ω, makes angles of constant sum with the lines joining the vertex of the cone, to an opposite pair of intersections of two of these common tangents (Chasles, *loc. cit.*, § 827, p. 528). Projecting on to an arbitrary plane we have the theorem that if P be a variable point of one of two conics having S, H as common foci, the Cayley separations PS, PH in regard to the other conic have a constant sum. An elementary proof can be given depending on the fact that if PS meet the other conic in S_1, S_2, and PH meet the other conic in H_1, H_2, then, with proper notation, each of S_1H_2, S_2H_1 passes through a fixed point of the line SH.

§ 7. This theorem for conics is a particular case of the following: Two conics V, W, have both double contact with a conic U, and also both have double contact with another conic K. From a point P of K a tangent PX is drawn to V, and also a tangent PY to W; then the Cayley separations PX, PY, taken in regard to U, have a constant sum (or difference) as P varies on K. Two tangents are possible from the point P to the conic V; but the separation PX is the same for both.

If V degenerate into the pair of tangents to U from a focus S of U, and W into the pair of tangents to U from the conjugate focus H, then the conic K, touching these four tangents, will be confocal with U, and the tangents PX, PY will become the lines PS, PH. Thus the theorem includes that of § 6.

§ 8. The proof of the general theorem of § 7 is, analytically, identical with that of the following theorem, of three dimensions, which leads, in § 9, to the theorem of § 4, and may thus be regarded as summarising all the analogous theorems here obtained:—If two quadrics V, W both have ring contact with a quadric U, and also both have ring contact with a quadric K, and PX, PY be tangents respectively to V and W from a point P of K, the sum (or difference) of the Cayley separations PX, PY, in regard to U, is independent of the position of P upon K. When V and W coincide the difference of the separations is zero for all positions of P and the quadric K is unnecessary.

The theorem is easy to prove. In order that two quadrics $V = 0$, $W = 0$ should both have ring contact with another quadric $U = 0$, they must, if $P = 0$, $Q = 0$ be suitable planes, be capable of the forms $V = U - P^2$, $W = U - Q^2$, and thus V, W must have two points of contact, there being an identity of the form

$$V - W = pq,$$

where $p = 0$, $q = 0$ are two planes. Any quadric having ring contact with both V and W is then capable of either of the identical forms

$$V + \tfrac{1}{4}(a^{-1}p - aq)^2 = 0, \quad W + \tfrac{1}{4}(a^{-1}p + aq)^2 = 0,$$

wherein a is a constant, and two such quadrics can be drawn through an arbitrary point. We may then suppose

$$U = V + \tfrac{1}{4}(a^{-1}p - aq)^2, \quad K = V + \tfrac{1}{4}(b^{-1}p - bq)^2,$$

where $K = 0$ is the quadric of the enunciation, and b is a constant. Thus we have the identity

$$U - K = \tfrac{1}{4}(a^{-2} - b^{-2})(p^2 - a^2b^2q^2),$$

involving in particular that U, K have two points of contact on the line joining the points of contact of V and W. Putting

$$P = \tfrac{1}{2}(a^{-1}p - aq), \quad Q = \tfrac{1}{2}(a^{-1}p + aq),$$

this is the same as

$$(1 - \sigma^2)(U - K) = P^2 + Q^2 + 2\sigma PQ,$$

where $\sigma = (a^2 + b^2)/(a^2 - b^2)$. This again, if U is not zero, is the same as

$$(P^2 - U)(Q^2 - U) - (PQ + \sigma U)^2 = (1 - \sigma^2)\,UK.$$

We remarked however above (§ 4), that if θ, ϕ be the Cayley separations PX, PY, taken in regard to U,

$$\cos\theta = \frac{P}{U^{\frac{1}{2}}}, \quad \cos\phi = \frac{Q}{U^{\frac{1}{2}}},$$

where the coordinates in U, P, Q are those of the point P. If this point be on the quadric $K = 0$, but not on $U = 0$, we thus get

$$\cos\theta\cos\phi + \sigma = \pm\sin\theta\sin\phi$$

showing that $\theta \pm \phi =$ constant, as was stated.

§ 9. Now suppose a skew quadrilateral $ABCD$ of which the sides AB, BC both touch the quadric V, say in X and Y, respectively, while the sides CD, DA both touch the quadric W, say in Z and T respectively. The quadrics V, W are supposed to have two points of contact, so that quadrics can be drawn having ring contact with both. Let U be one such; let K be another such passing through C, and let A be on K. Then, considering Cayley separations in regard to U, we have (BX), (BY) equal because V has ring contact with U, and also (DZ), (DT) equal because W has ring contact with U. By § 8 we also have $(AX) - (AT)$ equal to $(CY) - (CZ)$, if a proper sense be assigned to the separations involved.

We infer therefore that

$$(AB) - (AD) = (AX) + (XB) - [(AT) + (TD)] = (AX) - (AT) + (XB) - (TD) = (CY) - (CZ) + \epsilon\,(YB) - \zeta\,(ZD),$$

where ϵ, ζ are each ± 1. Without making the proper detailed examination, we shall put both ϵ and ζ equal to 1, so obtaining

$$(AB) - (AD) = (CB) - (CD).$$

This is verified (§ 4) in the particular case where the quadrics V, W coincide, there being then no need for the condition that A, C lie on the same quadric K having ring contact with V and W.

§ 10. A line joining a point of one focal conic to a point of another focal conic of a confocal system of quadrics is a particular case of a line touching two confocals of the system. And such a line is part of a continuous curve which on either of these two confocals may consist partly of arcs of the line of curvature which is the intersection of these two fundamental confocals, and partly of arcs of geodesics touching this line of curvature. As was recognised by Chasles this continuous curve has everywhere the geometrical property that if we take two other confocals of the system, the homography of the tangent planes drawn to one of these from a tangent line of the curve, in respect of the tangent planes drawn to the other, is the same for every point of the curve. That the analytic formulation may equally be regarded as uniform for all parts of the curve seems often to be unnoticed; it is recognised however by Staude in the papers above referred to. Let us consider the system of confocals

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}=1,$$

where $a>b>c$, using λ for ellipsoids, μ for hyperboloids of one sheet, ν for hyperboloids of two sheets. Suppose that the straight portion of the curve touches the confocals for which $\lambda=p$, $\lambda=q$, and denote by dw the Cayley separation of two consecutive points of the curve taken in regard to the confocal of parameter $\lambda=\theta$. Putting

$$F(x)=4(x+a)(x+b)(x+c)(x-p)(x-q),$$
$$L^2=f(\lambda),\quad M^2=f(\mu),\quad N^2=f(\nu),$$

the curve is such that

$$\int\frac{(\lambda-p)\,d\lambda}{L}+\int\frac{(\mu-p)\,d\mu}{M}+\int\frac{(\nu-p)\,d\nu}{N}=0,$$
$$\int\frac{(\lambda-q)\,d\lambda}{L}+\int\frac{(\mu-q)\,d\mu}{M}+\int\frac{(\nu-q)\,d\nu}{N}=0,$$

while, with $\Theta^2=F(\theta)$,

$$\int\frac{(\lambda-p)(\lambda-q)\,d\lambda}{(\lambda-\theta)L}+\int\frac{(\mu-p)(\mu-q)\,d\mu}{(\mu-\theta)M}+\int\frac{(\nu-p)(\nu-q)\,d\nu}{(\nu-\theta)N}$$
$$=2w\,.\,\frac{(\theta-p)(\theta-q)}{\Theta},$$

where $w=\int dw$. By supposing θ to increase indefinitely, and replacing $\theta^{\frac{1}{2}}dw$ by ds, we have the corresponding result when Euclidian distance is used.

In the notation of hyperelliptic functions (see *Multiply-periodic functions*, Cambridge, 1907, pp. 35, 36, using the p, q as a_1, a_2 are there used), we have

$$\frac{\Theta}{2(\theta-p)(\theta-q)}\int_{x_0}^{x}\frac{(x-p)(x-q)}{x-\theta}\frac{dx}{y}$$
$$=\zeta_1(u^{\theta,\infty})\,.\,u_1^{x,x_0}+\zeta_2(u^{\theta,\infty})\,.\,u_2^{x,x_0}+\tfrac{1}{2}\log\frac{\vartheta(u^{x,\theta}+k)}{\vartheta(u^{x,\phi}+k)},$$

where (ϕ) denotes the place conjugate to (θ), k is such that $\vartheta(k)$ vanishes identically, and when θ is large the significant terms of the functions ζ_1, ζ_2 are $\theta^{-\frac{1}{2}}pq$ and $\theta^{-\frac{1}{2}}(p+q)$.

Along a straight portion of the curve, joining, suppose, the points (λ_0,μ_0,ν_0), (λ,μ,ν), the places (λ), (μ), (ν) of the hyperelliptic construct are coresidual with the places (λ_0), (μ_0), (ν_0), and we can satisfy the identities

$$F(x)-[\psi(x)]^2=4(x-\lambda)(x-\mu)(x-\nu)(x-f_1)(x-f_2),$$
$$F(x)-[\psi_0(x)]^2=4(x-\lambda_0)(x-\mu_0)(x-\nu_0)(x-f_1)(x-f_2),$$

where

$$\psi(x) = 2(lx^2 + mx + n), \quad \psi_0(x) = 2(l_0x^2 + m_0x + n_0).$$

With this notation we find, for the Cayley separation of the two extreme points,

$$w = \tanh^{-1}\left(\frac{\psi(\theta)}{\Theta}\right) - \tanh^{-1}\left(\frac{\psi_0(\theta)}{\Theta}\right),$$

leading in particular, if r be the distance of these points, to $r = l - l_0$, and to

$$r = (\theta - \lambda_0)(\theta - \mu_0)(\theta - \nu_0)\frac{2\tanh w}{\Theta + \psi_0(\theta)\tanh w}.$$

The character of the symmetric functions of the places (λ), (μ), (ν), *regarded as functions of* w, along any portion of the curve, seems eminently worthy of investigation.

And it appears that the total value of w, along any closed portion of the continuous curve, is expressible by an aggregate of the periods of the integral

$$\frac{\Theta}{2(\theta - p)(\theta - q)}\int\frac{(x - p)(x - q)}{(x - \theta)y}\,dx,$$

where $y^2 = F(x)$, with integer coefficients; these will then be unaltered by any continuous small deformation of the arc of the curve. This remark appears to lead to all the known results.

In conclusion I should like to refer the reader to a most interesting note by Mr A. L. Dixon, *Messenger of Mathematics*, XXXII, 1903, 177.

On the construction of the ninth point of intersection of two plane cubic curves of which eight points are given. By Professor H. F. Baker.

[*Read* 3 May 1920.]

Cayley has collected, in a paper reprinted in Vol. IV of his *Papers*, pp. 495–504 (*Quart. J.*, v, 1862), the various solutions given of this problem, regarded as a problem of plane geometry, by Plücker, Weddle, Chasles and Hart, depending for the most part on the generation of a plane cubic curve (two points at a time) by the intersection of a pencil of lines and a homographic pencil of conics. So far as I have been able to notice, geometrical conceptions present themselves to an unbiassed child in the first instance as three dimensional, and he feels it to be an abstraction to regard plane geometry as self-contained; the discussion of the most natural Axioms of geometry seems also to point in this direction; and the most valuable part of a training in geometry would seem to lie in the cultivation of a faculty for visualisation of relations in space. However these things may be, it appears to me always to be an interesting extension when a property of space is shown to follow from a property in space of higher dimensions, this being generally accompanied by the removal of some artificiality. Thus, I regard the very simple example which now follows as being logically at least as fundamental as a proof in the plane.

Let A, B, C, M, N and P, Q, R be the eight given coplanar points. Take a point D outside the plane of these. There are ∞^1 quadric surfaces containing A, B, C and the lines DM, DN; let Ω be one of these (other than that consisting of the planes ABC, DMN). Let DP, DQ, DR meet this quadric again in P_1, Q_1, R_1. A definite twisted cubic curve can be drawn through D, A, P_1, Q_1, R_1 to have BC as a chord (see below). This cubic curve, meeting Ω in D, A, P_1, Q_1, R_1, meets Ω in a further point, say O_1. If DO_1 meet the original plane in O, this is the ninth point required.

For the space cubic is the intersection of two quadric surfaces drawn through D, A, P_1, Q_1, R_1, both having the line BC as a generator; denote these by U and V. The quartic space curve of intersection of U with Ω contains D, A, B, C, P_1, Q_1, R_1, and meets the generators DM, DN of Ω; this curve then projects from D on to the original plane into a cubic curve containing the eight given points A, B, C, P, Q, R, M, N. The curve of intersection of V with Ω projects from D into another cubic through these eight points. The point O_1, on the space cubic, lies on U

and V, and on Ω, and so projects from D into a point common to the two plane cubics. This justifies the statement.

Incidentally any two cubic curves in a plane are shown to be the projections of two quartic curves in space lying on the same quadric; and the plane problem is put in connexion with the space problem of finding the remaining eighth intersection of three quadrics with seven common points.

To construct a twisted cubic curve with five given points D, A, P_1, Q_1, R_1 to have a given line BC as chord, we may for instance first construct a quadric surface by the intersection of corresponding planes of two homographic axial pencils with DA, BC as axes, three pairs of corresponding planes being those containing P_1, Q_1, R_1, and then construct a quadric surface by the intersection of corresponding planes of two homographic axial pencils with DP_1, BC as axes, three pairs of corresponding planes being those containing A, Q_1, R_1. These quadric surfaces intersect in the cubic curve required.

It is seen that analytically each step requires only the solution of linear equations. Indeed, if the conic through A, B, C, M, N be written (referred to A, B, C, D) as $Ayz + Bzx + Cxy = 0$, the line MN being $x + y + z = 0$, we may take for Ω the quadric $t(x + y + z) = Ayz + Bzx + Cxy$. The general plane cubic curve through the five points A, B, C, M, N may be taken to be

$$(Ayz + Bzx + Cxy)(lx + my + nz) + (x + y + z)\,x(qy + rz) = 0,$$

and two cubics through these and P, Q, R may be found by solving for the ratios of l, m, n, q, r in the three equations obtained by substituting the coordinates of P, Q, R. Corresponding to two sets of ratios $l_1 : m_1 : n_1 : q_1 : r_1$, and $l_2 : m_2 : n_2 : q_2 : r_2$ so chosen, there are two quadric surfaces

$$t(l_1x + m_1y + n_1z) + x(q_1y + r_1z) = 0,$$
$$t(l_2x + m_2y + n_2z) + x(q_2y + r_2z) = 0,$$

which intersect in a cubic curve containing D, A, P_1, Q_1, R_1 and having BC for chord. The combination of these with the equation of Ω will lead to a linear equation for O_1, from which O is found. Or the solution may be stated, naturally enough, without reference to three dimensions.

On a proof of the theorem of a double six of lines by projection from four dimensions. By Professor H. F. Baker.

[*Read* 9 February 1920.]

The theorem in question is that if five lines in three dimensions, of which no two intersect, say a, b, c, d, e, have a common transversal, say f', and we take the five transversals other than f' of every four of these five given lines, the five new lines so obtained have also a common transversal. Namely if a' be the transversal, beside f', of b, c, d, e, and b' be the transversal, beside f', of a, c, d, e, and so on, so that we have the scheme

$$\begin{array}{ccccc} a & b & c & d & e \\ a' & b' & c' & d' & e' \quad f' \end{array}$$

in which every line intersects those not occurring in the same row or column with itself, but not the others, in general, then there is a transversal f of a', b', c', d', e'.

We see that the theorem is that if we take eight lines a, b, c, d and a', b', c', d', so related that a' meets b, c, d, while b' meets a, c, d, and c' meets a, b, d and d' meets a, b, c, and if e', f' be the two transversals of a, b, c, d and e, f be the two transversals of a', b', c', d', then the meeting of one of the two former, f', with one of the two latter, e, involves the meeting of the other, e', of the two former, with the remaining one, f, of the two latter. But the original relation of the eight lines a, b, c, d, a', b', c', d' has a certain artificiality; the object of the present note is to show that there is a simple figure in four dimensions, possessing perfect naturalness, being determinate when four arbitrary lines of that space are given, from which the figure in three dimensions may be derived by projection; and that the condition for this derivation is precisely the intersection of the two transversals e and f'. The naturalness of this figure lies in the fact that three lines in four dimensions have just one transversal.

§ 1. In order to show this, it is necessary to enter into some detail in regard to the elements of the geometry of four dimensions; this appears worth while for its own sake; and in order not to over-emphasize the importance of the theorem in three dimensions which is here made the excuse for this, we first give an elementary proof of this theorem, employing only three dimensions (*Proc. Roy. Soc.* A, LXXXIV, 1911, 597).

With the notation above, denote the respective intersections (b', c), (b, c'), (c', a), (c, a'), (a', b), (a, b'), (a, f'), (b, f'), (c, f') by A, A', B, B', C, C', U', V', W'. Let f be the transversal other than e of a', b', c', d', which we may represent by $f = (a', b', c', d')/e$,

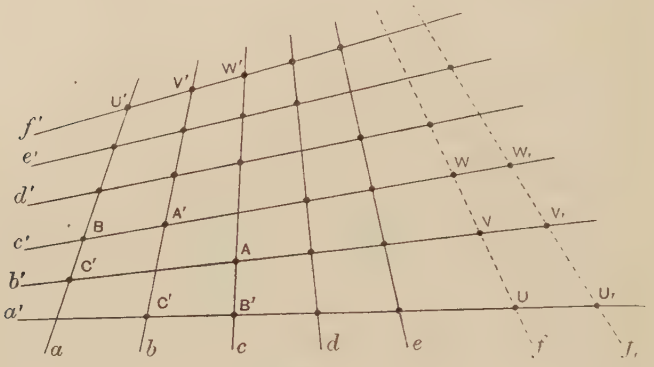


Fig. 1

and denote the points (a', f), (b', f), (c', f) respectively by U, V, W. Similarly let f_1 be the transversal other than d of (a', b', c', e'), which we may denote by $f_1 = (a', b', c', e')/d$; and let the points (a', f_1), (b', f_1), (c', f_1), be U_1, V_1, W_1.

Now take the lines

$$\left.\begin{matrix} a, b, c, f; e \\ a', b', c', f'; d' \end{matrix}\right\}.$$

The two quadric surfaces defined respectively as containing (b, c, e) and (b', c', d'), have, *both of them*, the two generators e and d', which are intersecting lines. The other common points of these two quadrics are then coplanar. Such points are A and A', respectively (b', c) and (b, c'), and U' or (a, f') and U or (a', f). Thus U lies on the plane A, A', U'). So, by considering the quadrics (c, a, e), (c', a', d'), we find that V lies on the plane (B, B', V'), and, by considering the quadrics (a, b, e), (a', b', d'), that W lies on the plane (C, C', W'). By taking the lines

$$\left.\begin{matrix} a, b, c, f_1; d \\ a', b', c', f'; e' \end{matrix}\right\},$$

and considering the pairs of quadrics

$$(b, c, d), (b', c', e'); \quad (c, a, d), (c', a', e'); \quad (a, b, d), (a', b', e')$$

we similarly show that U_1, V_1, W_1, lie respectively on the planes (A, A', U'), (B, B', V'), (C, C', W'), and therefore coincide

respectively with U, V, W, being the intersections of these planes respectively with the lines a', b', c'. Thus $f_1 = f$ is a common transversal of the lines a', b', c', d', e'; as was to be shown.

§ 2. Now take four arbitrary lines a, b, c, d in four dimensions, of which no two intersect. Two of these lines, determined by four points, two on each, determine a threefold space, defined by the four points, and this meets a third line in the four dimensional space in a point. From this point, in the threefold space, can be drawn an unique transversal to the two lines spoken of. Thus three lines in four dimensions, of which no two intersect, have an unique transversal. Let then a' be the transversal of b, c, d, and similarly b', c', d' the transversals respectively of c, a, d; a, b, d and a, b, c. Denote the points (b', c), (b, c'), (c', a), (c, a'), (a', b), (a, b') respectively by A, B, C, A', B', C' and the points (a, d'), (b, d'), (c, d'), (a', d), (b', d), (c', d) respectively by P, Q, R, P', Q', R'.

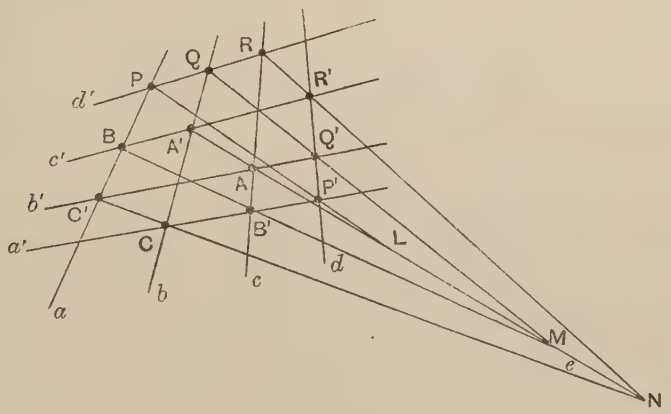


Fig. 2

In general use the word *plane* for the planar twofold space which is determined by three points, and the word *space*, or *threefold* for the planar threefold space determined by four points; as above remarked two lines determine a space, each line being determined by two points; reciprocally two spaces, in the most general case, intersect in a plane, there being a duality of properties in four dimensions wherein a space is reciprocal to a point and a plane to a line. The points A, A', being respectively on the lines $C'Q'$, BR', are in the space (a, d), and evidently are in the space (b, c); the points P, P', being on the lines QR, $B'C$ respectively, are in the space (b, c), and are evidently in the space (a, d). Thus the four points A, A', P, P' lie in a plane, which we

may denote by a, namely that common to the two spaces (b, c) and (a, d). We see how much more naturally this arises than the statement, to which it is evidently analogous, in the three dimensional figure considered in § 1. It follows that the lines AA' and PP' intersect one another, say in L. Similarly the plane, β, of intersection of the spaces (c, a), (b, d), contains the lines BB' and QQ', which then intersect, say in M; and the plane, γ, of intersection of the spaces (a, b), (c, d), contains the lines CC' and RR', intersecting, say, in N. The points L, M, N are however all in each of the spaces (a, a'), (b, b'), (c, c'), and so in a line, the intersection of these spaces. For instance the line AA' joins a point (A') of the line b, to a point (A) of the line b', and so is in (b, b'); and joins a point (A) of the line c, to a point (A') of the line c', and so is in (c, c'); thus L, on the line AA', is in the spaces (b, b'), (c, c'). But the line PP' joins a point (P) of the line a to a point (P') of the line a'; thus L is equally in the space (a, a'). Similarly both M and N are in the line of intersection of the spaces (a, a'), (b, b') and (c, c'). Thus the space (d, d') passes through the line of intersection of the spaces (a, a'), (b, b'), (c, c'); for we similarly show that each of L, M, N is in the space (d, d'). We denote this line by e; evidently its relation to the lines a', b', c', d' is exactly similar with its relation to the lines a, b, c, d; the plane, α, for example, defined as that common to the spaces (b, c), (a, d), is equally the plane common to the spaces (b', c'), (a', d'); and so on. It is usual to speak of e as the line *associated with* a, b, c, d; examination of the figure of fifteen lines and fifteen points which we have constructed will show that there is entire symmetry of mutual relation, and that we may speak equally well of any one of the five lines a, b, c, d, e as being associated with the other four; further e is also associated with a', b', c', d'; and indeed, taking any line of the figure, the eight lines of the figure which do not intersect it, consist of a set of four skew lines and their transversals, and the line in question is associated with either of these two sets of four. There are then $15 \cdot 2 \div 5 = 6$ ways of regarding the figure as depending upon a set of five associated lines.

§ 3. Consider now what planes exist meeting the lines a, b, c, d. In four dimensions an arbitrary plane does not meet an arbitrary line; two such elements which meet lie in a threefold space. It can be shown that a plane meeting a, b, c, d can be drawn through two arbitrary points, one on each of any two of these four lines, so that there are ∞^2 such planes. Further that every such plane also meets the associated line e. Further that two planes meeting a, b, c, d can be drawn through an arbitrary point of the four dimensional space, and, for instance, an infinity of such planes can be drawn through any point of the line e. Also, if the two

planes through an arbitrary point O, to meet a, b, c, d, meet the line e in T and U, then the two planes which can similarly be drawn through O to meet the lines a', b', c', d', meet the line e in the same two points T and U. In general two planes in four dimensions have only one point in common; when they have two points in common, the join of these points lies in both the planes which then both lie in the same threefold space. By what we have said there is a plane through OT intersecting a, b, c, d and also a plane through OT intersecting a', b', c', d', with a similar statement for planes through OU. Namely considering the two planes through O which meet a, b, c, d and also the two planes through O which meet a', b', c', d' either one of the former meets one of the latter in a line.

To prove these statements we may proceed as follows. The joining line of two points arbitrarily taken respectively, say, on the lines b and c, will meet the space (a, d) in a point, from which, in this space, a transversal can be drawn to a and d. Then the plane of the original join and this transversal is a plane, say ϖ, meeting the four lines a, b, c, d. The point of intersection of these two lines determining this plane ϖ is evidently on the plane, α, common to the spaces (b, c) and (a, d). Similarly the point of intersection of the plane ϖ with the plane, β, common to the spaces (c, a) and (b, d), is a point from which two transversals can be drawn respectively to the pairs of lines c, a and b, d; and the plane of these transversals is a plane through this point meeting the four lines a, b, c, d; conversely the join of the two points where the plane ϖ meets the lines c and a lies in the space (c, a), and so intersects the plane β, namely in the supposed unique point common to ϖ and β; this join is thus identical with the transversal drawn from the point (ϖ, β) to the lines c, a. There is thus an unique plane ϖ, meeting a, b, c, d, passing through any general point of the plane α, beside the plane α itself. It will follow from the general result enunciated above, to be proved below, that the plane ϖ', drawn through the same point of the plane α to meet the lines a', b', c', d', meets ϖ in a line intersecting the line e.

Take now any general point O, and a varying point P of the line d; a plane can be drawn through OP to meet the lines a, b, this being the plane containing OP and the common transversal of OP, a and b. Let this plane meet a, b respectively in P_1 and P_2. Thereby any position of P, on the line d, determines the position of P_1 on the line a. Conversely given O and P_1, a plane can be drawn through OP_1 to meet b and d, which, being unique, coincides with the former. Thus any position of P_1 on a determines the position of P on d. The correspondence being algebraic, it follows that P_1, P describe homographic ranges respectively on a and d. Using the line c instead of b, we obtain another range (P') on d,

also homographic with (P_1). Thence the ranges (P'), (P), on d, are homographic; and, if not coincident, they will have two common points, which may coalesce. When P has a position in which it coincides with P', there is a single plane containing O, P_1, P_2, P_3, P, where P_3 is the point of c on the plane OP, P'. Thus through the point O can be drawn, either an infinity of planes meeting all of a, b, c, d, or else *two*, which may however coincide.

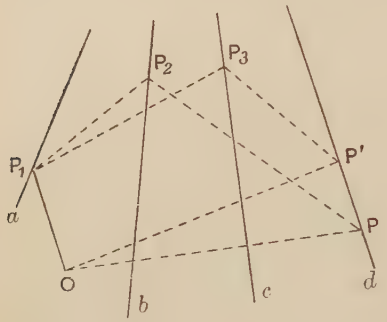


Fig. 3

When O is on the line e, the plane Od' meets a, b, c, and it also meets d because, as we have shown, e, d, d' are in a three dimensioned space. Equally the planes Oa', Ob', Oc' meet a, b, c, d. As there are thus more than two planes through O meeting a, b, c, d, it follows, by what we have shown, that there is an infinity; this is when O is anywhere on the line e. The aggregate of planes so obtained, by taking O to be every point of e, is identical with the aggregate of all planes meeting a, b, c, d, namely any plane meeting a, b, c, d can be identified with one of these; for taking O on e, and P on d, this P determines P_1, P_2, respectively on a, b, when regarded as belonging to one of the coincident ranges on e, and determines P_1, P_3, respectively on a, c, when regarded as belonging to the other range on e. Thus *every plane meeting a, b, c, d also meets e*, or more generally *five associated lines are such that every plane meeting four of them also meets the fifth**.

In general, as we have seen, from any point on a plane meeting a, b, c, d (and e), there can be drawn another such plane. If the point be on the conic through the five points in which the first

* The reader may compare the proofs of this result given by Segre, *Circolo Mat., Palermo*, II, 1888, 45, Alcune considerazioni....The elementary theorems here given for the geometry of four dimensions are of course well known; but I have thought that it was necessary for the purpose of this Note to supply demonstrations. The reader may consult Bertini, *Introduzione alla geometria projettiva degli iperspazi*, Pisa, 1907, a volume of 400 pages, p. 177. In English there is Mr Richmond's paper On the figure of six points in four dimensions, *Quart. Journ.*, XXXI, 1899; *Math. Annal.*, LIII, 1900 (see also *Trans. Camb. Phil. Soc.*, XV, 1894, 267), which deals with a diagram intimately related with that of the text, and Coolidge, *A treatise on the Circle and Sphere*, Oxford, 1916, p. 482, etc., where the lines of four dimensions are replaced by spheres. The origin of the five associated lines seems to be a result given by Stephanos, *Compt. rendus*, XCIII, 1881, p. 578. I have not seen it formally remarked that the property of the double six follows from the geometry of four dimensions; indeed the argument given in § 1 was invented in ignorance of this. The fifteen points and lines of our figure (Fig. 2) are the *diagonal* points and *transversal* lines of the figure considered by Mr Richmond. See also Hudson, *Kummer's Quartic Surface* (1905), Chap. xii.

plane meets a, b, c, d, e, the second plane coincides with the first. It is not necessary for our purpose to prove this.

§ 4. The theorem that two planes can be drawn from an arbitrary point O to meet the lines a, b, c, d is obvious from the theorem in three dimensions that four skew lines have two transversals, the proof of which also depends on the fact that two homographic ranges on a line have two common points. For, if we project a, b, c, d from O, on to an arbitrary threefold space Σ, the planes joining O to the two transversals of the four lines of Σ so obtained, all meet a, b, c, d. And, we now see, e projects into a fifth line meeting these two transversals. When O is on e, the projections in Σ of a, b, c, d are all met by the projections in Σ of a', b', c', d'; for the plane Od', for example, meets a, b, c, and meets d because e, d, d' are in the same three dimensional space; thus the projections in Σ of a, b, c, d are four generators of the same system of a quadric surface of which the projections of a', b', c', d' are generators of the other system. The planes from O each meeting a, b, c, d intersect the space Σ in lines all meeting the projections of a, b, c, d; that is, in lines which are generators of this quadric of the same system as a', b', c', d'. The planes from O meeting a', b', c', d' similarly give rise to generators of the system (a, b, c, d). Thus any plane from the point O meeting a, b, c, d meets any plane from O drawn to meet a', b', c', d' in a line through O; and every line drawn from O in a plane of the former system is the intersection with this plane of a plane of the second system. If O be a point of e lying on a plane drawn from a point H, not on e, to meet a, b, c, d (which therefore also meets e), the line HO lies in a definite plane meeting a', b', c', d'. Thus *either of the two planes of the first system, those meeting a, b, c, d, drawn from a point H, not on e, meets one of the two planes of the second system, those meeting a', b', c', d', drawn from H, in a line; and the two lines so arising intersect the line e.*

§ 5. Hence we can obtain from the four dimensional figure a figure in three dimensions with the characteristics of that used in proving the double six theorem.

If, in the four dimensions, ρ, σ be the planes drawn from an arbitrary point O to meet a', b', c', d', and ρ', σ' those meeting a, b, c, d, and if ρ and σ' meet in a line, as also ρ' and σ; and if we consider the intersections with an arbitrary threefold space Σ, of these four planes, and also of the planes joining O to $a, b, c, d, a', b', c', d'$, denoting these twelve lines respectively by $(\rho), \ldots, (a), \ldots$, then, arranged as follows:

$$\begin{array}{cccccc} (a) & (b) & (c) & (d) & (\rho) & (\sigma) \\ (a') & (b') & (c') & (d') & (\rho') & (\sigma'), \end{array}$$

these form a double six, any one of the lines meeting the five which do not lie in the same row or column with itself.

§ 6. Conversely we now proceed to show that if

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_1' & b_1' & c_1' & d_1' \end{matrix}$$

be eight lines in three dimensions such that no two of a_1, b_1, c_1, d_1 intersect, while d_1' intersects a_1, b_1, c_1, a_1' intersects b_1, c_1, d_1, etc., and if one of the two transversals, say l, of a_1', b_1', c_1', d_1', intersects one of the two transversals, say m', of a_1, b_1, c_1, d_1, then these lines may be obtained by projection from four dimensions; namely a_1, b_1, c_1, d_1, a_1', b_1', c_1', d_1' are projections of four lines a, b, c, d in space of four dimensions and of the transversals a', b', c', d' of threes of these, respectively, while l and m' are the intersections with the original three dimensional space of planes in four dimensions meeting respectively the set a', b', c', d' and the set a, b, c, d.

We give an analytical proof of this. And for this purpose first explain an analytical view of the theorems which have been given in §§ 2, 3, 4, which indeed renders these very obvious.

It is fundamental that a point may be represented by a single symbol, say P, the same point being equally represented by any numerical multiple of this, say mP, where m is an ordinary number. Then a space of r dimensions is one in which every $r + 2$ points, P_1, P_2, ..., P_{r+2}, are connected by a syzygy,

$$m_1P_1 + m_2P_2 + \ldots + m_{r+2}P_{r+2} = 0,$$

where m_1, ..., m_{r+2} are ordinary numbers; thus the space is determined by any $r + 1$ points of it, themselves not lying in a space of less than r dimensions; and, in terms of such $r + 1$ points, say A_1, ..., A_{r+1}, every other point of the space may be represented by a symbol $x_1A_1 + x_2A_2 + \ldots + x_{r+1}A_{r+1}$, where x_1, x_2, ..., x_{r+1} are ordinary numbers; whose ratios may be called the coordinates of this point, relatively to A_1, ..., A_{r+1}. Thus any point of a line determined by two points A, B, is representable by a symbol $mA + nB$, in which m, n are numbers; and any point of a plane determined by three points A, B, C, is representable by a symbol $xA + yB + zC$.

In our figure in four dimensions (§ 2), let the points

$$A = (b', c), \quad B = (c', a), \quad C = (a', b),$$
$$A' = (b, c'), \quad B' = (c, a'), \quad C' = (a, b'),$$

be regarded as fundamental. Being in four dimensions, they are connected by a syzygy; absorbing proper numerical multipliers in the symbols, this may be taken to be

$$A + B + C + A' + B' + C' = 0.$$

It is not then allowable to modify further these symbols by multiplication with numbers, except the same multiplier for all. And, the six points being supposed not to lie in a three dimensional space, there is no further syzygy connecting them.

Each of the points P, Q, R of our diagram (§ 2) is then expressible linearly by two of these six points, P by B and C', etc., while P, Q, R, being collinear, are themselves connected by a syzygy. This becomes then a syzygy for A, B, C, A', B', C', which must be the same as the fundamental syzygy. Whence it appears at once that, by absorbing proper numbers in the symbols P, Q, R, we can write

$$P = B + C', \quad Q = C + A', \quad R = A + B'.$$

By a similar argument we deduce for the points P', Q', R' of our diagram (§ 2)

$$P' = B' + C, \quad Q' = C' + A, \quad R' = A' + B.$$

From these two sets of equations we have however

$$P + P' = -(A + A');$$

there is then a point lying both on the line PP' and on the line AA', namely these lines intersect, say in the point L. So also M, N are the points $B + B'$, $C + C'$ respectively. And the identity

$$A + A' + B + B' + C + C' = 0$$

shows that L, M, N lie on a line, e.

A plane meeting the lines a, b, c can evidently be defined by three points, one on each of them, represented by symbols of the form

$$yB + zC', \quad zC + xA', \quad x_1A + y_1B'.$$

In order that the plane should also meet the line d, it is necessary and sufficient that numbers $\lambda, \mu, \nu, \rho, \sigma$ exist for which there is the syzygy

$$\lambda(yB + zC') + \mu(zC + xA') + \nu(x_1A + y_1B') + \rho P' + \sigma Q' = 0;$$

if herein P', Q' be replaced respectively by $B' + C$, $C' + A$, it must reduce to the fundamental syzygy. Thus we find at once that $x_1 = x$ and $y_1 = y$, and any one of the ∞^2 planes meeting a, b, c, d is the join of points respectively on the lines a, b, c, given by

$$yB + zC', \quad zC + xA', \quad xA + yB'.$$

This plane however contains the point represented by

$$x(yB + zC') + y(zC + xA') + z(xA + yB'),$$

which is

$$yzP' + zxQ' + xyR',$$

and is therefore its intersection with the line d. And the plane contains the point represented by

$$(yB + zC') + (zC + xA') + (xA + yB'),$$

which is $$x(A + A') + y(B + B') + z(C + C')$$
and is therefore its intersection with the line e.

Similarly it can be shown that a general plane meeting the lines a', b', c', d' and e is that joining the points, respectively on a', b', c', represented by
$$yB' + zC,\ zC' + xA,\ xA' + yB.$$
If the plane, meeting a, b, c, d, joining the points
$$yB + zC',\ zC + xA',\ xA + yB',$$
pass through an arbitrary point of the four dimensional space expressed by
$$\xi A + \eta B + \zeta C + \xi' A' + \eta' B' + \zeta' C',$$
it can be shown without difficulty that
$$\frac{\xi' - \xi}{x} + \frac{\eta' - \eta}{y} + \frac{\zeta' - \zeta}{z} = 0,$$
$$(\eta' - \zeta)x + (\zeta' - \xi)y + (\xi' - \eta)z = 0,$$
and also
$$\frac{(\eta - \zeta')(\eta' - \zeta)}{y - z} + \frac{(\zeta - \xi')(\zeta' - \xi)}{z - x} + \frac{(\xi - \eta')(\xi' - \eta)}{x - y} = 0;$$
to satisfy the condition it is necessary and sufficient, in fact, that numbers λ, μ, ν, ρ should be possible such that
$$\xi = \nu x + \rho, \quad \eta = \lambda y + \rho, \quad \zeta = \mu z + \rho,$$
$$\xi' = \mu x + \rho, \quad \eta' = \nu y + \rho, \quad \zeta' = \lambda z + \rho.$$

The first two of these equations determine the two planes meeting the lines a, b, c, d, e which pass through the arbitrary point in question. The last equation determines the two points, of the form
$$x(A + A') + y(B + B') + z(C + C'),$$
or
$$(x - z)(A + A') + (y - z)(B + B'),$$
where these planes meet the line e. As this last equation is unaltered by interchanging ξ, η, ζ respectively with ξ', η', ζ', it shows that the two planes drawn through the arbitrary point to meet a, b, c, d intersect the line e in the same two points as do the planes drawn through this point to meet a', b', c', d'.

§ 7. Now consider a figure in three dimensions consisting of four lines a, b, c, d, skew to one another, and four other lines, a', b', c', d', skew to one another, each meeting three of the former. As before, with the same notation, the figure consists of a skew hexagon $BC'AB'CA'$, of which one set of three alternate sides is intersected by a line respectively in P, Q, R, and the other set of alternate sides is intersected by another line in P', Q', R' respectively. But here the six points A, B, C, A', B', C', being in three

dimensions, are subject to two syzygies, which, without loss of generality, may be written

$$A + B + C + A' + B' + C' = 0,$$
$$aA + bB + cC + a'A' + b'B' + c'C' = 0.$$

The points P, Q, R, respectively in syzygy with (B, C'), with (C, A'), and with (A, B'), are themselves in syzygy. Thus, not only is P given, save for a multiplier, by an expression $\mu B + \nu' C'$, and Q by $\nu C + \lambda' A'$, and R by $\lambda A + \mu' B'$, but the multipliers $\mu, \nu', \nu, \lambda', \lambda, \mu'$ may be chosen so that we have

$$\mu B + \nu' C' + \nu C + \lambda' A' + \lambda A + \mu' B' = 0.$$

This must then be a consequence of the two fundamental syzygies; or, for a proper value of ρ, we must have

$$\lambda : \mu : \nu : \lambda' : \mu' : \nu' = a + \rho : b + \rho : c + \rho : a' + \rho : b' + \rho : c' + \rho.$$

Thence P, Q, R may be expressed by

$$P = bB + c'C' + \rho\ (B + C'),$$
$$Q = cC + a'A' + \rho\ (C + A'),$$
$$R = aA + b'B' + \rho\ (A + B').$$

Similarly P', Q', R' are expressible by symbols

$$P' = b'B' + cC + \rho'\ (B' + C),$$
$$Q' = c'C' + aA + \rho'\ (C' + A),$$
$$R' = a'A' + bB + \rho'\ (A' + B).$$

Suppose now that one of the two transversals, say u', of the four lines a, b, c, d, intersects one of the two transversals, say u, of the four lines a', b', c', d'. Then as before (§ 1), remarking that the two quadric surfaces determined respectively by the triads of lines (b, c, u), (b', c', u') contain respectively also the lines (a', d', u'), (a, d, u), we infer that the points A, A', P, P' are coplanar. And by similar reasoning, also because u, u' intersect one another, we infer that B, B', Q, Q' are coplanar, and also that C, C', R, R' are coplanar. Conversely let us assume only the first fact, that A, A', P, P' are coplanar. We can then prove that the numbers ρ, ρ', occurring respectively in the expressions of P, Q, R and of P', Q', R', are equal. For the fact that A, A', P, P' are coplanar involves the existence of a syzygy of the form

$$m\,[bB + c'C' + \rho\,(B + C')] + m'\,[b'B' + cC + \rho'\,(B' + C)] + pA + p'A' = 0;$$

for suitable values of m, m', p, p' and of another number k, this must be the same as

$$(a + k)\,A + (a' + k)\,A' + (b + k)\,B + (b' + k)\,B' + (c + k)\,C + (c' + k)\,C' = 0;$$

we may then take

$$p = a + k,\ p' = a' + k,\ m(b + \rho) = b + k,\ m'(b' + \rho') = b' + k,$$
$$m'(c + \rho') = c + k,\ m(c' + \rho) = c' + k;$$

these lead to $m(b - c') = b - c'$, $m'(b' - c) = b' - c$, and hence $m = m' = 1$, unless the lines d, d' have particular positions; whence $\rho = \rho' = k$.

Thus we have

$$P = bB + c'C' + \rho(B + C'),\ P' = b'B' + cC + \rho(B' + C),\ \text{etc.},$$

which give, in particular,

$$Q + Q' = aA + a'A' + cC + c'C' + \rho(C + C' + A + A')$$
$$= -[bB + b'B' + \rho(B + B')],$$

showing that the points B, B', Q, Q' are coplanar. And similarly the points C, C', R, R' are coplanar.

§ 8. Now take a point O, not in the original space of three dimensions which we have considered; with this original space the point O determines a four dimensional space containing both. Therein take points A_1, B_1, C_1, A_1', B_1', C_1' given by

$$A_1 = (a + \rho)A + lO, \qquad B_1 = (b + \rho)B + mO,$$
$$A_1' = (a' + \rho)A' + l'O, \quad B_1' = (b' + \rho)B' + m'O,$$
$$C_1 = (c + \rho)C + nO,$$
$$C_1' = (c' + \rho)C' + n'O,$$

wherein l, m, n, l', m', n' are arbitrary numbers save for the single relation

$$l + m + n + l' + m' + n' = 0.$$

Then we have

$$A_1 + B_1 + C_1 + A_1' + B_1' + C_1' = 0.$$

Take also P_1, Q_1, R_1, P_1', Q_1', R_1' given by

$$P_1 = P + (m + n')O, \quad Q_1 = Q + (n + l')O, \quad R_1 = R + (l + m')O,$$
$$P_1' = P' + (m' + n)O, \quad Q_1' = Q' + (n' + l)O, \quad R_1' = R' + (l' + m)O;$$

then we have

$$P_1 = B_1 + C_1', \quad Q_1 = C_1 + A_1', \quad R_1 = A_1 + B_1',$$
$$P_1' = B_1' + C_1, \quad Q_1' = C_1' + A_1, \quad R_1' = A_1' + B_1.$$

The original figure in three dimensions is thus the projection from O of the figure now formed in four dimensions, and this is exactly such a figure as that we considered originally.

Geometrically, what is arbitrary in the four dimensional space is the point O, and five of the six points taken on the lines OA, OB, OC, OA', OB', OC'. These being taken, it is no doubt possible to complete the construction without use of the symbols. These seem however to add to clearness.

On transformations with an absolute quadric. By Professor H. F. Baker.

[*Read* 9 February 1920.]

We consider homographic (linear) transformations of projective space which leave unaltered a given quadric, sometimes called the Absolute. We suppose the two generators, of either system, of the quadric, which are unchanged by the transformation, to be distinct. Denoting then by DC and AB the two diagonals of the skew quadrilateral formed by these four generators, it is known that the transformation may be represented (a) by rotations about DC and AB of suitable amplitudes, whose order is indifferent, (b) by a "half turn" about an arbitrary line meeting DC and AB, followed by a half turn about another appropriately chosen line meeting DC and AB, or preceded by such a half turn of appropriate axis, (c) by a "right vector" and a "left vector" together, whose order is indifferent. The object of the present note is to mention another mode of decomposition of the transformation. For this purpose we define inversion, in regard to a point O and a plane ϖ, as the process of passing from a point P to a point P' on the line OP such that P, P' are harmonically separated by O and ϖ. We consider only cases in which ϖ is the polar plane of O in regard to the absolute quadric. We also define harmonic inversion in regard to two given skew lines, as the process of passing from any point P to a point P' on the transversal drawn from P to the lines, such that P, P' are harmonically separated by these. When the lines are polars of one another in regard to the absolute quadric, the process is the same as a half turn about either of them, and is obtainable as the sequence of two inversions about any two points taken on either of the lines so as to be conjugate to one another in regard to the quadric.

Thus if H, K be two arbitrary points respectively on AB, DC, and H_1, K_1 be two appropriately chosen points on these lines respectively, it follows from the decomposition (b) referred to above that the complete transformation can be represented by the sequence of four inversions

$$(H_1)\,(K_1)\,(H)\,(K),$$

wherein, since H, K_1 are conjugate points, the process represented by $(K_1)\,(H)$ is the same as that represented by $(H)\,(K_1)$. The whole is then equivalent to

$$(H_1)\,(H)\,.\,(K_1)\,(K).$$

Conversely, it is an easy matter to prove directly that the two inversions $(H_1)(H)$, about points on AB, are together equivalent to a rotation about the line DC, of amplitude equal to twice the Cayley separation of H and H_1, in regard to the quadric, with a similar fact for $(K_1)(K)$. This gives an elementary theory of the transformation.

In Euclidian space, a general movement of a rigid body is thus obtainable by a succession of four inversions, two of these about points on the central axis, and two on the line at infinity of the plane at right angles to this. Taking the central axes to be $x = 0$, $y = 0$, the two former may be taken to be reflexions in planes $z = k$, $z = k_1$, and the two latter to be reflexions in the planes $x = 0$, $x = y \tan \frac{1}{2}\theta$. By the former we obtain

$$z + z_1 = 2k, \quad z' + z_1 = 2k_1,$$

giving $z' = z + 2(k_1 - k)$; by the latter we obtain

$$x_1 = -x, \quad y_1 = y, \quad (x' - x_1)\sin\tfrac{1}{2}\theta + (y' - y_1)\cos\tfrac{1}{2}\theta = 0,$$
$$(x' + x_1)\cos\tfrac{1}{2}\theta - (y' + y_1)\sin\tfrac{1}{2}\theta = 0,$$

giving

$$x' + x_1\cos\theta - y_1\sin\theta = 0, \quad y' - y_1\cos\theta - x_1\sin\theta = 0.$$

Thus altogether

$$x' = x\cos\theta + y\sin\theta, \quad y' = -x\sin\theta + y\cos\theta, \quad z' = z + 2(k_1 - k).$$

On a set of transformations of rectangular axes. By Professor H. F. Baker.

[*Read* 9 February 1920.]

In a paper in the *Acta Mathematica*, xxv, 1902, 291–296, Dr Burnside has greatly simplified, by geometrical considerations, results obtained in the same Journal, xxiv, 1901, 123–158, by Lipschitz, for the relations connecting the four rotations changing one system of orthogonal axes into another. In a paper, *Proceedings Lond. Math. Soc.*, ix, 1910, 197, I have incidentally noticed a theorem which is intimately connected with these results, and may be made to include them. We may associate with a rotation in Euclidian space about an axis through the origin, of direction cosines l, m, n, through an angle θ, a point of projective space, of coordinates (a, b, c, d) given by

$$a = l \sin \tfrac{1}{2}\theta, \quad b = m \sin \tfrac{1}{2}\theta, \quad c = n \sin \tfrac{1}{2}\theta, \quad d = \cos \tfrac{1}{2}\theta;$$

and we may call this point, which determines the rotation, the *representative point* of the rotation. The theorem referred to is then the following: Let Ω, Ω' be any two congruent figures upon a sphere; let Ω_1, Ω_2, Ω_3 be the figures obtained from Ω by reflexion in, or rotations of amplitude π about, the respective coordinate axes; let Ω_1', Ω_2', Ω_3' be similarly derived from Ω'; there is then a rotation changing any one of Ω_1, Ω_2, Ω_3, Ω into any one of Ω_1', Ω_2', Ω_3', Ω'. *The representative points of these sixteen rotations form a Kummer configuration.* In other words they lie in sixes upon sixteen conics, whose planes are in sixes tangent planes of sixteen quadric cones whose vertices are the points; or again, they can be arranged in twenty ways so as to form the vertices of a set of four tetrahedra of which set every two are mutually inscribed. As will be seen, the theorem is the same as that, if a point (a, b, c, d) be represented by the quaternion

$$P = ai + bj + ck + d,$$

the sixteen points

$$\begin{array}{cccc} iPi, & iPj, & iPk, & iP \\ jPi, & jPj, & jPk, & jP \\ kPi, & kPj, & kPk, & kP \\ Pi, & Pj, & Pk, & P \end{array}$$

form a Kummer system, the four tetrahedra formed each by the points in any row being mutually inscribed in pairs, as are the four tetrahedra formed each by the points in any column. This

last statement is easily verifiable, and it is sufficient therefore to reduce the original statement to this. By direct algebra this can be done by using the well-known fact that the rotation of representative point (a, b, c, d) is given by

$$x' = ux + hy + g_1 z, \quad y' = h_1 x + vy + fz, \quad z' = gx + f_1 y + wz,$$

where the nine coefficients are given respectively by

$$\begin{array}{lll} u,\ h,\ g_1 = d^2 + a^2 - b^2 - c^2, & 2(ab - cd), & 2(ca + bd) \\ h_1,\ v,\ f & 2(ab + cd), & d^2 + b^2 - c^2 - a^2, \quad 2(bc - ad) \\ g,\ f_1,\ w & 2(ca - bd), & 2(bc + ad), \quad d^2 + c^2 - a^2 - b^2 \end{array}$$

each of the elements on the right being divided by $a^2 + b^2 + c^2 + d^2$, which is unity. Using the representative points given by the above scheme to calculate the rotations, it is at once seen that they have the interpretations assigned to them in the statement of the theorem.

In particular the representative points given by the forms

$$Pi,\ Pj,\ Pk,\ P$$

are respectively

$$(d, c, -b, -a), \quad (-c, d, a, -b), \quad (b, -a, d, -c), \quad (a, b, c, d),$$

and if these belong to rotations of respective amplitudes $\theta_1, \theta_2, \theta_3, \theta$ about axes of direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3), (l, m, n), we have

$$\begin{array}{llll} d = l_1 \sin \tfrac{1}{2}\theta_1, & c = m_1 \sin \tfrac{1}{2}\theta_1, & -b = n_1 \sin \tfrac{1}{2}\theta_1, & -a = \cos \tfrac{1}{2}\theta_1, \\ -c = l_2 \sin \tfrac{1}{2}\theta_2, & d = m_2 \sin \tfrac{1}{2}\theta_2, & a = n_2 \sin \tfrac{1}{2}\theta_2, & -b = \cos \tfrac{1}{2}\theta_2, \\ b = l_3 \sin \tfrac{1}{2}\theta_3, & -a = m_3 \sin \tfrac{1}{2}\theta_3, & d = n_3 \sin \tfrac{1}{2}\theta_3, & -c = \cos \tfrac{1}{2}\theta_3, \\ a = l \sin \tfrac{1}{2}\theta, & b = m \sin \tfrac{1}{2}\theta, & c = n \sin \tfrac{1}{2}\theta, & d = \cos \tfrac{1}{2}\theta, \end{array}$$

from which follow, if the axes be OD_1, OD_2, OD_3, OD,

$$-bc = \cos D_2D_3 \sin \tfrac{1}{2}\theta_2 \sin \tfrac{1}{2}\theta_3 = -\cos \tfrac{1}{2}\theta_2 \cos \tfrac{1}{2}\theta_3,$$

$$\cos D_2D_3 = -\cot \tfrac{1}{2}\theta_2 \cot \tfrac{1}{2}\theta_3,$$

$$ad = \cos DD_1 \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta_1 = -\cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta_1,$$

$$\cos DD_1 = -\cot \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta_3,$$

and hence

$$\cos DD_1 \cos D_2D_3 = \cos DD_2 \cos D_3D_1 = \cos DD_3 \cos D_1D_2,$$

in accordance with the fact that the plane containing any two of these four axes is at right angles to the plane containing the other two; as also

$$\tan^2 \tfrac{1}{2}\theta_1 = -\cos D_2D_3 / \cos D_1D_2 \cos D_1D_3,$$

these being equations noticed by Dr Burnside *loc. cit.*, p. 294.

But the definition of the representative point of a rotation which has been given is, like this proof of the theorem, both metrical and analytical, and it is desirable to alter it in both respects. A point of view which is not metrical is to regard the transformation

$$x' = ux + hy + g_1 z,\quad y' = h_1 x + vy + fz,\quad z' = gx + f_1 y + wz, \qquad \text{(I)},$$

as that unique homographic transformation of the plane of the homogeneous variables x, y, z which leaves the conic

$$x^2 + y^2 + z^2 = 0$$

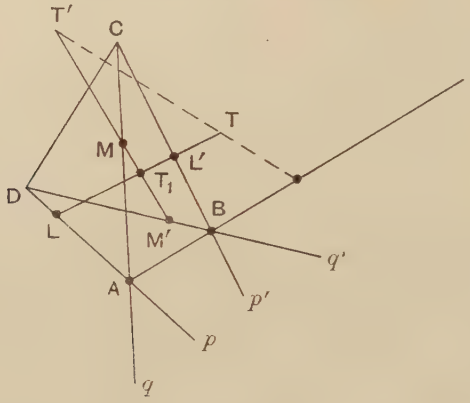
unaltered, and changes the self-polar triangle $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ into the self-polar triangle (u, h_1, g), (h, v, f_1), (g_1, f, w). If the vertices A, B, C of the former be joined to the corresponding vertices A', B', C' of the latter, by lines forming a triangle D, E, F, of which E, F will be collinear with A, A', etc., and we take on the line A, A', E, F the points P, P' harmonic in regard both to A, A' and E, F; and similarly take Q, Q' on the line B, B', F, D harmonic in regard both to B, B' and F, D; and take R, R' on the line C, C', D, E harmonic in regard both to C, C' and D, E; then it can be shown that the six points P, P', Q, Q', R, R' lie in threes upon four straight lines, which are in fact

$$dx + cy - bz = 0,\quad dy + az - cx = 0,$$
$$dz + bx - ay = 0,\quad ax + by + cz = 0.$$

This gives a geometrical interpretation, which is not metrical, of a, b, c, d.

A much better point of view is however as follows. The rotation, expressed by four equations of which three are those marked (I) above, and the fourth is $t' = t$, may be regarded as a homographic transformation of projective space (x, y, z, t), leaving the quadric whose equation is $x^2 + y^2 + z^2 + t^2 = 0$ (or indeed any quadric $x^2 + y^2 + z^2 + Mt^2 = 0$) unaltered. And it may be regarded as compounded from two transformations P, Q, taken in either order, of which P is such as to leave every generator of the quadric of one system, say the q-system, unaltered, while it interchanges the generators of the other system, say the p-system, among themselves; and Q has a similar meaning with the systems of generators interchanged. In the transformation P there will be two particular generators of the p-system, say p and p', which remain unaltered; and the transformation may be described geometrically as changing any point T of space, not necessarily on the quadric, into a point T' of the transversal drawn from T to p, p', such that the homography $(T'T,\ pp')$ is constant, equal to $e^{i\theta}$ say, the transformation Q having a similar meaning in regard to two generators q, q' of the q-system and *having the same value for the corresponding homo-*

graphy; so that P, Q may be described as *equal* transformations of the p-kind and q-kind respectively (or if q' be interchanged with q as *equal* but *opposite* transformations).



If two lines p, q meet in A, two others p', q' meet in B, while p, q' meet in D and p', q meet in C, and on the transversal TLL', of p, p', be taken T_1 so that $(T_1T, LL') = \lambda$, and on the transversal T_1MM', of q, q', be taken T' so that $(T'T_1, MM') = \lambda$, it is at once shown that, referred to $ABCD$, the coordinates of T', (x', y', z', t'), are expressible in terms of (x, y, z, t), the coordinates of T, by the formulae

$$x' = x, \quad y' = \lambda^2 y, \quad z' = \lambda z, \quad t' = \lambda t.$$

These are the equations of a rotation round DC, of which every point remains unaltered, as does every plane through AB. If the planes joining DC to T', T meet AB in U', U we have

$$(U'U, AB) = \lambda^2.$$

The two processes may be taken in the reverse order. Any quadric containing the lines p, q, p', q', whose equation is $xy = Mzt$, is unaltered by the composite transformation. The transformation here taken first is $x_1 = x$, $y_1 = \lambda y$, $z_1 = \lambda z$, $t_1 = t$, which changes the p, q-generators, expressed respectively by $z = px$, $y = Mpt$ and $y = qz$, $qx = Mt$, into the generators p_1, q_1 given by $p_1 = \lambda p$, $q_1 = q$. The transformation here taken second is

$$x' = x_1, \quad y' = \lambda y_1, \quad z' = z_1, \quad t' = \lambda t_1,$$

which changes p_1, q_1 into p', q' given by $p' = p_1$, $q' = \lambda q_1$.

Conversely any homographic transformation of space which leaves every point of a line DC unchanged, and leaves also two points A, B, not on DC, both unchanged, will leave the lines

joining A, B to an arbitrary point D of DC both unchanged, and also the lines joining A, B to a further arbitrary point C of this line; referred to A, B, C, D, such a transformation will be expressible by equations $x' = m\lambda^{-1}x$, $y' = m\lambda y$, $z' = z$, $t' = t$. If it be further restricted to be such as to leave any (and therefore every) quadric $xy = Mzt$ through AD, BD, AC, BC, unchanged, we may take $m = 1$. Then any conic through A, B touching the planes ADC, BDC is also unchanged, and the equations of transformation $x' = \lambda^{-1}x$, $y' = \lambda y$, $z' = z$, $t' = t$ are those just considered. By the component transformation $x_1 = x$, $y_1 = \lambda y$, $z_1 = \lambda z$, $t_1 = t$, any point O of DC is changed to a point O_1 of DC given by

$$(O_1O, DC) = \lambda.$$

In more general terms, if we write, using matrices,

$$P = \begin{vmatrix} d, & -c, & b, & a \\ c, & d, & -a, & b \\ -b, & a, & d, & c \\ -a, & -b, & -c, & d \end{vmatrix}, \quad Q = \begin{vmatrix} d, & -c, & b, & -a \\ c, & d, & -a, & -b \\ -b, & a, & d, & -c \\ a, & b, & c, & d \end{vmatrix},$$

denoting by P', Q' what P, Q become by substituting a', b', c', d' for a, b, c, d, we find $PQ' = Q'P$, in particular $PQ = QP$, and

$$PQ = \begin{vmatrix} u, & h, & g_1, & 0 \\ h_1, & v, & f, & 0 \\ g, & f_1, & w, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix}.$$

It is then easy to verify that the transformation

$$(x', y', z', t') = P.(x, y, z, t)$$

changes the p, q-generators of $x^2 + y^2 + z^2 + t^2 = 0$ expressed by

$$x + iy = p\,(z + it), \qquad x - iy = -p^{-1}(z - it);$$
$$x + iy = q\,(z - it), \qquad x - iy = -q^{-1}(z + it)$$

into p', q' given by

$$p' = \frac{-p\,(c - id) + a + ib}{p\,(a - ib) + c + id}, \quad q' = q;$$

this, with $a = l \sin \tfrac{1}{2}\theta$, $b = m \sin \tfrac{1}{2}\theta$, $c = n \sin \tfrac{1}{2}\theta$, $d = \cos \tfrac{1}{2}\theta$, is equivalent with

$$\frac{p' - \dfrac{l + im}{1 + n}}{p' + \dfrac{l + im}{1 - n}} = e^{i\theta} \frac{p - \dfrac{l + im}{1 + n}}{p + \dfrac{l + im}{1 - n}},$$

the stationary values of p being those of the generators passing

through the points where the conic $t = 0$, $x^2 + y^2 + z^2 = 0$ is met by the line $ax + by + cz = 0$. The transformation

$$(x', y', z', t') = Q\,(x, y, z, t)$$

makes precisely the same changes in the parameters q, p respectively.

Clearly the *representative point* (a, b, c, d) *is the point to which the vertex* $(0, 0, 0, 1)$ *is changed by the transformation* P. This definition is relative to the position chosen for this vertex upon the axis of the original transformation, but this is immaterial in discussing the composition of rotations about different axes passing through the same vertex.

In particular the representative points for rotations of amplitude π respectively about the lines joining (0001) to (1000), to (0100) and to (0010), are (1000), (0100), (0010), the corresponding transformations P being

$$i = \begin{vmatrix} 0, 0, & 0, 1 \\ 0, 0, & -1, 0 \\ 0, 1, & 0, 0 \\ -1, 0, & 0, 0 \end{vmatrix}, \quad j = \begin{vmatrix} 0, & 0, 1, 0 \\ 0, & 0, 0, 1 \\ -1, & 0, 0, 0 \\ 0, & -1, 0, 0 \end{vmatrix}, \quad k = \begin{vmatrix} 0, -1, & 0, 0 \\ 1, & 0, & 0, 0 \\ 0, & 0, & 0, 1 \\ 0, & 0, & -1, 0 \end{vmatrix},$$

by which the point (x, y, z, t) becomes changed respectively to

$$(x' = t, y' = -z, z' = y, t' = -x), \quad (x' = z, y' = t, z' = -x, t' = -y),$$
$$(x' = -y, y' = x, z' = t, t' = -z).$$

We at once find

$$i^2 = j^2 = k^2 = -\omega, \; jk = -kj = i, \; ki = -ik = j, \; ij = -ji = k,$$

and the matrix of the general transformation P can be written in terms of the matrices i, j, k in the form

$$P = ai + bj + ck + d\omega,$$

where ω is the matrix of the identical transformation. For the generators of the p-system, the transformations i, j, k lead respectively to $p' = p^{-1}$, $p' = -p^{-1}$, $p' = -p$, that is they are harmonic inversions in the pairs of generators of the p-system passing through the intersections of the conic $x^2 + y^2 + z^2 = 0, t = 0$ respectively with the lines $x = 0$, $y = 0$, $z = 0$. By the transformation P, these three pairs of generators are changed to the generators passing through the intersection of the conic respectively with the lines

$$ux + h_1 y + gz = 0, \quad hx + vy + f_1 z = 0 \text{ and } g_1 x + fy + wz = 0,$$

as is easy to see. For instance the generators $p = 1$, $p = -1$, are changed respectively to the generators

$$p = (u + ih_1)/(1 + g), \quad p = -(u + ih_1)/(1 - g).$$

Further a succession of any number of rotations about axes through the vertex (0, 0, 0, 1), namely a transformation representable in the form

$$P_1Q_1 \,.\, P_2Q_2 \,.\, P_3Q_3 \ldots$$

is, in virtue of the commutative property of any p-transformation with any q-transformation, capable of being regarded as

$$(P_1P_2P_3 \ldots)\,(Q_1Q_2Q_3 \ldots)$$

wherein the first factor is a p-transformation, and the second factor a q-transformation. The laws of composition of the rotations are then precisely the same as those of the associated p-transformations—and the representative point for a composite transformation can be obtained by the composition of the p-transformations.

This suffices to reduce the theorem we have stated to the theorem that the representative points of the sixteen transformations referred to are those of the transformations

$$(i, j, k, \omega)\; P\; (i, j, k, \omega),$$

where P is any p-transformation. For we can pass from any figure on a sphere to any congruent figure by an appropriate rotation PQ.

But it is at once obvious that the representative point of a composite transformation PP' may be obtained by forming the product of the symbols associated therewith

$$(ai + bj + ck + d\omega)\,(a'i + b'j + c'k + d'\omega)$$

by means of the multiplication rules for i, j, k, ω given above. In fact this product gives

$$(ad' + a'd + bc' - b'c)\,i + (bd' + b'd + ca' - c'a)\,j$$
$$+ (cd' + c'd + ab' - a'b)\,k + (dd' - aa' - bb' - cc')\,\omega,$$

or say

$$Ai + Bj + Ck + D\omega,$$

while the product of the two matrices P, P' is at once verified to be the same function of A, B, C, D as is P of a, b, c, d.

The theorem stated is thus proved.

Remark. Any rotation is thus associated with a quaternion, whose vector coefficients give the direction of the axis of the rotation, the amplitude of this being *twice* the angle whose cotangent is $d\,(a^2 + b^2 + c^2)^{-\frac{1}{2}}$. In particular the symbol i is associated with a rotation about the axis of x of amplitude π.

If two rotations of amplitudes θ, θ' about axes (l, m, n), (l', m', n') be equivalent to a rotation of amplitude ϕ about an axis (λ, μ, ν), we have such equations as

$$\lambda \sin \tfrac{1}{2}\phi = l \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta' + l' \sin \tfrac{1}{2}\theta' \cos \tfrac{1}{2}\theta + (mn' - m'n) \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta',$$

$$\cos \tfrac{1}{2}\phi = \cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta' - \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta' \,(ll' + mm' + nn').$$

The latter of these expresses Hamilton's law that a rotation of amount $2C$ about the vertex C of a spherical triangle ABC, followed by a rotation of amount $2B$ about the vertex B, gives a rotation about the vertex A of amount $2\pi - 2A$. The former expresses that if O be the pole of the arc BC, and P be any point of the sphere,

$$\cos PO \sin a \sin B \sin C$$
$$= \cos PA \sin A - \cos PB \sin B \cos C - \cos PC \sin C \cos B.$$

The transformation here denoted by P is that called by Clifford a *right* (or *left*) vector. It can also be represented in the form

$$P = \cos \tfrac{1}{2}\theta \,.\, \omega + \sin \tfrac{1}{2}\theta \,.\, \varpi,$$

where
$$\varpi, = \begin{vmatrix} 0, & -n, & m, & l \\ n, & 0, & -l, & m \\ -m, & l, & 0, & n \\ -l, & -m, & -n, & 0 \end{vmatrix},$$

is such that $\varpi^2 = -\omega$; this is the same as $P = e^{\frac{1}{2}\varpi\theta}$.

On the generation of sets of four tetrahedra of which any two are mutually inscribed. By C. V. Hanumanta and Professor H. F. Baker.

[*Read* 8 March 1920.]

If from a point P the transversal be drawn to two given skew lines, the point P' of this transversal harmonically separated from P by these lines may be said to be obtained from P by harmonic inversion in regard to them. If O and ϖ be a given point and a given plane, and on the line joining O to an arbitrary point P there be taken the point P' harmonically separated from P by the point O and the plane ϖ, we may speak of P' as obtained from P by harmonic inversion in regard to O and ϖ; in particular when ϖ is the polar plane of O in regard to a given quadric it may be sufficient to speak of P' as the inverse of P in regard to O, this use of the term including the ordinary use when inversion in regard to a circle is spoken of.

When two tetrahedra $ABCD$, $A_1B_1C_1D_1$ are such that the points A, B, C, D, A_1, B_1, C_1, D_1 lie respectively on the planes $B_1C_1D_1$, $C_1A_1D_1$, $A_1B_1D_1$, $A_1B_1C_1$, BCD, CAD, ABD, ABC, they will be said to be mutually inscribed. When this is so the two transversals of the four lines AA_1, BB_1, CC_1, DD_1 are generators of a certain quadric in regard to which both the tetrahedra are self-polar (the two transversals of the four lines BC, AD, B_1C_1, A_1D_1 being generators of this quadric of the other system). If another tetrahedron $A_2B_2C_2D_2$, self-polar in regard to the same quadric, be in- and circumscribed to $ABCD$, and such that the two transversals of AA_2, BB_2, CC_2, DD_2 belong to the same system of generators of this quadric as do the transversals of AA_1, ..., DD_1, then $ABCD$, $A_2B_2C_2D_2$ may be said to be mutually inscribed *in the same sense* as are $ABCD$, $A_1B_1C_1D_1$. It is well known that there exist systems of four tetrahedra of which every two are mutually inscribed in the same sense.

The object of the present note is to point out that *any such four tetrahedra may be regarded as all derived from a single other tetrahedron, by inversion of this in the vertices of a certain further tetrahedron, taken in turn*, each of these vertices being associated with the opposite face of this tetrahedron in this process of inversion.

More precisely we may state this result thus. Let a tetrahedron $XYZT$ be self-polar in regard to a certain quadric. Denote the two systems of generators of this quadric as the p-system and the q-system. There exist linear transformations of space changing any

point of the quadric into another point lying on the same q-generator; such a transformation will interchange the p-generators among themselves, save for two p-generators, each of which will be unchanged. We may call such a transformation a p-transformation. Let $ABCD$ be the tetrahedron arising from $XYZT$ by any such transformation; let $PQRS$ be the tetrahedron arising from $ABCD$ by inversion from the point T. Thus $PQRS$ may also be described as arising from $XYZT$ by the q-transformation which effects the same transformation, of the points of the section of the quadric by the plane XYZ, as does the transformation by which $ABCD$ is derived from $XYZT$; in other words if the parameters for the p, q-generators be chosen so as to be the same for the two generators which intersect at a point of the section by the plane XYZ, then $PQRS$ is derived from $XYZT$ by the q-transformation expressed by the same equation, connecting the parameters of the generators, as is the p-transformation by which $ABCD$ is derived from $XYZT$.

Then take the inverses of $PQRS$ respectively from X, Y, Z, T, and let them be $D_1C_1B_1A_1$, $C_2D_2A_2B_2$, $B_3A_3D_3C_3$, $ABCD$. The tetrahedra $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, $ABCD$ are then such that every two are mutually inscribed. And conversely given any such four tetrahedra, the transformation by which $XYZT$, $PQRS$ are determined is definite. The tetrahedron $XYZT$ is unique, being defined by the fact that the edges YZ, XT are the diagonals of the skew quadrilateral formed by (1) the two q-generators meeting the edges BC, AD, (2) the two p-generators which are the common transversals of AA_1, BB_1, CC_1, DD_1, while the other two pairs of edges are similarly obtainable. The tetrahedron $PQRS$ has however been obtained from $XYZT$ and $ABCD$ in an unsymmetrical way, and there are three other possibilities. Let $P_1Q_1R_1S_1$ be the tetrahedron obtained from $ABCD$ by inversion from X. Then $D_1C_1B_1A_1$, $C_2D_2A_2B_2$, $B_3A_3D_3C_3$, $ABCD$ are obtainable from $P_1Q_1R_1S_1$ by inversion respectively from $TZYX$. So if $P_2Q_2R_2S_2$ be obtained from $ABCD$ by inversion from Y, the same four tetrads of points are obtained by inversion of $P_2Q_2R_2S_2$ respectively from $ZTXY$; and if $P_3Q_3R_3S_3$ be obtained from $ABCD$ by inversion from Z, the same four tetrads are obtained by inversion of $P_3Q_3R_3S_3$ respectively from $YXTZ$. But $P_1Q_1R_1S_1$, $P_2Q_2R_2S_2$, $P_3Q_3R_3S_3$ are obtainable from $PQRS$ by harmonic inversion in two lines in each case, respectively YZ, TX; ZX, TY and XY, TZ.

Taking then the first case, where $PQRS$ is used, since inversions in the vertices T, X, in succession, in either order, are together equivalent to harmonic inversion in the opposite edges YZ, TX, we may also say that *the tetrads* $D_1C_1B_1A_1$, $C_2D_2A_2B_2$, $B_3A_3D_3C_3$ *are obtainable from* $ABCD$ *by harmonic inversion respectively in the pairs of edges* YZ, TX; ZX, TY; XY, TZ. And we have remarked

above that given the four tetrahedra $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, $ABCD$, there is a simple construction for $XYZT$.

The sets $PQRS$, $S_1R_1Q_1P_1$, $R_2S_2P_2Q_2$, $Q_3P_3S_3R_3$ are related to one another precisely as are $ABCD$, $D_1C_1B_1A_1$, $C_2D_2A_2B_2$, $B_3A_3D_3C_3$, and, with the substitution of one set of generators for the other, have the same relation to $XYZT$.

These results were obtained geometrically, and appear to constitute a simple geometrical construction for such a set of four mutually inscribed tetrahedra. But they can be readily verified analytically from the formulae by which the configuration is most usually treated.

On the reduction of homography to movement in three dimensions. By Professor H. F. Baker.

[*Read* 9 February 1920.]

It is known, and was assumed by Poncelet, that a self-homography of a Euclidian metrical plane is reducible, by a "movement" of one of the figures, to a perspectivity, called by Poncelet a homology, that is a transformation in which the join of two corresponding points passes through a fixed point, while corresponding lines meet on a fixed line. It is also known that this is not true for Euclidian metrical space of three dimensions. See H. J. S. Smith, *Proc. Lond. Math. Soc.*, II, 1866–1869, pp. 196–248; Chasles, *Géom. Sup.*, 1880, pp. 375–381; Salmon, *Higher Plane Curves*, 1879, p. 298.

The question arises what is the nearest analogous proposition for three dimensions. In the projective plane the reduction can also be made (instead of by a movement), by means of a transformation keeping a given arbitrary conic unaltered. In the present note we prove that in projective space of three dimensions a general homography is reducible, by means of a transformation leaving an arbitrary quadric unaltered, applied to one of the figures, to a transformation which we may call an axis-range perspectivity: namely one in which every point of a certain range is unaltered and every plane passing through a certain axis is unaltered. This seems to add something to what is known, and to be a very natural generalisation of the results in a plane.

§ 1. Let S, S_0 be two quadrics with a common self-polar tetrahedron; let the two systems of generators of S_0 be called the axes and transversals, respectively. There are four axes of S_0 which touch the curve of intersection of S and S_0, and are generators of the developable surface of common tangent planes of S and S_0; there are also four transversals of S_0 with the same property. The eight points of contact of these lines with the curve (S, S_0) are on the curve upon S where it is touched by the common tangent planes of S and S_0; they are then the common points of S, S_0, S_0', where S_0' is the polar reciprocal of S_0 in regard to S.

§ 2. We now consider a certain porismatic relation between the two quadrics. From an axis of S_0, which we distinguish by the parameter θ, two tangent planes can be drawn to S, each of which will contain a transversal of S_0, say these are distinguished by the

parameters p, q. There will thus be a rational equation connecting θ and p, of the second order in p, equally satisfied by θ and q. Through the transversal p can be drawn, beside the plane (θ, p), another tangent plane to S, which, as it contains p, will contain an axis of S_0, say ϕ_1; thus the previous equation connecting (θ, p) is of the second order in θ, and is equally satisfied by ϕ_1 and p. Similarly through q can be drawn, beside (θ, q), another tangent plane to S, which, containing q, will contain an axis of S_0, say ϕ_2, and the relation connecting (θ, q) is equally satisfied by ϕ_2 and q. There is thus a single relation of the second order in each of θ, p which is equally satisfied by (θ, q), (ϕ_1, p), (ϕ_2, q). As in other cases it may happen that $\phi_1 = \phi_2$ for all values of θ, provided a certain relation hold between the quadrics.

To find a sufficient condition for this, we may consider the particular case when the axis θ coincides with the identical axis $\phi_1 = \phi_2$, or the transversal p touches S; then, this line p, being a generator of S_0, is a chord of the curve (S, S_0), and therefore, in the particular case considered, is a tangent line of the curve. Thus a sufficient condition is that the tangent planes of S at two of the four points referred to above, on the curve (S, S_0), associated with the transversals of S_0, should intersect on an axis of S_0.

More generally, if S_0 be $x^2 + y^2 + z^2 + t^2 = 0$, its generators of the two systems being of the respective forms

$$\left.\begin{aligned} x + iy &= i\theta \quad (z + it) \\ x - iy &= i\theta^{-1}\,(z - it) \end{aligned}\right\}, \qquad \left.\begin{aligned} x + iy &= ip \quad (z - it) \\ x - iy &= ip^{-1}\,(z + it) \end{aligned}\right\},$$

the plane containing these,

$$x + iy + \theta p\,(x - iy) - \theta i\,(z + it) - pi\,(z - it) = 0,$$

touches S, say $ax^2 + by^2 + cz^2 + dt^2 = 0$, provided

$$\theta^2 p^2\,(a^{-1} - b^{-1}) + \theta^2\,(d^{-1} - c^{-1}) + p^2\,(d^{-1} - c^{-1}) + 2\theta p\,(a^{-1} + b^{-1} - c^{-1} - d^{-1}) + a^{-1} - b^{-1} = 0.$$

Hence, for the porismatic relation in question, by considering this equation, we see that we require that the discriminantal equation $|\, S_0 - \lambda S \,| = 0$, which with our coordinates is

$$(1 - \lambda a)\,(1 - \lambda b)\,(1 - \lambda c)\,(1 - \lambda d) = 0,$$

should have the sum of two of its roots equal to the sum of the other two. Writing then as usual (Salmon, *Solid Geometry*, 1882, p. 173)

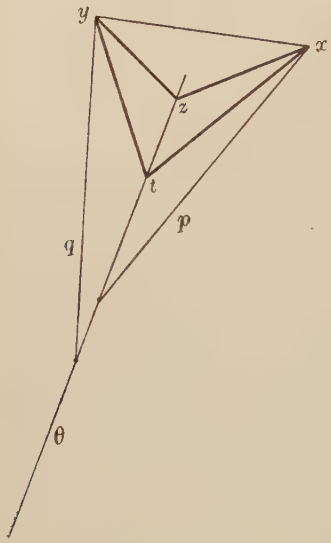
$$|\, S_0 - \lambda S \,| = \Delta_0 - \lambda\Theta_0 + \lambda^2\Phi - \lambda^3\Theta + \lambda^4\Delta,$$

the condition is

$$\Theta_0^3 - 4\Theta_0\Delta_0\Phi + 8\Delta_0^2\Theta = 0.$$

(Cf. Purser, *Quart. J. of Math.*, VIII, 149, and Salmon, *loc. cit.*, p. 181.)

This is then the condition for the existence of tetrahedra with faces touching S of which two pairs of opposite edges lie on S_0, or equally, for the existence of tetrahedra with vertices on S_0, of which two pairs of opposite edges lie on S. Or say, it is the condition for S to be tetrahedrally inscribed in S_0, or for S_0 to be tetrahedrally circumscribed to S.



Supposing the quadric S to be in the specified relation to S_0, there will, corresponding to each of the three ways of pairing the roots of the equation $|S_0 - \lambda S| = 0$, be two figures such as that here indicated. Here p, q are two transversals of S_0, touching S, say at the points x, y; θ is an axis of S_0 associated with p, q; and the lines xz, xt and yz, yt are the two generators of S at the points x, y. The tetrahedron of which the vertices are the points (θ, p), (θ, q), each taken doubly, of which the faces are the planes (θ, p), (θ, q), each taken doubly, of which the edges are the lines p, q and the line θ taken four times over, has its edges p, q, θ, θ as generators of S_0 while its faces touch S; the tetrahedron x, y, z, t has its vertices on S_0, and two pairs of opposite edges of it are generators of S.

Referred to (x, y, z, t) the quadric S may be taken to be

$$xy - zt = 0,$$

and the quadric S_0 to be

$$2hxy + x(gz + ut) + y(fz + vt) = 0.$$

Then the equation $|S_0 - \lambda S| = 0$ has the roots

$$\lambda_1, \lambda_2 = h \pm \{h^2 - [(fu)^{\frac{1}{2}} + (gv)^{\frac{1}{2}}]^2\}^{\frac{1}{2}},$$
$$\lambda_3, \lambda_4 = h \pm \{h^2 - [(fu)^{\frac{1}{2}} - (gv)^{\frac{1}{2}}]^2\}^{\frac{1}{2}}$$

for which $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$. The axes of S_0 are given by

$$(h - \theta)x + fz + vt = 0, \quad (h + \theta)y + gz + ut = 0,$$

and the transversals by

$$x = py, \quad hx + hpy + (f + gp)z + (v + up)t = 0;$$

the plane

$$(h - \theta)x + fz + vt + p[(h + \theta)y + gz + ut] = 0,$$

containing one of each, touches the quadric S if

$$p\theta^2 + p^2gu + p\,(fu + gv - h^2) + fv = 0;$$

this is the form of the (2, 2) relation in this case, the two axes θ, ϕ of S_0 being θ, $-\theta$. For the axis which is the line zt we have $\theta = \infty$ and the two corresponding values of p, q are $p = 0$, $q = \infty$. It is then material to remark that the transformation

$$x_1 = \frac{h - \theta_0}{h + \theta_0}x, \quad y_1 = \frac{h + \theta_0}{h - \theta_0}y, \quad z_1 = z, \quad t_1 = t$$

changes the particular axis of S_0 determined by θ_0 into the axis determined by $-\theta_0$, and changes the plane (θ_0, p), or

$$(h - \theta_0)\,x + fz + vt + p\,[(h + \theta_0)\,y + gz + ut] = 0,$$

into the plane $(-\theta_0, p)$, for all values of p. This transformation is one which leaves the quadric S, or $xy - zt = 0$, unaltered, and leaves every point of the line zt, which is a generator of S_0, unaltered, as well as every plane through the line xy. It is thus a transformation of the kind which we have called an axis-range perspective, or, in a usual phraseology, in regard to S as the absolute quadric, it is what is called a "rotation" round the line zt.

Thus we may say: *If a quadric S be tetrahedrally inscribed in a quadric S_0, it is possible to find a self-transformation of S, which is a rotation about a generator of S_0, such as to change any axis of S_0 into its associated axis, and to change any plane through this into the plane through the associated axis containing the same transversal of S_0 as the former.*

It is easy to write down a corresponding result for the plane.

§ 3. Now consider any homography in space, say

$$x' = a_1x + b_1y + c_1z + d_1t, \ \ldots, \ t' = a_4x + b_4y + c_4z + d_4t,$$

which we denote, in matrix notation, by

$$(x', y', z', t') = \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} (x, y, z, t),$$

or also by $(x') = \Theta\,(x)$. Take an arbitrary quadric S; this is changed by the transformation into another quadric, and is itself obtainable from another by the transformation; regarded in these two aspects let it be denoted respectively by σ and ρ', the quadric into which it is changed by the transformation being σ', and that from which it may be supposed to arise being ρ. Denote the developable surface formed by the common tangent planes of ρ and σ by (ρ, σ), and that formed by the common tangent planes of ρ' and σ'

by (ρ', σ'); the former is changed to the latter by the transformation $\mathfrak{H}$. Let l be *the line of intersection of any two planes of the developable* (ρ, σ), and l' the line arising therefrom by the transformation, the intersection of the two planes of (ρ', σ') which correspond to the planes of (ρ, σ) taken. To the pencil of all possible planes through l will correspond a homographic pencil of corresponding planes through l'; and the locus of the line of intersection of corresponding planes of these two axial pencils, will be another quadric, say S_0. Evidently, by the construction, S is tetrahedrally inscribed in S_0, in the sense explained in § 2.

Taking then such a tetrahedron of reference as used in § 2, let the lines l, l' be respectively those there associated with the parameters θ_0 and $-\theta_0$; then the homography $\mathfrak{H}$ changes the plane (θ_0, p), of equation

$$(h - \theta_0)\, x + fz + vt + p\,[(h + \theta_0)\, y + gz + ut] = 0,$$

into the plane $(-\theta_0, p)$, whose equation is obtainable by change of the sign of θ_0, *for all values of* p. When p satisfies the equation

$$p\theta_0{}^2 + p^2 gu + p\,(fu + gv - h^2) + fv = 0,$$

the planes (θ_0, p), $(-\theta_0, p)$ both touch S, which is σ and ρ'. Thus the plane (θ_0, p) touches the quadric ρ, and $(-\theta_0, p)$ touches σ'.

Now with the above form for the homography $\mathfrak{H}$, the above equation for the plane (θ_0, p), which the transformation changes into $(-\theta_0, p)$, is, for all values of p, the same as

$$\begin{aligned}&(h + \theta_0)\,(a_1x + b_1y + c_1z + d_1t) + f\,(a_3x + b_3y + c_3z + d_3t)\\ &\quad + v\,(a_4x + b_4y + c_4z + d_4t) + p\,[(h - \theta_0)\,(a_2x + b_2y + c_2z + d_2t)\\ &\quad + g\,(a_3x + b_3y + c_3z + d_3t) + u\,(a_4x + b_4y + c_4z + d_4t)] = 0;\end{aligned}$$

the coefficients a_i, b_j, ... in $\mathfrak{H}$ must therefore be such that

$$\begin{aligned}&\frac{(h+\theta_0)\,a_1 + fa_3 + va_4}{h - \theta_0} = \frac{(h+\theta_0)\,b_1 + fb_3 + vb_4}{0} = \frac{(h+\theta_0)\,c_1 + fc_3 + vc_4}{f}\\ &= \frac{(h+\theta_0)\,d_1 + fd_3 + vd_4}{v} = \frac{(h-\theta_0)\,a_2 + ga_3 + ua_4}{0} = \frac{(h-\theta_0)\,b_2 + gb_3 + ub_4}{h + \theta_0}\\ &= \frac{(h-\theta_0)\,c_2 + gc_3 + uc_4}{g} = \frac{(h-\theta_0)\,d_2 + gd_3 + ud_4}{u},\end{aligned}$$

and there will be no loss of generality in supposing each of these fractions unity. With the ordinary notation of matrices these equations are then the same as

$$\begin{vmatrix} h+\theta_0, & 0, & f, & v \\ 0, & h-\theta_0, & g, & u \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = \begin{vmatrix} h-\theta_0, & 0, & f, & v \\ 0, & h+\theta_0, & g, & u \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix},$$

and enable us to infer that the matrix of the homography has the form

$$\mathfrak{H} = \begin{vmatrix} h+\theta_0, & 0, & f, & v \\ 0, & h-\theta_0, & g, & u \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix}^{-1} \begin{vmatrix} h-\theta_0, & 0, & f, & v \\ 0, & h+\theta_0, & g, & u \\ a, & b, & c, & d \\ a', & b', & c', & d' \end{vmatrix},$$

where $a, b, c, d, a', b', c', d'$ are written respectively for

$$a_3, b_3, c_3, d_3, a_4, b_4, c_4, d_4.$$

If then we put

$$m = \frac{h-\theta_0}{h+\theta_0}, \qquad \mu = \begin{vmatrix} m, & 0, & 0, & 0 \\ 0, & m^{-1}, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix},$$

we have

$$\mathfrak{H}\mu^{-1} = \begin{vmatrix} h+\theta_0, & 0, & f, & v \\ 0, & h-\theta_0, & g, & u \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix}^{-1} \begin{vmatrix} h+\theta_0, & 0, & f, & v \\ 0, & h-\theta_0, & g, & u \\ am^{-1}, & bm, & c, & d \\ a'm^{-1}, & b'm, & c', & d' \end{vmatrix}.$$

If the transformation associated with the matrix μ change the point (x, y, z, t) to (x_1, y_1, z_1, t_1), so that $(x_1) = \mu(x)$, we have from $(x') = \mathfrak{H}(x)$, for the change from (x_1) to (x'), the equations expressed by $(x') = \mathfrak{H}\mu^{-1}(x_1)$, which, by what we have just seen, are the same as

$$\begin{aligned} (h+\theta_0)\,x' + fz' + vt' &= (h+\theta_0)\,x_1 + fz_1 + vt_1, \\ (h-\theta_0)\,y' + gz' + ut' &= (h-\theta_0)\,y_1 + gz_1 + ut_1, \\ z' &= am^{-1}x_1 + bmy_1 + cz_1 + dt_1, \\ t' &= a'm^{-1}x_1 + b'my_1 + c'z_1 + d't_1. \end{aligned}$$

Without entering into general statements about the minors of a matrix, we proceed now to show in detail in an elementary way that these last are the equations of what we have called an axis-range perspective. For this purpose, write

$$\begin{aligned} &F = \frac{f}{h+\theta_0}, \quad V = \frac{v}{h+\theta_0}, \quad G = \frac{g}{h-\theta_0}, \quad U = \frac{u}{h-\theta_0}, \\ &C = c - am^{-1}F - bmG, \quad D = d - am^{-1}V - bmU, \\ &C' = c' - a'm^{-1}F - b'mG, \quad D' = d' - a'm^{-1}V - b'mU, \end{aligned}$$

and also

$$\xi' = x' + Fz' + Vt', \quad \xi_1 = x_1 + Fz_1 + Vt_1,$$
$$\eta' = y' + Gz' + Ut', \quad \eta_1 = y_1 + Gz_1 + Ut_1;$$

then the equations of the transformation $\Im\mu^{-1}$ take the forms

$$\xi' = \xi_1,\ \eta' = \eta_1,\ z' = am^{-1}\xi_1 + bm\eta_1 + Cz_1 + Dt_1,$$
$$t' = a'm^{-1}\xi_1 + b'm\eta_1 + C'z_1 + D't_1;$$

assuming that the roots α, β of the equation in α,

$$(C - \alpha)(D' - \alpha) - C'D = 0,$$

are different from one another, and from unity, we can take p, p' and q, q' so that

$$\frac{pC + p'C'}{p} = \frac{pD + p'D'}{p'} = \alpha, \quad \frac{qC + q'C'}{q} = \frac{qD + q'D'}{q'} = \beta,$$

and thence, by

$$Z' = pz' + p't', \quad Z_1 = pz_1 + p't_1,$$
$$T' = qz' + q't', \quad T_1 = qz_1 + q't_1,$$

the last two equations of $\Im\mu^{-1}$ are replaced by

$$Z' = A\xi_1 + B\eta_1 + \alpha Z_1, \quad T' = A'\xi_1 + B'\eta_1 + \beta T_1,$$

where

$$A = (pa + p'a')\, m^{-1}, \quad B = (pb + p'b')\, m,$$
$$A' = (qa + q'a')\, m^{-1}, \quad B' = (qb + q'b')\, m,$$

so that, finally, if we put

$$\zeta' = Z' - \frac{A}{1-\alpha}\xi' - \frac{B}{1-\alpha}\eta', \quad \zeta_1 = Z_1 - \frac{A}{1-\alpha}\xi_1 - \frac{B}{1-\alpha}\eta_1,$$

$$\tau' = T' - \frac{A'}{1-\beta}\xi' - \frac{B'}{1-\beta}\eta', \quad \tau_1 = T_1 - \frac{A'}{1-\beta}\xi_1 - \frac{B'}{1-\beta}\eta_1,$$

the equations of $\Im\mu^{-1}$ are

$$\xi' = \xi_1,\ \eta' = \eta_1,\ \zeta' = \alpha\zeta_1,\ \tau' = \beta\tau_1$$

which are the equations of an axis-range perspective in which every point of the line $\zeta = 0$, $\tau = 0$ is unchanged, and every plane through the line $\xi = 0$, $\eta = 0$ is unchanged. The latter line is the axis given in the notation first used by the parameter $-\theta_0$, belonging to the quadric S_0; the former line, we easily see, is given by

$$am^{-1}x' + bmy' + (c - 1)z' + dt' = 0,$$
$$a'm^{-1}x' + b'my' + c'z' + (d' - 1)t' = 0.$$

The transformation $(x_1) = \mu(x)$, equivalent to

$$x_1 = mx,\ y_1 = m^{-1}y,\ z_1 = z,\ t_1 = t,$$

is evidently one which leaves unaltered the fundamental quadric S whose equation was taken to be $xy - zt = 0$.

The result then is: if we apply to the figure for which the coordinates (x) have been used the transformation μ, by $(x_1) = \mu(x)$, this being a transformation leaving the arbitrary quadric S unaltered, the relation between the figure (x_1), and the figure (x'), which arose from (x) by $(x') = \vartheta(x)$, which is expressed by

$$(x') = \vartheta\mu^{-1}(x_1),$$

is that the figure (x') arises from the figure (x_1) by an axis-range perspectivity. The axis of this is an arbitrary "line in two planes" of a certain developable determined by the arbitrary quadric and the original transformation ϑ.

On the transformation of the equations of electrodynamics in the Maxwell and in the Einstein forms. By Professor H. F. Baker.

[*Read* 9 February 1920.]

The present note, which arose from reading the paper of Mr W. J. Johnston in the *Proceedings of the Royal Society*, A, xcvi, 1919, 331, and the paper of M. Th. de Donder, *Archives du Musée Teyler* (Haarlem), iii, 1917, 80–179, has a very humble purpose. (1) It is shown that the noncommutative "imaginaries" used by Mr Johnston may be interpreted as aggregates of ordinary numbers, much in the same way as the complex numbers of ordinary analysis may be interpreted as aggregates of real numbers. It is shown indeed how to interpret a system of any number of units obeying the same laws of combination. (2) By this interpretation a form of Maxwell's equations of electrodynamics is reached from which the ordinary Lorenz transformation is obvious at sight. This form of the equations is equivalent to a solution of the electrodynamic equations in terms of arbitrary functions. (3) The equations given by M. Th. de Donder, and stated by him to be the equations of electrodynamics in the Einstein field, are then considered, and their invariance under a general transformation is established. The purpose of this section is to show how simply this invariance follows by the use of notation which is familiar in other applications. The result of course includes the case of Maxwell's equations.

§ 1. It is a familiar fact that the so-called complex numbers of ordinary analysis are couplets of two *real* numbers (x, y), subject to the laws

(i) if m be a real number, $m(x, y) = (mx, my)$,

(ii) $(x, y) + (x', y') = (x + x', y + y')$,

(iii) $(x, y)(x', y') = (xx' - yy', xy' + x'y)$,

those of the numbers which we ordinarily call real being couplets $(x, 0)$, and those which we call pure imaginaries being couplets $(0, y)$. By means of these rules any number (x, y) is expressible in the form $x[1] + y[i]$, where $[1]$, $[i]$ stand for $(1, 0)$ and $(0, 1)$; and $[i][i] = -[1]$. It is this last equation which we ordinarily write $i^2 = -1$.

We may similarly have systems of more than two numbers, subject to laws of computation, addition (and subtraction), multiplication (and possibly division). In particular consider systems,

built with the numbers of ordinary analysis (including complex numbers), each system consisting of n^2 numbers ($n = 2, 3, \ldots$). These we may conveniently arrange in the form of a square; and for purposes of explanation it will be sufficient to write down only two rows and columns. Suppose then

$$(a) = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix},$$

where a_{11}, a_{12}, ... are numbers of ordinary analysis, and introduce the rules (i) when m is a number of ordinary analysis $m\,(a)$ shall mean the same as (a) with each element a_{rs} replaced by ma_{rs}. (ii) If (a), (b) be squares of the same number of rows and columns $(a) + (b)$ shall mean the square of which the general element is $a_{rs} + b_{rs}$; thus $(a) + (b)$ is the same as $(b) + (a)$. (iii) $(a)\,(b)$ shall mean the square whose general element c_{rs} is formed by combining the rth row of (a) with the sth column of b, in such a way that

$$c_{rs} = a_{r1}b_{1s} + a_{r2}b_{2s};$$

thus $(a)\,(b)$ is not generally equal to $(b)\,(a)$. (iv) The zero square shall mean that in which every element is the number 0; denoting this zero square by 0, we shall then have $(a) + 0 = (a)$, and $(a)\,0 = 0\,(a) = 0$ whatever (a) may be. (v) The unit square shall mean that in which all the elements are zero except those, a_{rr}, in the principal diagonal, each of these being unity. Denoting this by E we shall then have $E\,(a) = (a)\,E = (a)$, whatever (a) may be. Addition of such squares is then not only associative, but is also commutative. It is easy to prove that multiplication is associative also; but in general it is not commutative.

Of such squares, of two rows and columns, that given by

$$\begin{pmatrix} x, & -y \\ y, & x \end{pmatrix}$$

is easily seen to obey all the laws of the complex number (x, y), the multiplication being commutative. This square may then equally be used to represent the numbers of ordinary analysis. Putting

$$(1) = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, \quad (i) = \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}$$

this square is $x\,(1) + y\,(i)$.

Now denote the two units of ordinary analysis by 1, ϵ, where, for convenience of notation, ϵ is used instead of the ordinary i, so that $\epsilon^2 = -1$. And consider the squares of two rows and columns expressed by

$$E = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, I = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, J = \begin{pmatrix} \epsilon, & 0 \\ 0, & -\epsilon \end{pmatrix}, K = \begin{pmatrix} 0, & -\epsilon \\ -\epsilon, & 0 \end{pmatrix}.$$

By the laws above given we shall then find

$$E^2 = E, \quad EI = IE = I, \text{ etc.}$$

$$I^2 = J^2 = K^2 = -E, \quad JK = -KJ = I,$$

$$KI = -IK = J, \qquad IJ = -JI = K,$$

so that I, J, K, E behave precisely as the units of the theory of quaternions. Other binary squares can be found with the same results of composition; for instance the work below suggests that $I' = -I$, $J' = -K$, $K' = -J$ might have been taken. Adopting however the above, the usual quaternion $xI + yJ + zK + tE$ is represented by

$$\begin{pmatrix} 0, & x \\ -x, & 0 \end{pmatrix} + \begin{pmatrix} \epsilon y, & 0 \\ 0, & -\epsilon y \end{pmatrix} + \begin{pmatrix} 0, & -\epsilon z \\ -\epsilon z, & 0 \end{pmatrix} + \begin{pmatrix} t, & 0 \\ 0, & t \end{pmatrix},$$

or

$$\begin{pmatrix} t + \epsilon y, & x - \epsilon z \\ -x - \epsilon z, & t - \epsilon y \end{pmatrix},$$

and the formula of multiplication of two quaternions may be obtained by multiplying two such squares according to the rule given above.

Now consider squares of four rows and columns. But for brevity, instead of writing four rows and columns, introduce symbols to stand as above each for a square of two rows and columns. Namely, using the letter O for the square of two rows and columns of which each element is zero, let

$$e = \begin{pmatrix} O, & -E \\ E, & O \end{pmatrix}, \; i = \begin{pmatrix} O, & I \\ I, & O \end{pmatrix}, \; j = \begin{pmatrix} O, & J \\ J, & O \end{pmatrix}, \; k = \begin{pmatrix} O, & K \\ K, & O \end{pmatrix}, \; u = \begin{pmatrix} E, & O \\ O, & E \end{pmatrix},$$

where E, I, J, K are as above, E being the unit square of two rows and columns, and u the unit square of four rows and columns.

Then by the rules above given it is easy to compute that

$$e^2 = i^2 = j^2 = k^2 = -u,$$

$$ei = -ie = \begin{pmatrix} -I, & O \\ O, & I \end{pmatrix}, \; ej = -je = \begin{pmatrix} -J, & O \\ O, & J \end{pmatrix}, \; ek = -ke = \begin{pmatrix} -K, & O \\ O, & K \end{pmatrix},$$

$$jk = -kj = \begin{pmatrix} I, & O \\ O, & I \end{pmatrix}, \quad ki = -ik = \begin{pmatrix} J, & O \\ O, & J \end{pmatrix}, \; ij = -ji = \begin{pmatrix} K, & O \\ O, & K \end{pmatrix},$$

together with

$$ijk = \begin{pmatrix} O, & -E \\ -E, & O \end{pmatrix}, \; ejk = \begin{pmatrix} O, & -I \\ I, & O \end{pmatrix}, \; eki = \begin{pmatrix} O, & -J \\ J, & O \end{pmatrix}, \; eij = \begin{pmatrix} O, & -K \\ K, & O \end{pmatrix}$$

and

$$eijk = \begin{pmatrix} E, & O \\ O, & -E \end{pmatrix};$$

and, the multiplication being associative, it is not necessary to remark such equations as

$$ijk = -jik = jki = -ikj, \text{ etc.}$$

The squares e, i, j, k thus obey the laws of combination of the symbols occurring in Mr Johnston's paper above referred to, denoted by him by o, i, j, k.

If we consider a square of which every element is ϵ times the corresponding element in $eijk$, and denote this by m, we find at once

$$m^2 = -u, \quad em = -me, \quad im = -mi, \quad jm = -mj, \quad km = -mk,$$

and e, i, j, k, m are a system of five squares obeying the same laws of combination as the three quaternion squares I, J, K. We have $eijkm = -\epsilon u$, and save for multiplication with -1 or ϵ, or both, there are sixteen squares arising by multiplications of u, e, i, j, k, m, namely these six and those obtainable by the ten products of two of e, i, j, k, m.

As was remarked to me by Dr W. Burnside*, F.R.S., the above work is capable of generalisation.

If $e_1, e_2, \ldots, e_{2r-1}$ be squares each of m rows and columns obeying the laws $e_i^2 = -u$, $e_i e_j = -e_j e_i$, where u is the unit square of m rows and columns, then the $2r+1$ squares of each $2m$ rows and columns given by

$$E = \begin{pmatrix} 0, & -u \\ u, & 0 \end{pmatrix}, \quad E_i = \begin{pmatrix} 0, & e_i \\ e_i, & 0 \end{pmatrix}, \ (i = 1, 2, \ldots, (2r-1)), \quad F = \mu E E_1 \ldots E_{2r-1},$$

where $\mu, = (-1)^{\frac{1}{4}[1+(-1)^r]}$, is 1 or ϵ according as r is odd or even, obey exactly the same laws—namely if U be the unit square of $2m$ rows and columns

$$E^2 = E_i^2 = -U, \quad EE_i = -E_i E, \quad E_i E_j = -E_j E_i.$$

For instance for $r = 0$, $m = 1$, this gives

$$\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, \quad \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}$$

as two squares of which the square of the latter is the negative of the former. These are the two units (usually denoted by 1, i) of ordinary analysis. Or if, denoting these, as we have done, by 1, ϵ, we take $r = 1$, $m = 1$, the theorem gives the binary squares

$$\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}, \quad \begin{pmatrix} 0, & \epsilon \\ \epsilon, & 0 \end{pmatrix}, \quad \begin{pmatrix} -\epsilon, & 0 \\ 0, & \epsilon \end{pmatrix}, \quad \text{with } \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},$$

say, I, J, K, U, as units satisfying

$$I^2 = J^2 = K^2 = -U, \quad JK = -KJ = I,$$
$$KI = -IK = J, \quad IJ = -JI = K,$$

which are the laws of Hamilton's quaternions.

* Dr Burnside writes: "In group notation it really comes to this:—The operations $S_1, S_2, \ldots, S_n$, such that

$$S_1^2 = S_2^2 = \ldots = S_n^2 = T, \quad T^2 = E, \quad S_i S_j = S_j S_i T$$

generate a group G_n of order 2^{2n+1}. In this group E, T are the only invariant operations if n is even; when n is odd, the factor group $G_n \mid \{S_1 S_2 \ldots S_n\}$ is identical in type with G_{n-1}."

§ 2. With the symbols e, i, j, k above explained, satisfying the equations

$$e^2 = i^2 = j^2 = k^2 = -u,\ ei = -ie,\ \ldots,\ jk = -kj,\ \ldots,\ eu = ue = e,\ \ldots;$$

we clearly have

$$\begin{aligned}(e\tau + i\xi + j\eta + k\zeta)\,(e\phi + iF + jG + kH)\\ = -u\,(\tau\phi + \xi F + \eta G + \zeta H)\\ + jk\,(\eta H - \zeta G) + ki\,(\zeta F - \xi H) + ij\,(\xi G - \eta F)\\ + ei\,(\tau F - \xi\phi) + ej\,(\tau G - \eta\phi) + ek\,(\tau H - \zeta\phi),\end{aligned}$$

and if we denote this by

$$uW + jkL + kiM + ijN + eiX + ejY + ekZ,$$

we further have the identity (A)

$$\begin{aligned}(e\tau + i\xi + j\eta + k\zeta)\,(uW + jkL + kiM + ijN + eiX + ejY - ekZ)\\ = e\,(\tau W + \xi X + \eta Y + \zeta Z) + i\,(\xi W + \eta N - \zeta M - \tau X)\\ + j\,(\eta W - \xi N + \zeta L - \tau Y) + k\,(\zeta W + \xi M - \eta L - \tau Z)\\ + e_1\,(\xi L + \eta M + \zeta N) + i_1\,(-\eta Z + \zeta Y + \tau L)\\ + j_1\,(-\zeta X + \xi Z + \tau M) + k_1\,(-\xi Y + \eta X + \tau N),\end{aligned}$$

where $\quad e_1 = ijk,\ \ i_1 = ejk,\ \ j_1 = eki,\ \ k_1 = eij.$

If now we interpret ξ, η, ζ, τ as symbols of differentiation in regard respectively to x, y, z, t, the coefficients of $e, i, \ldots, e_1, \ldots$ arising in these equations are the combinations occurring in the equations of electrodynamics considered by Mr Johnston in the paper referred to, the F, G, H, X, Y, Z being certain constant multiples of those usually denoted by these symbols, except that W, which vanishes in the electrodynamic case, is here retained, and t is a certain imaginary multiple of the time.

However, a symbol $e\tau + i\xi + j\eta + k\zeta$, if we use the interpretation of e, i, j, k above developed, is the same as

$$\tau\begin{pmatrix} O, & -E \\ E, & O \end{pmatrix} + \xi\begin{pmatrix} O, & I \\ I, & O \end{pmatrix} + \eta\begin{pmatrix} O, & J \\ J, & O \end{pmatrix} + \zeta\begin{pmatrix} O, & K \\ K, & O \end{pmatrix},$$

or

$$\begin{pmatrix} O, & -\tau E + \xi I + \eta J + \zeta K \\ \tau E + \xi I + \eta J + \zeta K, & O \end{pmatrix},$$

or, taking E, I, J, K as above, is the same as

$$\begin{vmatrix} 0, & 0, & -\tau + \epsilon\eta, & \xi - \epsilon\zeta \\ 0, & 0, & -\xi - \epsilon\zeta, & -\tau - \epsilon\eta \\ \tau + \epsilon\eta, & \xi - \epsilon\zeta, & 0, & 0 \\ -\xi - \epsilon\zeta, & \tau - \epsilon\eta, & 0, & 0 \end{vmatrix},$$

where 0 is the zero of ordinary analysis. The above operational equations may then be obtained by multiplications of such squares of four rows.

But when ξ, η, ζ, τ stand for

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t},$$

if we take x_1, y_1, z_1, t_1 so that

$$x = x_1 + z_1, \quad \epsilon z = x_1 - z_1, \quad t = t_1 + y_1, \quad \epsilon y = t_1 - y_1,$$

we have

$$\frac{\partial}{\partial x_1} = \xi - \epsilon\zeta, \quad \frac{\partial}{\partial y_1} = \tau + \epsilon\eta, \quad \frac{\partial}{\partial z_1} = \xi + \epsilon\zeta, \quad \frac{\partial}{\partial t_1} = \tau - \epsilon\eta,$$

and, denoting these by $\xi_1, \eta_1, \zeta_1, \tau_1$, the above square is

$$\begin{vmatrix} 0, & 0, & -\tau_1, & \xi_1 \\ 0, & 0, & -\zeta_1, & -\eta_1 \\ \eta_1, & \xi_1, & 0, & 0 \\ -\zeta_1, & \tau_1, & 0, & 0 \end{vmatrix}.$$

Again

$$uW + jkL + kiM + ijN + eiX + ejY + ekZ$$

is

$$W \begin{pmatrix} E, & O \\ O, & E \end{pmatrix} + L \begin{pmatrix} I, & O \\ O, & I \end{pmatrix} + M \begin{pmatrix} J, & O \\ O, & J \end{pmatrix} + N \begin{pmatrix} K, & O \\ O, & K \end{pmatrix}$$
$$+ X \begin{pmatrix} -I, & O \\ O, & I \end{pmatrix} + Y \begin{pmatrix} -J, & O \\ O, & J \end{pmatrix} + Z \begin{pmatrix} -K, & O \\ O, & K \end{pmatrix},$$

or

$$\begin{Bmatrix} EW + (L - X)\, I + (M - Y)\, J + (N - Z)\, K, & O, \\ O, & WE + (L + X)\, I + (M + Y)\, J + (N + Z)\, K \end{Bmatrix},$$

which, if we take E, I, J, K as above, is the same as

$$\begin{vmatrix} W + (M - Y)\,\epsilon, & L - X - (N - Z)\,\epsilon, & 0, & 0 \\ -(L - X) - (N - Z)\,\epsilon, & W - (M - Y)\,\epsilon, & 0, & 0 \\ 0, & 0, & W + (M + Y)\,\epsilon, & L + X - (N + Z)\,\epsilon \\ 0, & 0, & -(L + X) - (N + Z)\,\epsilon, & W - (M + Y)\,\epsilon \end{vmatrix}.$$

Now put

$$\alpha = L + X + \epsilon\,(N + Z), \quad \gamma = L - X + \epsilon\,(N - Z),$$
$$\alpha' = L + X - \epsilon\,(N + Z), \quad \gamma' = L - X - \epsilon\,(N - Z),$$
$$\beta = W + \epsilon\,(M + Y), \quad \delta = W + \epsilon\,(M - Y),$$
$$\beta' = W - \epsilon\,(M + Y), \quad \delta' = W - \epsilon\,(M - Y),$$

so that $$\beta + \beta' = \delta + \delta'.$$

Then the last square is

$$\begin{vmatrix} \delta, & \gamma', & 0, & 0 \\ -\gamma, & \delta', & 0, & 0 \\ 0, & 0, & \beta, & \alpha' \\ 0, & 0, & -\alpha, & \beta' \end{vmatrix},$$

and the equations of electrodynamics are expressed by the vanishing of the product

$$\begin{vmatrix} 0, & 0, & -\tau_1, & \xi_1 \\ 0, & 0, & -\zeta_1, & -\eta_1 \\ \eta_1, & \xi_1, & 0, & 0 \\ -\zeta_1, & \tau_1, & 0, & 0 \end{vmatrix} \begin{vmatrix} \delta, & \gamma', & 0, & 0 \\ -\gamma, & \delta', & 0, & 0 \\ 0, & 0, & \beta, & \alpha' \\ 0, & 0, & -\alpha, & \beta' \end{vmatrix};$$

this is equivalent to the vanishing of the two products

$$\begin{pmatrix} \tau_1, & -\xi_1 \\ \zeta_1, & \eta_1 \end{pmatrix} \begin{pmatrix} \beta, & \alpha' \\ -\alpha, & \beta' \end{pmatrix}, \quad \begin{pmatrix} \eta_1, & \xi_1 \\ -\zeta_1, & \tau_1 \end{pmatrix} \begin{pmatrix} \delta, & \gamma' \\ -\gamma, & \delta' \end{pmatrix},$$

and the equations are therefore

$$\left.\begin{aligned} &\frac{\partial \alpha}{\partial x_1} + \frac{\partial \beta}{\partial t_1} = 0, \quad \frac{\partial \alpha'}{\partial z_1} + \frac{\partial \beta'}{\partial y_1} = 0, \quad \frac{\partial \gamma}{\partial t_1} + \frac{\partial \delta}{\partial z_1} = 0, \quad \frac{\partial \gamma'}{\partial y_1} + \frac{\partial \delta'}{\partial x_1} = 0 \\ &\frac{\partial \alpha}{\partial y_1} - \frac{\partial \beta}{\partial z_1} = 0, \quad \frac{\partial \alpha'}{\partial t_1} - \frac{\partial \beta'}{\partial x_1} = 0, \quad \frac{\partial \gamma}{\partial x_1} - \frac{\partial \delta}{\partial y_1} = 0, \quad \frac{\partial \gamma'}{\partial z_1} - \frac{\partial \delta'}{\partial t_1} = 0 \end{aligned}\right\} \text{(I)}.$$

These results are easily shown directly to be the same as the ordinary forms (putting $W = 0$, $\beta' = -\beta$, $\delta' = -\delta$). They involve that each of the functions $\alpha, \beta, \alpha', \beta', \gamma, \delta, \gamma', \delta'$ satisfies the equation

$$\frac{\partial^2 V}{\partial x_1 \partial z_1} + \frac{\partial^2 V}{\partial y_1 \partial t_1} = 0,$$

or

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial t^2} = 0;$$

and they are equivalent also to

$$\alpha = \int\left(-\frac{\partial \beta}{\partial t_1}\,dx_1 + \frac{\partial \beta}{\partial z_1}\,dy_1\right) + A\,(z_1, t_1),$$

$$\alpha' = \int\left(-\frac{\partial \beta'}{\partial y_1}\,dz_1 + \frac{\partial \beta'}{\partial x_1}\,dt_1\right) + A'\,(x_1, y_1),$$

$$\gamma = \int\left(-\frac{\partial \delta}{\partial z_1}\,dt_1 + \frac{\partial \delta}{\partial y_1}\,dx_1\right) + C\,(y_1, z_1),$$

$$\gamma' = \int\left(-\frac{\partial \delta'}{\partial x_1}\,dy_1 + \frac{\partial \delta'}{\partial t_1}\,dz_1\right) + C'\,(x_1, t_1),$$

where A, C, A', C' are arbitrary functions of their respective arguments. The expressions under the integral signs are perfect differentials in virtue of the differential equation satisfied by $\beta, \delta, \beta', \delta'$.

The equations are also equivalent to

$$U\alpha = V\beta, \qquad R\delta' = S\gamma',$$
$$U\beta' = -V\alpha', \; R\gamma = -S\delta,$$

together with those obtained by putting $-\epsilon$ for ϵ, where

$$U = \frac{\partial}{\partial x_1} + \epsilon \frac{\partial}{\partial y_1}, \quad V = \epsilon\left(\frac{\partial}{\partial z_1} + \epsilon \frac{\partial}{\partial t_1}\right),$$
$$R = \frac{\partial}{\partial x_1} + \epsilon \frac{\partial}{\partial t_1}, \quad S = \epsilon\left(\frac{\partial}{\partial z_1} + \epsilon \frac{\partial}{\partial y_1}\right).$$

What is to be remarked here however is that the equations (I) are evidently unaltered by replacing

$$\alpha, \alpha', \gamma, \gamma', y_1, t_1$$

respectively by $\sigma\alpha, \sigma^{-1}\alpha', \sigma^{-1}\gamma, \sigma\gamma', \sigma y_1, \sigma^{-1}t_1$,

where σ is an arbitrary constant. These are the equations of the Lorenz transformation of the older relativity theory. If, putting $W = 0$, we equate to zero the coefficients of i, j, k, i_1, j_1, k_1, in the identity (A) at the beginning of this article (§ 2), and write

$$\tau = \frac{\sqrt{\mu K}}{ic} \frac{\partial}{\partial t}, \quad L = mP, \quad M = mQ, \quad N = mR, \quad m = -ic\sqrt{\frac{\mu}{K}},$$

where $i = \sqrt{-1}$, we obtain the six familiar equations

$$\frac{K}{c^2} \nabla (X, Y, Z) = \text{curl } (P, Q, R),$$
$$-\mu \nabla (P, Q, R) = \text{curl } (X, Y, Z),$$

and the above results furnish a general *solution* of these, in terms of two arbitrary functions satisfying the above potential equation.

§ 3. We now pass to the equations described by M. Th. de Donder as the equations of electrodynamics in the Einstein field (*loc. cit.*, p. 93). Here instead of one system of two sets each of three quantities (X, Y, Z), (L, M, N), which interchange, we have two systems each of six, connected together in a way which shall be explained, with coefficients which are quadratic functions of the coefficients in a certain Absolute quadric. These two systems, imitating the notation of the author, are denoted respectively by $(M_{23}, M_{31}, M_{12}, M_{14}, M_{24}, M_{34})$ and $(N_{23}, N_{31}, N_{12}, N_{14}, N_{24}, N_{34})$, and symbols M_{32}, M_{41}, etc., are occasionally used where

$$M_{32} = -M_{23}, \quad M_{41} = -M_{14}, \text{ etc.}$$

The differential equations in question, writing ξ, η, ζ, τ for

$$\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}, \frac{\partial}{\partial t'}$$

respectively, are

$$\begin{aligned} -\eta M_{34} &+ \zeta M_{24} - \tau M_{23} = \rho V_x, \\ \xi M_{34} &- \zeta M_{14} - \tau M_{31} = \rho V_y, \\ -\xi M_{24} + \eta M_{14} &- \tau M_{12} = \rho V_z, \\ \xi M_{23} + \eta M_{31} &+ \zeta M_{12} = \rho, \end{aligned}$$

$$\begin{aligned} -\eta N_{34} &+ \zeta N_{24} - \tau N_{23} = 0, \\ \xi N_{34} &- \zeta N_{14} - \tau N_{31} = 0, \\ -\xi N_{24} + \eta N_{14} &- \tau N_{12} = 0, \\ \xi N_{23} + \eta N_{31} &+ \zeta N_{12} = 0. \end{aligned}$$

If herein we replace M_{23}, M_{31}, M_{12}, M_{14}, M_{24}, M_{34} respectively by X, Y, Z, $-L$, $-M$, $-N$ and N_{23}, N_{31}, N_{12}, N_{14}, N_{24}, N_{34} by any constant multiples respectively of $-L$, $-M$, $-N$, X, Y, Z the functions on the left become, save for W, the coefficients of

$$e, i, j, k, -e_1, i_1, j_1, k_1,$$

in the identity (A) of § 2.

Our object is to show that with any transformation of x, y, z, t leaving a certain quadric differential form in dx, dy, dz, dt unaltered, and appropriate corresponding transformations of the elements M_{ij}, N_{ij} of the two systems, and of V_x, ..., ρ, the equations change into others of the same form. In the Maxwell form of the equations the differential form is simply $-(dx^2 + dy^2 + dz^2 + dt^2)$.

In the paper referred to M. Th. de Donder bases his work, after Hargreaves, *Trans. Camb. Phil. Soc.*, XXI, 25 Aug. 1908, upon the transformation of differential (integral) forms, which he compounds together by rules which appear to have a certain artificiality. In what follows this is replaced by the use of Grassman's alternate units*. For the actual transformation here considered M. Th. de Donder does not give the proof in the paper in question (*loc. cit.*, p. 147), but refers to *Mém. Acad. roy. de Belgique*, I, 1904. But more, in the present note, full use is made of the theory of matrices, which not only brings out the identity of much of the algebra with what is familiar in other work, but is also very much briefer, as the reader may easily see by comparison with the original paper. A short account of the use of matrices, under the name of squares, is given in § 1 of this note.

* Mr Bateman, *Proceedings of the Lond. Math. Soc.*, VIII, 1909, p. 245, to whom M. Th. de Donder refers, has already suggested that Grassman's units might be used.

We have quantities M_{ij}, N_{ij}, for $i, j = 1, 2, 3, 4$, such that $M_{ii} = N_{ii} = 0$, $M_{ij} = -M_{ji}$, $N_{ij} = -N_{ji}$. We consider two matrices

$$m = \begin{vmatrix} 0, & M_{12}, & M_{13}, & M_{14} \\ M_{21}, & 0, & M_{23}, & M_{24} \\ M_{31}, & M_{32}, & 0, & M_{34} \\ M_{41}, & M_{42}, & M_{43}, & 0 \end{vmatrix}, \quad n = \begin{vmatrix} 0, & N_{12}, & N_{13}, & N_{14} \\ N_{21}, & 0, & N_{23}, & N_{24} \\ N_{31}, & N_{32}, & 0, & N_{34} \\ N_{41}, & N_{42}, & N_{43}, & 0 \end{vmatrix},$$

of which the second is determined from the former by an equation given later, with the help of a certain quadric. The determinants of m, n (supposed not zero) are respectively μ^2, ν^2, where

$$\mu = M_{23}M_{14} + M_{31}M_{24} + M_{12}M_{34}, \quad \nu = N_{23}N_{14} + N_{31}N_{24} + N_{12}N_{34}.$$

It will appear later, from the equation connecting m, n, that $\nu = -\mu$. Connected with m is a certain other matrix

$$\begin{vmatrix} 0, & M_{34}, & -M_{24}, & M_{23} \\ -M_{34}, & 0, & M_{14}, & M_{31} \\ M_{24}, & -M_{14}, & 0, & M_{12} \\ -M_{23}, & -M_{31}, & -M_{12}, & 0 \end{vmatrix}.$$

Multiplying this last with m we find that in the product every element vanishes except those in the diagonal, each of which is $-\mu$. Replacing the unit matrix by unity, as usual, we therefore say that the last written matrix is equal to $-\mu m^{-1}$. The corresponding matrix derived from n is therefore equal to $-\nu n^{-1}$.

We consider a certain quadric form in four variables, whose coefficients a, h, ... will later be regarded as functions of variables x, y, z, t, and put Δ for the matrix

$$\Delta = \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix}.$$

The determinant of this matrix being denoted by δ, we put

$$\epsilon = (-\delta)^{\frac{1}{2}}.$$

Then the coefficients N_{ij} are defined by the equations expressed by

$$\epsilon n = \mu \Delta m^{-1} \Delta$$

which is the same as

$$\epsilon m = \mu \Delta n^{-1} \Delta,$$

so that M_{ij} are the same functions of N_{ij} as are N_{ij} of M_{ij}. At greater length these equations are the same as

$$\epsilon n = -\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix} \begin{vmatrix} 0, & M_{34}, & -M_{24}, & M_{23} \\ -M_{34}, & 0, & M_{14}, & M_{31} \\ M_{24}, & -M_{14}, & 0, & M_{12} \\ -M_{23}, & -M_{31}, & -M_{12}, & 0 \end{vmatrix} \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix},$$

which give for instance

$$\begin{aligned} -\epsilon N_{23} &= (hw - gv)\, M_{23} + (bw - fv)\, M_{31} + (fw - cv)\, M_{12} \\ &\quad + (bc - f^2)\, M_{14} + (fg - ch)\, M_{24} + (hf - bg)\, M_{34}, \\ -\epsilon N_{14} &= (ad - u^2)\, M_{23} + (hd - uv)\, M_{31} + (gd - wu)\, M_{12} \\ &\quad + (hw - gv)\, M_{14} + (gu - aw)\, M_{24} + (av - hu)\, M_{34}, \\ -\epsilon N_{12} &= (av - hu)\, M_{23} + (hv - bu)\, M_{31} + (gv - fu)\, M_{12} \\ &\quad + (hf - bg)\, M_{14} + (gh - af)\, M_{24} + (ab - h^2)\, M_{34}. \end{aligned}$$

The reader will compare the formulae for the polar line of a line in regard to a quadric, Salmon's *Solid Geometry* (1882, p. 60), and may verify that, the coordinates of a line being l, m, n, l', m', n', where the line is given by $l't + mz - ny = 0$, $lt + m'y + n'z = 0$, the polar line $(\lambda\mu\nu\lambda'\mu'\nu')$ of $(lmnl'm'n')$ in regard to the quadric $(a, b, c, d, f, g, h, u, v, w \mathbin{\text{\textbar\!\textbar}} x, y, z, t)^2 = 0$ is given by

$$\begin{vmatrix} 0, & -\nu, & \mu, & \lambda' \\ \nu, & 0, & -\lambda, & \mu' \\ -\mu, & \lambda, & 0, & \nu' \\ -\lambda', & -\mu', & -\nu', & 0 \end{vmatrix} = \Delta \begin{vmatrix} 0, & -n', & m', & l \\ n', & 0, & -l', & m \\ -m', & l', & 0, & n \\ -l, & -m, & -n, & 0 \end{vmatrix} \Delta.$$

Thus if $M_{23}, M_{31}, M_{12}, M_{14}, M_{24}, M_{34}$ were proportional to $l, m, n, -l', -m', -n'$, then $N_{23}, N_{31}, N_{12}, N_{14}, N_{24}, N_{34}$ would be proportional to $\lambda, \mu, \nu, -\lambda', -\mu', -\nu'$. We assume however that $M_{23}M_{14} + M_{31}M_{24} + M_{12}M_{34}$, and the corresponding expression for N_{ij}, are not zero. The above expression of n in terms of m gives, on taking the determinants of both sides, $\epsilon^4\nu^2 = \mu^4\delta^2\,\mu^{-2}$, or, from $\epsilon^4 = \delta^2, \nu^2 = \mu^2$. The explicit forms for N_{ij} show that $\nu = -\mu$.

We consider now a transformation, from the variables (x, y, z, t), upon which $a, h, g, \ldots, M_{ij}, N_{ij}$, depend to variables (x', y', z', t'); we write

$$\begin{aligned} dx &= p_{11}dx' + p_{12}dy' + p_{13}dz' + p_{14}dt', \\ dy &= p_{21}dx' + p_{22}dy' + p_{23}dz' + p_{24}dt', \end{aligned}$$

and so on, or more briefly, with the ordinary matrix notation,

$$(dx, dy, dz, dt) = p\,(dx', dy', dz', dt'),$$

the converse equations being written

$$(dx', dy', dz', dt') = p'\,(dx, dy, dz, dt).$$

Thus the matrix p' is the inverse of p; so that if P_{rs} be the cofactor of p_{rs} in the determinant, ϖ, of p, we have $p_{sr}' = P_{rs}/\varpi$. With this transformation the quadric differential form

$$(abcduvwfgh\backslash dx, dy, dz, dt)^2$$

changes into another such form which is written

$$(a', \ldots, h'\backslash dx', dy', dz', dt')^2.$$

Using Δ' for the matrix of this latter, we thus have

$$\Delta' = \bar{p}\Delta p, \quad \Delta = \bar{p}'\Delta' p',$$

where, as usual, $\bar{p}$ denotes the matrix obtained from p by interchange of rows and columns.

Corresponding to this transformation, we introduce new functions M_{ij}' .... These may be defined by the fact that if

$$(dx, dy, dz, dt) = p\,(dx', dy', dz', dt'),$$
$$(\delta x, \delta y, \delta z, \delta t) = p\,(\delta x', \delta y', \delta z', \delta t'),$$

then

$$\Sigma\,\{M_{23}\,(dy\delta z - dz\delta y) + M_{14}\,(dx\delta t - dt\delta x)\}$$
$$= \Sigma\,\{M_{23}'\,(dy'\delta z' - dz'\delta y') + M_{14}'\,(dx'\delta t' - dt'\delta x')\}.$$

Thus, in matrix notation

$$-m\,(dx, dy, dz, dt\backslash\delta x, \delta y, \delta z, \delta t)$$
$$= -m'\,(dx', dy', dz', dt')\,(\delta x', \delta y', \delta z', \delta t'),$$

leading to $$m' = \bar{p}mp, \quad m = \bar{p}'m'p'.$$

If, as before, the determinants of p, m, Δ be ϖ, μ^2, $-\epsilon^2$, and the determinants of m', Δ' be similarly μ'^2, $-\epsilon'^2$, these equations, with $\Delta' = \bar{p}\Delta p$, give $\epsilon'^2 = \varpi^2\epsilon^2$, $\mu'^2 = \varpi^2\mu^2$ and hence, with a proper sign for ϵ', which is $(-\delta')^{\frac{1}{2}}$, we have $\epsilon'/\mu' = \epsilon/\mu$.

The matrix n is replaced after the transformation by a matrix n' connected with m' as was n with m, namely by

$$\epsilon' n' = \mu'\Delta'\,(m')^{-1}\Delta'$$
$$= \mu'\bar{p}\Delta p \,.\, p^{-1}m^{-1}\bar{p}^{-1} \,.\, \bar{p}\Delta p$$
$$= \mu'\bar{p}\Delta m^{-1}\Delta p$$
$$= \mu'\,\frac{\epsilon}{\mu}\,\bar{p}np,$$

so that $$n' = \bar{p}np, \quad n = \bar{p}'n'p'.$$

The matrix n is thus transformed by the same rule as was m.

We now introduce Grassman's units e_1, e_2, e_3, e_4, obeying the laws

$$e_i^2 = 0, \quad e_ie_j = -e_je_i,$$

so that, whatever a, b, c, d, a', b', c', d' may be, we have

$$(e_1a + e_2b + e_3c + e_4d)^2 = 0,$$

$$(e_1a + e_2b + e_3c + e_4d)\,(e_1a' + e_2b' + e_3c' + e_4d') = -\,(e_1a' + e_2b' + e_3c' + e_4d')\,(e_1a + e_2b + e_3c + e_4d),$$

while

$$(e_1a + e_2b + e_3c + e_4d)\,(e_1a' + e_2b' + e_3c' + e_4d')\,(e_1a'' + e_2b'' + e_3c'' + e_4d'') = e_2e_3e_4\Delta_1 + e_3e_1e_4\Delta_2 + e_1e_2e_4\Delta_3 + (-e_1e_2e_3)\,\Delta_4,$$

in which Δ_1, Δ_2, Δ_3, Δ_4 denote the determinants obtained from the array

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix}$$

by omitting in turn the first, second, third, fourth columns, and prefixing respectively the signs $+$, $-$, $+$, $-$.

We shall put

$$E_1 = e_2e_3e_4, \quad E_2 = e_3e_1e_4, \quad E_3 = e_1e_2e_4, \quad E_4 = -e_1e_2e_3,$$

and shall also introduce the four units given by

$$(e_1', e_2', e_3', e_4') = p'\,(e_1, e_2, e_3, e_4),$$

so that $\quad e_1' = p_{11}'e_1 + p_{12}'e_2 + p_{13}'e_3 + p_{14}'e_4$, etc.

These equations are equivalent with

$$e_1 = p_{11}e_1' + p_{12}e_2' + p_{13}e_3' + p_{14}e_4', \text{ etc.},$$

and we may write the relations $e' = p'e$, $e = pe'$. The four combinations

$$E_1' = e_2'e_3'e_4', \quad E_2' = e_3'e_1'e_4', \quad E_3' = e_1'e_2'e_4', \quad E_4' = -e_1'e_2'e_3'$$

are then linear functions of E_1, E_2, E_3, E_4; we find in fact

$$(E_1', E_2', E_3', E_4') = \varpi^{-1}\bar{p}\,(E_1, E_2, E_3, E_4).$$

The laws of transformation from $(e_1, \ldots, e_4)$ to $(e_1', \ldots, e_4')$ are the same as those from $(dx, \ldots, dt)$ to $(dx', \ldots, dt')$. Thus the equations

$$m' = \bar{p}mp, \quad n' = \bar{p}np,$$

lead to

$$m'e'^2 = me^2, \quad n'e'^2 = ne^2,$$

where as usual me^2 is the notation for the quadric form

$$m\,(e_1e_2e_3e_4)\,(e_1e_2e_3e_4), = -\,\Sigma M_{rs}e_re_s.$$

For instance

$$me^2 = m\,(e)\,(e) = m\,(pe')\,(pe') = mp\,(e')\,.\,p\,(e') = \bar{p}\,.\,mp\,(e')\,(e') = m'\,(e')\,(e') = m'e'^2.$$

Next let $\xi, \eta, \ldots, \xi', \eta', \ldots$ denote respectively

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \ldots.$$

We then have

$$(\xi', \eta', \zeta', \tau') = \bar{p}\,(\xi, \eta, \zeta, \tau).$$

Hence
$$e'\xi', = e_1'\,\xi' + e_2'\eta' + e_3'\zeta' + e_4'\tau',$$
$$= \xi' e' = \bar{p}\xi \,.\, e' = pe'\xi = e\xi.$$

Now consider the differential equations. They are capable of very succinct expression. Put

$$M_1 = -\eta M_{34} + \zeta M_{24} - \tau M_{23}, \quad M_2 = -\zeta M_{14} + \xi M_{34} - \tau M_{31},$$
$$M_3 = -\xi M_{24} + \eta M_{14} - \tau M_{12}, \quad M_4 = \xi M_{23} + \eta M_{31} + \zeta M_{12},$$

these being the forms occurring on the left in these equations (p. 174). Denote the row of these four quantities by

$$m^{(1)} = (M_1, M_2, M_3, M_4).$$

The expression $e\xi \,.\, me^2$, whose meaning is

$$-(e_1\xi + e_2\eta + e_3\zeta + e_4\tau)$$
$$(e_2e_3M_{23} + e_3e_1M_{31} + e_1e_2M_{12} + e_1e_4M_{14} + e_2e_4M_{24} + e_3e_4M_{34}),$$

is at once found on evaluation to be equal to

$$E_1M_1 + E_2M_2 + E_3M_3 + E_4M_4,$$

which we can represent by $Em^{(1)}$. In other words we have

$$e\xi \,.\, me^2 = Em^{(1)}.$$

Thus the original differential equations are

$$\left.\begin{aligned} e\xi \,.\, me^2 &= \rho\,(E_1V_x + E_2V_y + E_3V_z + E_4) \\ e\xi \,.\, ne^2 &= 0 \end{aligned}\right\} \qquad \text{(II)}.$$

In this form it is easy to show that they are unaltered by the transformation, provided suitable values be adopted for the new values of ρ, V_x, V_y, V_z.

For we have shown that the operators $e\xi$, $e'\xi'$ are equal, and that me^2, ne^2 are equal respectively to $m'e'^2$, $n'e'^2$. Also that

$$(E_1', \ldots, E_4') = \frac{1}{\varpi}\,\bar{p}\,(E_1, \ldots, E_4).$$

Thus, as $e'\xi' \,.\, m'e'^2$ is identically equal to

$$E_1'M_1' + \ldots + E_4'M_4',$$

that is to
$$\frac{1}{\varpi}\,\bar{p}\,(E)\,(M') = \frac{1}{\varpi}\,E \,.\, pM',$$

we have $(\varpi M_1, \varpi M_2, \varpi M_3, \varpi M_4) = p\,(M_1', M_2', M_3', M_4')$,

these being the equations (228), p. 146 of M. Th. de Donder's paper, in a form which to the present writer at least seems much more intelligible.

We shall then naturally take ρ', $V_{x'}$, $V_{y'}$, $V_{z'}$ so that

$$\rho'\,(E_1'V_{x'} + \ldots + E_4') = \rho\,(E_1V_x + \ldots + E_4),$$

or $$\rho'\,(E')\,(V') = \rho\,(E)\,(V),\ \frac{1}{\varpi}\rho' \cdot \bar{p}\,(E)\,(V') = \rho\,(E)\,(V)$$

or $$\rho'\,(E)\,(pV') = \varpi\rho\,(E)\,(V),$$

namely, so that $$\varpi\rho\,(V) = \rho'\,(pV'),$$

or $$\rho'\,(p_{11}V_{x'} + p_{12}V_{y'} + p_{13}V_{z'} + p_{14}) = \varpi\rho V_x,$$

$$\rho'\,(p_{41}V_{x'} + p_{42}V_{y'} + p_{43}V_{z'} + p_{44}) = \varpi\rho,$$

which give, for instance,

$$V_x = (p_{11}V_{x'} + \ldots + p_{14})/(p_{41}V_{x'} + \ldots + p_{44}),$$

and this, comparing with

$$\delta x/\delta t = (p_{11}\delta x' + \ldots + p_{14}\delta t')/(p_{41}\delta x' + \ldots + p_{44}\delta t'),$$

is in accordance with the view which regards V_x as a velocity.

Remark. We may evidently use Grassman's units in § 2 instead of the quaternion units, the Maxwell equations being a particular case of those considered here.

On the stability of periodic motions in general dynamics. By Professor H. F. Baker.

[*Read* 9 February 1920.]

The question of the stability of periodic motions has been discussed by Lord Kelvin (*Coll. Papers*, IV, pp. 484–515), by Sir George Darwin in his paper on Periodic Orbits (*Acta Math.*, XXI, 1897), and by Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, I, 226, etc. The stability in question is what for distinction from the secular or final stability, we may call the conventional or instantaneous stability; the solution of a differential equation whose coefficients are periodic functions of the independent variable t being expressed by a sum of terms such as $e^{\alpha t}[\phi + \phi_1 t + \ldots + \phi_r t^r]$, where $\phi, \phi_1, \ldots, \phi_r$ are periodic functions of t, the motion associated therewith is called stable or not according to the character of the exponent α. In the present note I am concerned only with the development of a regular algebraic calculus for the determination of α. The method employed has already been applied to the differential equation

$$\frac{d^2x}{dt^2} + (n^2 + 4nH + 8n\lambda k_1 \cos 2t + 8n\lambda^2 k_2 \cos 4t + \ldots)\, x = 0,$$

which has so great an importance in Astronomical and other investigations. Here λ is a small number, so that the series multiplying x converges, and n is an integer. There is no difficulty when $n^2 + 4nH$ is not near an integer. If however H be small difficulty arises. The solution is a sum of terms of the form $e^{i(n+2q)t}\phi$, where ϕ is periodic in t, and the stability, in the sense considered, depends on the sign of the real quantity q^2. For $n = 1$, the value of q^2 is positive, and the motion stable, so long as H does not lie between the values

$$-k_1\lambda - \tfrac{1}{2}k_1{}^2\lambda^2 + (\tfrac{1}{4}k_1{}^3 - k_1k_2)\,\lambda^3 + \ldots,$$
$$k_1\lambda - \tfrac{1}{2}k_1{}^2\lambda^2 - (\tfrac{1}{4}k_1{}^3 - k_1k_2)\,\lambda^3 + \ldots,$$

which it is seen are two small values on either side of zero unless $k_1 = 0$. For $n = 2$, there is similarly stability so long as H does not lie between

$$-(\tfrac{2}{3}k_1{}^2 - k_2)\,\lambda^2 \text{ and } (\tfrac{10}{3}k_1{}^2 - k_2)\,\lambda^2,$$

which again, generally include zero in their range (unless

$$\tfrac{2}{3}k_1{}^2 < k_2 < \tfrac{10}{3}k_1{}^2).$$

But for $n = 3$, there is stability unless H lie between

$$\tfrac{3}{4}k_1^2\lambda^2 - P\lambda^3, \quad \tfrac{3}{4}k_1^2\lambda^2 + P\lambda^3, \quad (P = \tfrac{9}{4}k_1^3 - 3k_1k_2 + k_3),$$

which limits do not include zero unless $k_1 = 0$. For greater values of n the range of values for H within which stability fails is that between two values of the form

$$\frac{2n}{n^2-1}\,k_1^2\lambda^2 + M\lambda^3, \quad \frac{2n}{n^2-1}\,k_1^2\lambda^2 + N\lambda^3.$$

(See *Royal Soc. Phil. Trans.*, A, CCXVI, 1916, 184.)

In the following we consider a system of equations

$$\frac{dx_r}{dt} = \frac{\partial F}{\partial y_r}, \quad \frac{dy_r}{dt} = -\frac{\partial F}{\partial x_r}, \qquad (r = 1, 2, \ldots, n),$$

wherein the function F is expressible as a convergent power series in a small parameter μ,

$$F = F_0 + \mu F_1 + \ldots,$$

whose coefficients are analytical functions in

$$x_1, \ldots, x_n, \quad y_1, \ldots y_n,$$

having no singularities in the range of values here considered, and are periodic, of period 2π, in each of $y_1, \ldots, y_n$ separately. Thus F does not contain t explicitly. *It is further assumed that F_0 is a function of $x_1, \ldots, x_n$ only.* This is an important limitation, suggested by the use of Delaunay's variables in the problem of three bodies. We shall for the most part limit ourselves to the case when $n = 2$, so that there will be *four* differential equations.

If $x_1^0 \ldots x_n^0$ be initial values for $x_1 \ldots x_n$, the quantities

$$n_r = -\frac{\partial F_0}{\partial x_r},$$

for $x_s = x_s^0$, may be called the conventional mean motions. By Poincaré's theory there is a periodic motion provided certain restrictions for the initial circumstances are introduced, when the ratios $n_1 : n_2 : \ldots$ are commensurable—that is when there are $(n-1)$ identities $n_1\alpha_2 - n_2\alpha_1 = 0$, $n_1\alpha_3 - n_3\alpha_1 = 0$, ... with integer coefficients $\alpha_1, \alpha_2, \ldots$. By appropriate linear transformation of the variables $x_1, x_2, \ldots, y_1, y_2, \ldots$, these conditions are reducible to the simpler forms $n_1 = 0$, $n_2 = 0, \ldots$, these being $n - 1$ in number. More precisely, noticing for the sake of comparison that, for $\mu = 0$, the differential equations are satisfied by

$$x_1 = x_1^0, \quad x_2 = x_2^0, \quad y_1 = n_1t + \varpi_1; \quad y_2 = n_2t + \varpi_2,$$

where ϖ_1, ϖ_2 are arbitrary, it can be shown that, for small values of μ, there is a periodic solution, reducing for $t = 0$ to

$$x_1 = x_1^0 + A_1, \quad x_2 = x_2^0 + A_2, \quad y_1 = \varpi_1 + B_1, \quad y_2 = \varpi_2 + B_2,$$

where, corresponding to $n_1 = 0$, ${x_1}^0$, ${x_2}^0$ are such as to satisfy

$$\frac{\partial F_0}{\partial x_1} = 0, \text{ for } x_1 = {x_1}^0, x_2 = {x_2}^0,$$

ϖ_1 is such that the function $[F_1]$, of ${x_1}^0$, ${x_2}^0$, ϖ_1, obtained by picking out the part of F_1 depending on x_1, x_2, y_1 only and substituting ${x_1}^0$, ${x_2}^0$, ϖ, for x_1, x_2, y_1, has a maximum or minimum value, namely ϖ_1 is such that

$$\frac{\partial [F_1]}{\partial \varpi_1} = 0,$$

and A_1, A_2, B_1 are determined, namely by linear equations whose determinant is

$$\left| \begin{matrix} \dfrac{\partial^2 F_0}{\partial {x_1}^{02}} & \dfrac{\partial^2 F_0}{\partial {x_1}^0 \partial {x_2}^0} \\ \dfrac{\partial^2 F_0}{\partial {x_1}^0 \partial {x_2}^0} & \dfrac{\partial^2 F_0}{\partial {x_2}^{02}} \end{matrix} \right| , \quad \frac{\partial^2 [F_1]}{\partial {\varpi_1}^2} ,$$

so that neither of these factors may be zero, while $\varpi_2 + B_2$ is arbitrary. In other words, the origin of time-reckoning is arbitrary, and either the initial value of x_2 or its corresponding mean motion.

If $x_1 = \phi_1(t)$, $x_2 = \phi_2(t)$, $y_1 = \psi_1(t)$, $y_2 = \psi_2(t)$ be the equations of such a periodic motion, and we substitute in the differential equations

$$\frac{dx_r}{dt} = \frac{\partial F}{\partial y_r}, \quad \frac{dy_r}{dt} = -\frac{\partial F}{\partial x_r},$$

supposing

$$F = F_0 + \mu F_1,$$

values

$$x_r = \phi_r(t) + \xi_r, \quad y_r = \psi_r(t) + \eta_r,$$

and retain only first powers of the small increments ξ_r, η_r, we have a system of linear differential equations with periodic coefficients (of period $2\pi/n_2$)

$$\frac{d\xi_1}{dt} = (y_1, x_1)\,\xi_1 + (y_1, x_2)\,\xi_2 + (y_1, y_1)\,\eta_1 + (y_1, y_2)\,\eta_2,$$

$$\frac{d\xi_2}{dt} = (y_2, x_1)\,\xi_1 + (y_2, x_2)\,\xi_2 + (y_2, y_1)\,\eta_1 + (y_2, y_2)\,\eta_2,$$

$$\frac{d\eta_1}{dt} = -(x_1, x_1)\,\xi_1 - (x_1, x_2)\,\xi_2 - (x_1, y_1)\,\eta_1 - (x_1, y_2)\,\eta_2,$$

$$\frac{d\eta_2}{dt} = -(x_2, x_1)\,\xi_1 - (x_2, x_2)\,\xi_2 - (x_2, y_1)\,\eta_1 - (x_2, y_2)\,\eta_2,$$

where (y_r, x_s) denotes the value of $\dfrac{\partial^2 F}{\partial y_r \partial x_s}$ for

$$x_r = \phi_r(t), \quad y_r = \psi_r(t);$$

these values of x_r, y_r are of the forms ${x_r}^0 + \mu X_r$, $n_r t + \varpi_r + \mu Y_r$. Since F_0 does not contain y_1, y_2 all of the sixteen coefficients are

of the form μU, where U is periodic in t, having μ as a factor, except the three

$$(x_1, x_1), \quad (x_1, x_2), \quad (x_2, x_2);$$

any one of these, say (x_r, x_s), is of the form $(x_r^0, x_s^0) + \mu\{x_r, x_s\}$, where the notation $\{\}$ must be observed. The matrix of four rows and columns formed by the sixteen coefficients on the right, if we retain only to the first power of μ, may then be represented by

$$\begin{pmatrix} 0, & 0 \\ -H_0, & 0 \end{pmatrix} + \mu \begin{pmatrix} (y, x), & (y, y) \\ -\{x, x\}, & -(x, y) \end{pmatrix},$$

where the 0 denotes a matrix of two rows and columns whose elements are all zero, H_0 denotes the value for $x_1 = x_1^0$, $x_2 = x_2^0$, of the matrix

$$H_0 = \begin{vmatrix} \frac{\partial^2 F_0}{\partial x_1^2}, & \frac{\partial^2 F_0}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F_0}{\partial x_2 \partial x_1}, & \frac{\partial^2 F_0}{\partial x_2^2} \end{vmatrix},$$

(y, x) denotes for $x_r = x_r^0$, $y_r = n_r t + \varpi_r$, the matrix

$$(y, x) = \begin{vmatrix} \frac{\partial^2 F_1}{\partial y_1 \partial x_1}, & \frac{\partial^2 F_1}{\partial y_1 \partial x_2} \\ \frac{\partial^2 F_1}{\partial y_2 \partial x_1}, & \frac{\partial^2 F_1}{\partial y_2 \partial x_2} \end{vmatrix},$$

with similar meanings for (y, y) and (x, y), but $\{x, x\}$ is such that $\mu\{x, x\}$

$$= \begin{vmatrix} \frac{\partial^2 F_0}{\partial x_1^2} - \left(\frac{\partial^2 F_0}{\partial x_1^2}\right), & \frac{\partial^2 F_0}{\partial x_1 \partial x_2} - \left(\frac{\partial^2 F_0}{\partial x_1 \partial x_2}\right) \\ \frac{\partial^2 F_0}{\partial x_1 \partial x_2} - \left(\frac{\partial^2 F_0}{\partial x_1 \partial x_2}\right), & \frac{\partial^2 F_0}{\partial x_2^2} - \left(\frac{\partial^2 F_0}{\partial x_2^2}\right) \end{vmatrix} + \mu \begin{vmatrix} \frac{\partial^2 F_1}{\partial x_1^2}, & \frac{\partial^2 F_1}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F_1}{\partial x_1 \partial x_2}, & \frac{\partial^2 F_1}{\partial x_2^2} \end{vmatrix},$$

wherein $\left(\frac{\partial^2 F_0}{\partial x_1^2}\right)$ is the value of $\frac{\partial^2 F_0}{\partial x_1^2}$ for $x_1 = x_1^0$, $x_2 = x_2^0$, etc., and in the first matrix we are to substitute $x_r = x_r^0 + \mu X_r$, and retain only to the first power of μ, while in the second matrix we are to put x_r^0, $n_r t + \varpi_r$ for x_r, y_r.

If the matrix of coefficients as so explained be denoted by $u + \mu v$ we require, to solve these equations, to compute what, in the notation of the paper referred to, is denoted by $\Omega(u + \mu v)$, which by a theorem there given (p. 159) is equal to

$$\Omega(u)\,\Omega[(\Omega(u))^{-1}\mu v\,\Omega(u)].$$

In case, as here,

$$u = \begin{pmatrix} 0, & 0 \\ -H_0, & 0 \end{pmatrix},$$

we have $u^2 = 0$, and hence

$$\Omega(u) = 1 + \begin{pmatrix} 0, & 0 \\ -H_0 t, & 0 \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix},$$

whose inverse is $\begin{pmatrix} 1, & 0 \\ tH_0, & 1 \end{pmatrix}$.

Thus we require, to first power of μ,

$$\begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix} \Omega \left[\mu \begin{pmatrix} 1, & 0 \\ tH_0, & 1 \end{pmatrix} \begin{pmatrix} (y,x), & (y,y) \\ -\{x,x\}, & -(x,y) \end{pmatrix} \begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix} \right],$$

or

$$\begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix} \Omega \left[\mu \begin{pmatrix} A, & B \\ C, & D \end{pmatrix} \right],$$

where $A = (y,x) - t(y,y)H_0$,

$B = (y,y)$,

$C = -\{x,x\} + t[H_0(y,x) + (x,y)H_0] - t^2 H_0 (y,y) H_0$,

$D = -(x,y) + tH_0(y,y)$.

Here the matrizant Ω, to first power of μ, is given by the definition

$$\Omega\{\mu \begin{pmatrix} A, & B \\ C, & D \end{pmatrix}\} = 1 + \mu \begin{pmatrix} QA, & QB \\ QC, & QD \end{pmatrix},$$

and the product is

$$\begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix} + \mu \begin{pmatrix} QA, & QB \\ -tH_0 \cdot QA + QC, & -tH_0 \cdot QB + QD \end{pmatrix},$$

that is

$$\begin{pmatrix} 1 + \mu QA, & \mu QB \\ -tH_0(1 + \mu QA) + QC, & 1 - \mu t H_0 \cdot QB + \mu QD \end{pmatrix}.$$

In the paper referred to the rule reached for determining the characteristic exponents was (p. 163) to pick out the coefficient of t in the matrix such as this, as it occurs outside trigonometrical signs, then put $t = 0$, and from this form a determinantal equation. That rule was founded on the assumption (p. 162) that a certain matrix, $\Omega_0^{\omega}(u)$, had, in case of equal roots, only linear invariant factors. It will be proved below that for the application of the rule the assumption is unnecessary; this is important in the present case since two of the characteristic exponents are necessarily zero, as we shall see. Of the quantities A, B, C, D, which are given explicitly above, only parts are material for the result. Denote then provisionally the coefficient of t in the matrix last written by

$$\begin{pmatrix} \mu\alpha, & \mu\beta \\ -H_0 + \mu\gamma, & \mu\delta \end{pmatrix},$$

where under the trigonometrical signs t is put zero; the determinantal equation is then

$$\begin{vmatrix} \mu\alpha - \rho, & \mu\beta \\ -H_0 + \mu\gamma, & \mu\delta - \rho \end{vmatrix} = 0.$$

As we have computed A, B, C, D only to the first power of μ, we retain only to first power of μ in this determinant. It is convenient to multiply by the determinant

$$\begin{vmatrix} -\rho, & 0 \\ H_0, & -\rho \end{vmatrix},$$

which for n variables x and n variables y would be equal to ρ^{2n}, and for $n = 2$ is equal to ρ^4. Then the above determinant becomes

$$\begin{vmatrix} -\rho(\mu\alpha - \rho), & -\rho\mu\beta \\ \mu(H_0\alpha - \rho\gamma), & \rho^2 - \mu\rho\delta + \mu H_0\beta \end{vmatrix},$$

and the evaluation to the first power of μ is extremely simple, namely in general it is

$$(\rho^2)^{2n} + (\rho^2)^{2n-1}\mu M,$$

where M is the sum of the diagonal elements of the two matrices

$$-\rho\alpha, \quad -\rho\delta + H_0\beta,$$

which in general are each of n rows and columns.

Removing the factor ρ^{2n}, which was introduced, and remembering that the roots ρ are in pairs, equal, and opposite in sign (cf. p. 165 of the paper referred to), we infer that the sum of the diagonal elements of α and δ is zero, and to our approximation the equation reduces to

$$\rho^{2n} + \rho^{2n-2}\mu\Sigma(H_0\beta)_{ii} = 0$$

which however, as was remarked above, divides by ρ^2. Thus we see that for $n = 2$, ρ^2/μ is developable as a power series in μ, and further that the sign of ρ^2 is that of the negative of the sum of the diagonal elements of the matrix, of two rows and columns, expressed by $H_0\beta$.

Now consider the meaning of β. We have

$$B = \begin{pmatrix} \dfrac{\partial^2 F_1}{\partial y_1^2}, & \dfrac{\partial^2 F_1}{\partial y_1 \partial y_2} \\ \dfrac{\partial^2 F_1}{\partial y_1 \partial y_2}, & \dfrac{\partial^2 F_1}{\partial y_2^2} \end{pmatrix}$$

where we are to put

$$x_1 = x_1^0, \quad y_1 = y_1^0, \quad y_1 = n_1 t + \varpi_1, \quad y_2 = n_2 t + \varpi_2,$$

and, as was said, we have $n_1 = 0$. In general F_1 has a form

$$F_1 = A_{00} + \Sigma A_{p_1 p_2} \cos(p_1 y_1 + p_2 y_2 + h),$$

wherein A_{00}, $A_{p_1 p_2}$, h are functions of x_1, x_2. Let $[F_1]$ be the part of F_1 which is a function of y_1 only,

$$[F_1] = A_{00} + \Sigma A_{p_1, 0} \cos(p_1 y_1 + h),$$

which for $y_1 = \varpi_1$, $y_2 = n_2 t + \varpi_2$, does not contain t, and let F_1' be the remaining part of F_1.

We have

$$\frac{\partial^2 F_1}{\partial y_r \partial y_s} = \frac{\partial^2 [F_1]}{\partial y_r \partial y_s} + \frac{\partial^2 F_1'}{\partial y_r \partial y_s},$$

and $\dfrac{\partial^2 [F_1]}{\partial y_r \partial y_s}$ vanishes unless $r = s = 1$, while $Q \dfrac{\partial^2 F'}{\partial y_r \partial y_s}$ is an entirely periodic function not having the factor t outside trigonometrical terms. Thus in QB the only term having the factor t arises from $Q \dfrac{\partial^2 [F_1]}{\partial y_1^2}$; this is

$$-Q\Sigma A_{p_1,\,0}\, p_1^2 \cos(p_1\varpi_1 + h) = -t\Sigma A_{p_1,\,0}\, p_1^2 \cos(p_1\varpi_1 + h) = t\frac{\partial^2 [F_1]}{\partial \varpi_1^2}$$

so that

$$\beta = \begin{vmatrix} \dfrac{\partial^2 [F_1]}{\partial \varpi_1^2}, & 0 \\ 0, & 0 \end{vmatrix},$$

and

$$\Sigma (H_0 \beta)_{ii} = \frac{\partial^2 F_0}{\partial x_1^{02}} \frac{\partial^2 [F_1]}{\partial \varpi_1^2},$$

which gives, for the characteristic exponents which do not vanish,

$$\rho = \pm \sqrt{\mu} \left\{ -\frac{\partial^2 F_0}{\partial x_1^{02}} \cdot \frac{\partial^2 [F_1]}{\partial \varpi_1^2} \right\}^{\frac{1}{2}}.$$

And it will be recalled that

$$\frac{\partial F_0}{\partial x_1^0} = 0, \quad \frac{\partial [F_1]}{\partial \varpi_1} = 0.$$

In order to obtain the corresponding expression when there are three variables x, and three variables y, it is necessary to carry the approximations as far as μ^2. Poincaré's corresponding work is in *Méth. Nouv.*, I, pp. 201–226.

Note. It has been remarked that two of the characteristic exponents are zero, and that the condition that, in case of equal roots, the matrix $\Omega_0^{\omega}(u)$ should have linear invariant factors, is not necessary for the application of the rule above. In fact as t does not occur explicitly in F, the existence of a periodic solution $x_r = \phi_r(t)$, $y_r = \psi_r(t)$ of the original equations, involves the existence also of a solution $x_r = \phi_r(t + h)$, $y_r = \psi_r(t + h)$, in which h is arbitrary. This involves the existence of a solution

$$\xi_r = \frac{\partial \phi_r (t + h)}{\partial h}, \quad \eta_r = \frac{\partial \psi_r (t + h)}{\partial h}$$

of the equations of variation, where after differentiation h is to be put zero. This is however a purely periodic solution with zero characteristic exponent. The nature of the equations of variation

however involves that the characteristic exponents are in pairs of equal values of opposite sign. There is therefore also another solution of vanishing characteristic exponent.

Now suppose that in the argument of p. 162 of the paper referred to the matrix $\Omega_0^\omega(u)$ has not linear invariant factors. It will be sufficient to take one case, and to suppose that

$$\Omega_0^\omega(u) = k \begin{vmatrix} e^{ic_1\omega}, & ., & . \\ ., & e^{ic_2\omega}, & i\omega e^{ic_2\omega} \\ ., & ., & e^{ic_2\omega} \end{vmatrix} k^{-1}.$$

Noticing then, multiplying the matrices, that

$$\begin{vmatrix} X^{-1}, & ., & . \\ ., & Y^{-1}, & -itY^{-1} \\ ., & ., & Y^{-1} \end{vmatrix} \begin{vmatrix} X', & ., & . \\ ., & Y', & it'Y' \\ ., & ., & Y' \end{vmatrix}$$

$$= \begin{vmatrix} X'X^{-1}, & ., & . \\ ., & Y'Y^{-1}, & i(t'-t)\,Y'Y^{-1} \\ ., & ., & Y'Y^{-1} \end{vmatrix},$$

of which a particular case is for $X' = X$, $Y' = Y$, $t' = t$, we have

$$\Omega_0^{\omega+t}(u) = \Omega_\omega^{\omega+t}(u)\,\Omega_0^\omega(u) = \Omega_0^t(u)\,\Omega_0^\omega(u),$$

because u is periodic, and hence, by the form assumed for $\Omega_0^\omega(u)$,

$$\Omega_0^{\omega+t}(u) \,.\, k \begin{vmatrix} e^{-ic_1(\omega+t)}, & ., & . \\ ., & e^{-ic_2(\omega+t)}, & -i(\omega+t)\,e^{-ic_2(\omega+t)} \\ ., & ., & e^{-ic_2(\omega+t)} \end{vmatrix}$$

is equal to

$$\Omega_0^t(u)\,k \begin{vmatrix} e^{-ic_1t}, & ., & . \\ ., & e^{-ic_2t}, & -ite^{-ic_2t} \\ ., & ., & e^{-ic_2t} \end{vmatrix},$$

showing that the matrix

$$Q_0^t(u) = \Omega_0^t(u)\,k \begin{vmatrix} e^{-ic_1t}, & ., & . \\ ., & e^{-ic_2t}, & -ite^{-ic_2t} \\ ., & ., & e^{-ic_2t} \end{vmatrix} k^{-1}$$

has the period ω, and is therefore such that

$$Q_0^\omega(u) = Q_0^0(u) = 1.$$

This gives

$$\Omega_0^t(u) = Q_0^t(u)\,k \begin{vmatrix} e^{ic_1 t}, & . , & . \\ . , & e^{ic_2 t}, & ite^{ic_2 t} \\ . , & . , & e^{ic_2 t} \end{vmatrix} k^{-1}.$$

In this the terms involving t outside trigonometrical signs are

$$iQ_0^t(u)\,k \begin{vmatrix} c_1, & . , & . \\ . , & c_2, & 1 \\ . , & . , & c_2 \end{vmatrix} k^{-1},$$

and putting therein $t = 0$, we have the determinantal equation for the characteristic exponents

$$\begin{vmatrix} ic_1 - \rho, & . , & . \\ . , & ic_2 - \rho, & i \\ . , & . , & ic_2 \end{vmatrix} = 0.$$

The form of this equation establishes the result in question.

If we put $Q_0^t(u)\,k = \Phi(t)$, and, denoting the initial values of the dependent variables by x_1^0, x_2^0, x_3^0, put

$$k^{-1}(x_1^0, x_2^0, x_3^0) = (z_1^0, z_2^0, z_3^0),$$

the general solution $\Omega_0^t(u) \,.\, (x_1^0, x_2^0, x_3^0)$ becomes

$$\Phi(t) \,.\, \begin{vmatrix} e^{ic_1 t}, & . , & . \\ . , & e^{ic_2 t}, & ite^{ic_2 t} \\ . , & . , & e^{ic_2 t} \end{vmatrix} (z_1^0, z_2^0, z_3^0),$$

or

$$\begin{vmatrix} \Phi_{11}e^{ic_1 t}, & \Phi_{12}e^{ic_2 t}, & \Phi_{12}ite^{ic_2 t} + \Phi_{13}e^{ic_2 t} \\ \Phi_{21}e^{ic_1 t}, & \Phi_{22}e^{ic_2 t}, & \Phi_{22}ite^{ic_2 t} + \Phi_{23}e^{ic_2 t} \\ \Phi_{31}e^{ic_1 t}, & \Phi_{32}e^{ic_2 t}, & \Phi_{32}ite^{ic_2 t} + \Phi_{33}e^{ic_2 t} \end{vmatrix} (z_1^0, z_2^0, z_3^0),$$

so that

$$\begin{aligned} x_1 &= z_1^0 e^{ic_1 t}\Phi_{11} + z_2^0 e^{ic_2 t}\Phi_{12} + z_3^0 e^{ic_2 t}(\Phi_{13} + it\Phi_{12}), \\ x_2 &= z_1^0 e^{ic_1 t}\Phi_{21} + z_2^0 e^{ic_2 t}\Phi_{22} + z_3^0 e^{ic_2 t}(\Phi_{23} + it\Phi_{22}), \\ x_3 &= z_1^0 e^{ic_1 t}\Phi_{31} + z_2^0 e^{ic_2 t}\Phi_{32} + z_3^0 e^{ic_2 t}(\Phi_{33} + it\Phi_{32}), \end{aligned}$$

where the Φ's are periodic functions, and z_1^0, z_2^0, z_3^0 are arbitrary constants of integration.

On the stability of rotating liquid ellipsoids. By Professor H. F. Baker.

[*Read* 9 February 1920.]

The present note deals only with the paedagogic problem of reducing the *algebraic* treatment of the stability of rotating ellipsoids and spheroids, for ellipsoidal or spheroidal displacements only, to the simplest possible terms. It is a modification of what is given on pp. 69–71 of Mr Hargreaves paper on the Domains of steady motion for a liquid ellipsoid, and the oscillations of the Jacobian figure, *Camb. Phil. Trans.*, XXII, 1914. Mr Hargreaves refers to the paper of C. O. Meyer, *Crelle*, XXIV, 1842, for one of the identities he uses, but does not do himself justice in that he refrains from pointing out that Meyer's work is vitiated in an important point by a mistake of algebra.

§ 1. Supposing a, b, c to be real positive quantities in descending order of magnitude, put

$$f(x) = (x+a)(x+b)(x+c),\ f'(x) = \frac{df(x)}{dx},\ y^2 = f(x);$$

then we can verify without difficulty that

$$\int_0^\infty \frac{x\,dx}{y^3} = \frac{3}{4}\int_0^\infty \frac{x^2 f'(x)}{y^5}\,dx,\quad \int_0^\infty \frac{x^2 dx}{y^3} = \int_0^\infty \frac{x^2\,[3f(x) - xf'(x)]}{y^5}\,dx.$$

Next put, y being throughout taken positive,

$$\phi = \int_0^\infty \frac{dx}{y},\quad a+b = \frac{1}{h},\quad abc = p,$$

so that

$$f(x) = (x+c)\left(x^2 + \frac{x}{h} + \frac{p}{c}\right) = x^3 + x^2\left(c + \frac{1}{h}\right) + x\left(\frac{p}{c} + \frac{c}{h}\right) + p;$$

in this regard p as a constant, so that $f(x)$ is regarded as depending on the two variable quantities h and c; so considered, denote it by P, and write

$$P_1 = \frac{\partial P}{\partial h},\quad P_2 = \frac{\partial P}{\partial c},\quad P_{11} = \frac{\partial^2 P}{\partial h^2},\ \text{etc.}$$

and correspondingly write

$$\phi_1 = \frac{\partial \phi}{\partial h},\quad \phi_{12} = \frac{\partial^2 \phi}{\partial h \partial c},\ \text{etc.}$$

Then if ξ, η be arbitrary we have

$$\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2 = -\frac{1}{2}\int_0^\infty \frac{P_{11}\xi^2 + 2P_{12}\xi\eta + P_{22}\eta^2}{y^3}\,dx + \frac{3}{4}\int_0^\infty \frac{(P_1\xi + P_2\eta)^2}{y^5}\,dx;$$

substituting the forms of P_{11}, etc., and utilising the two identities remarked at starting which here take the forms

$$\int_0^\infty \frac{x\,dx}{y^3} = \frac{3}{4}\int_0^\infty x^2\left[3x^2 + 2x\left(c + \frac{1}{h}\right) + \frac{p}{c} + \frac{c}{h}\right]\frac{dx}{y^5},$$

$$\int_0^\infty \frac{x^2 dx}{y^3} = \int_0^\infty x^2\left[x^2\left(c + \frac{1}{h}\right) + 2x\left(\frac{p}{c} + \frac{c}{h}\right) + 3p\right]\frac{dx}{y^5},$$

we have

$$\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2 = -\int_0^\infty x^2 Q\,\frac{dx}{y^5},$$

where

$$Q = Ux^2 + 2Vx + W,$$

in which, if we put

$$t = \frac{ph}{c^2},\quad u = ch,\quad \xi_1 = c\xi,\quad \eta_1 = h\eta,$$

the values of U, V, W are given by

$$\frac{4h^4}{3}Uc^2 = (\xi_1^2 - \xi_1\eta_1 + \eta_1^2) + \tfrac{1}{3}\xi_1^2(10u + 1) + \eta_1^2 u(3t - u - 1),$$

$$\frac{4h^4}{3}Vc = t(\xi_1^2 - \xi_1\eta_1 + \eta_1^2) + \tfrac{1}{3}\xi_1^2(3u + t + 4) + \eta_1^2 u(2t - 1),$$

$$\frac{4h^4}{3}W = (3t - 1)(\xi_1^2 - \xi_1\eta_1 + \eta_1^2) + \xi_1^2(2t + 1).$$

Hence as c, h, p are positive, if ξ, η *be any real quantities*, each of U, V, W is necessarily positive provided $2t > 1$. For this, being

$$\frac{2}{c} > \frac{1}{a} + \frac{1}{b},\quad \text{or } c < 2ab/(a + b),$$

involves, because of $4a^2b^2/(a + b)^2 < ab$, also $c^2 < ab$, and hence

$$t > u,\quad 3t - u - 1 = t - u + 2t - 1 > 0.$$

Thus for all real values of ξ, η, the quadratic form in ξ, η denoted by Q is necessarily positive, and the form

$$\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2$$

necessarily negative, the sign of y being taken positive as stated.

§ 2. Now consider a mass of homogeneous (incompressible) fluid, of mass M, whose surface has the equation

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

rotating with angular velocity ω about the axis of z. Let I, μ, $-W$ be respectively the moment of inertia, moment of momentum and potential energy of gravitation and

$$H = \frac{\mu^2}{2I} - W;$$

we have

$$I = \frac{M}{5}(a+b), \quad \mu = \frac{M}{5}(a+b)\,\omega, \quad W = \frac{3}{10}M^2\phi,$$

and if we put

$$K = \frac{10}{3M^2}.H, \quad \sigma = \frac{25}{3M^3}\mu^2, \quad h = \frac{1}{a+b},$$

we have

$$K = \sigma h - \phi.$$

We are concerned to make H a minimum when μ and M, and therefore also p, $= abc$, are retained constant, and hence to make K a minimum regarded as a function of h and c, retaining σ as constant.

The necessary conditions of rotational equilibrium

$$\frac{\partial K}{\partial h} = 0, \quad \frac{\partial K}{\partial c} = 0,$$

give

$$\phi_1 = c, \quad \phi_2 = 0,$$

and the condition of stability is that

$$\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2$$

should be negative for all real values of ξ, η.

We have

$$\phi_2 = -\frac{1}{2}\int_0^\infty \frac{x\,dx}{y^3}\left(x + \frac{1}{h} - \frac{p}{c^2}\right),$$

and, in the familiar way, this vanishes for a proper value of p, which is such that

$$p > \frac{c^2}{h}.$$

As this involves also $p > \frac{1}{2}c^2/h$, or, in a previous notation $2t > 1$, we see from § 1 that *the condition of stability is satisfied for the ellipsoidal forms of rotational equilibrium when ellipsoidal displacements are considered.*

§ 3. Consider a series of ellipsoidal shapes satisfying the conditions $\phi_1 = \sigma$, $\phi_2 = 0$, for all of which abc is the same. By $\phi_2 = 0$ we can then regard c as a function of h, subject to

$$\frac{dc}{dh} = -\frac{\phi_{12}}{\phi_{22}}.$$

For increments δh, δc, p being constant, the increment of K, to the second order, with constant σ, is then

$$\delta K = \sigma\delta h - (\phi_1\delta h + \phi_2\delta c) - \tfrac{1}{2}(\phi_{11}\delta h^2 + 2\phi_{12}\delta h\delta c + \phi_{22}\delta c^2)$$

$$= -\tfrac{1}{2}\left(\phi_{11} - \frac{\phi_{12}^2}{\phi_{22}}\right)\delta h^2,$$

while $$\delta\sigma = \phi_{11}\delta h + \phi_{12}\delta c = \left(\phi_{11} - \frac{\phi_{12}^2}{\phi_{22}}\right)\delta h,$$

and, from $\omega^2 = 3M\sigma h^2 = 3Mh^2\phi_1$, we have

$$\delta\left(\frac{\omega^2}{3M}\right) = (\phi_{11}\delta h + \phi_{12}\delta c)\,h^2 + 2\phi_1 h\delta h$$

$$= (\phi_{11}h^2 + 2\phi_1 h)\,\delta h - \frac{\phi_{12}^2}{\phi_{22}}\,h^2\delta h.$$

But, from § 1 above, we have

$$\phi_{22} < 0, \quad \phi_{11}\phi_{22} - \phi_{12}^2 > 0,$$

while

$$\phi_{11}h^2 + 2\phi_1 h = h^2\left[-\frac{1}{2}\int_0^\infty \frac{P_{11}}{y^3}\,dx + \frac{3}{4}\int_0^\infty \frac{P_1^2}{y^5}\,dx\right] - h\int_0^\infty \frac{P_1}{y^3}\,dx,$$

wherein $$P_{11} = 2\,\frac{x^2 + xc}{h^3}, \quad P_1 = -\frac{x^2 + xc}{h^2} = -\frac{1}{2}\,hP_{11},$$

so that $$\phi_{11}h^2 + 2\phi_1 h, \; = \frac{3}{4}\,h^2\int_0^\infty \frac{P_1^2}{y^5}\,dx,$$

is necessarily positive.

Thus we see that, *as h diminishes, for a series of shapes of rotational equilibrium, K constantly increases, σ also increases, but ω diminishes.*

§ 4. We can however show that as $a - b$ increases numerically, h does diminish, that is that $a + b$ increases. [The numerical tables show that along the series of ellipsoids a increases, but b and c both diminish.]

For

$$\tfrac{1}{2}\delta[(a-b)^2] = \tfrac{1}{2}\delta\left[(a+b)^2 - 4\,\frac{p}{c}\right] = -\frac{1}{h^3}\,\delta h + \frac{2p}{c^2}\,\delta c,$$

while, along the series considered, $\delta c = -\phi_{12}\delta h/\phi_{22}$; so that

$$\tfrac{1}{2}\delta\,[(a-b)^2] = -\left[\frac{\phi_{22}}{h^3} + \frac{2p}{c^2}\,\phi_{12}\right]\frac{\delta h}{\phi_{22}}.$$

We proved however (§ 1) that

$$-(\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2) = P\,(c^2\xi^2 - ch\xi\eta + h^2\eta^2) + Q\xi^2 + R\eta^2,$$

wherein P, Q, R are positive, and ξ, η are arbitrary. This gives

$$-(\phi_{22}\eta + \phi_{12}\xi) = P\,(-\tfrac{1}{2}ch\xi + h^2\eta) + R\eta;$$

hence, replacing ξ, η respectively by $2p/c^2$ and $1/h^3$, we infer

$$-\left(\frac{\phi_{22}}{h^3} + \frac{2p}{c^2}\,\phi_{12}\right) = P\left(-\frac{ph}{c} + \frac{1}{h}\right) + \frac{R}{h^3},$$

wherein

$$\frac{1}{h} - \frac{ph}{c}, = a + b - \frac{ab}{a+b},$$

is necessarily positive.

Thus, as ϕ_{22} is negative, the equation

$$\delta h = -\tfrac{1}{2}\,\phi_{22}\left(\frac{\phi_{22}}{h^3} + \frac{2p}{c^2}\,\phi_{12}\right)^{-1}\delta\,[(a-b)^2]$$

shows that δh has a sign opposite to that of $\delta\,[(a-b)^2]$.

We have thus proved that *as the axes* $2\sqrt{a}$, $2\sqrt{b}$ *become more unequal, the energy* H, *the moment of inertia* $\frac{1}{5}M/h$, *and the angular momentum* μ, *all constantly increase, while the angular velocity* ω *constantly diminishes.*

In Mr Hargreaves' notation (*Camb. Phil. Trans.*, XXII, 1914, 61), if momentarily m be used for the whole mass, instead of M,

$$\frac{L}{3m/2} = -h^2\phi_{11},\quad \frac{M}{3m/2} = -c^2\phi_{22},\quad \frac{N}{3m/2} = ch\phi_{12}.$$

He obtains $L > 2N$, $M > N$. The preceding work establishes $M > 2N$.

§ 5. For the series of *spheroids* of rotational equilibrium, the variables h, c are not appropriate, for a reason which will appear. Writing $a + x$, $b + x$ respectively for a, b and $h + \xi$, $c + \eta$ for h, c, we have at once, x, y being small,

$$\xi = -h^2\,(x+y) + h^3\,(x+y)^2 + \text{etc.},$$

$$\eta = -c\left(\frac{x}{a} + \frac{y}{b}\right) + c\left(\frac{x^2}{a^2} + \frac{xy}{ab} + \frac{y^2}{b^2}\right) + \text{etc.},$$

and hence

$$-\sigma\xi + \phi_1\xi + \phi_2\eta + \tfrac{1}{2}\,(\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2) = \Psi_1 + \Psi_2 + \text{etc.},$$

where $$\Psi_1 = (\sigma - \phi_1)\, h^2\,(x+y) - \phi_2 c\left(\frac{x}{a} + \frac{y}{b}\right),$$

and

$$\Psi_2 = (-\sigma + \phi_1)\, h^3\,(x+y)^2 + \phi_2 c\left(\frac{x^2}{a^2} + \frac{xy}{ab} + \frac{y^2}{b^2}\right)$$

$$+ \tfrac{1}{2}\left[\phi_{11} h^4\,(x+y)^2 + 2\phi_{12} c h^2\,(x+y)\left(\frac{x}{a} + \frac{y}{b}\right) + \phi_{22} c^2\left(\frac{x}{a} + \frac{y}{b}\right)^2\right].$$

If herein we put $a = b$ we obtain, for an arbitrary ellipsoidal variation from the spheroid, to terms of the second order, of the function $\psi = -\sigma h + \phi$, which is a constant negative multiple of the energy H, the following

$$\delta\psi = \left[(\sigma - \phi_1)\, h^2 - \phi_2 \frac{c}{a}\right](x+y)$$

$$+ (-\sigma + \phi_1)\, h^3\,(x+y)^2 + \frac{c\phi_2}{4a^2}\left[3\,(x+y)^2 + (x-y)^2\right]$$

$$+ \tfrac{1}{2}\,(x+y)^2\left[\phi_{11} h^4 + \frac{2ch^2}{a}\,\phi_{12} + \frac{c^2}{a^2}\,\phi_{22}\right].$$

Thus we obtain, as determining the angular velocity or momentum necessary for the rotational equilibrium of the spheroid, the single equation

$$\sigma = \phi_1 + \phi_2 \frac{c}{ah^2}$$

(explaining the reason for the change of variables), and thence

$$\delta\psi = \frac{c\phi_2}{4a^2}\,(x-y)^2$$

$$+ \left[\tfrac{1}{2}\phi_{11} h^4 + \phi_{12}\frac{ch^2}{a} + \tfrac{1}{2}\phi_{22}\frac{c^2}{a^2} + \frac{c}{4a^2}\,\phi_2\right](x+y)^2.$$

We have however shown that

$$\tfrac{1}{2}\phi_{11} h^4 + \phi_{12} h^2 \frac{c}{a} + \tfrac{1}{2}\,\phi_{22}\frac{c^2}{a^2}$$

is negative so long as $\dfrac{2}{c} > \dfrac{1}{a} + \dfrac{1}{b}$, and hence, for $a = b$, so long as $a > c$. For the value of ϕ_2, putting $a = b$, $a = c\sec^2\alpha$, $t = \tan\alpha$, we easily compute

$$\phi_2 = \frac{1}{8t^4 c^{\frac{3}{2}}}\left\{\frac{\alpha}{t}\,(3 + 14t^2 + 3t^4) - (3 + 13t^2)\right\},$$

which is negative for values of α ranging from $\alpha = 0$, when the spheroid is a sphere, to the value for which

$$\frac{\alpha}{t}(3 + 14t^2 + 3t^4) - (3 + 13t^2) = 0$$

(about 54° 22′, corresponding to a meridian eccentricity $e = \cdot 8127$, and a value of $\frac{\omega^2}{2\pi\rho}$ equal to $\cdot 187$). For this value $\phi_2 = 0$, and, by what we have previously shown, the spheroid is a particular one of the ellipsoids of rotational equilibrium.

Thus $\delta\psi$ is negative, the proof being valid for the spheroid for which $\phi_2 = 0$, and *the spheroids lying between the sphere and this ellipsoid are therefore stable for ellipsoidal displacements.*

§ 6. The coefficient of $(x + y)^2$ in the above expression for $\delta\psi$ remains negative however even after α has passed the value (about 54° 22′) for which $\phi_2 = 0$, and ϕ_2 has become positive, indeed up to $\alpha = \frac{\pi}{2}$. For from

$$\sigma = \phi_1 + \phi_2 \frac{c}{ah^2},$$

using $c = 4ph^2$, and hence $\delta c = 8ph\,\delta h$, we get

$$\delta\sigma = \frac{2\delta h}{h^4}\left[\tfrac{1}{2}\phi_{11}h^4 + \phi_{12}h^2\frac{c}{a} + \tfrac{1}{2}\phi_{22}\frac{c^2}{a^2} + \frac{c}{4a^2}\phi_2\right],$$

leading, for $a = b$, to

$$\delta\psi = \frac{(x-y)^2}{4}\frac{c\phi_2}{a^2} - \frac{(x+y)^2}{16a^2}\frac{d\sigma}{da};$$

on the other hand $\sigma = \phi_1 + \phi_2 \frac{c}{ah^2}$

gives $\sigma = 2a(a-c)\int_0^\infty \frac{x\,dx}{(x+a)^2(x+c)^{\frac{3}{2}}},$

and hence

$$\frac{d\sigma}{da} = \int_0^\infty \left\{x(2a + c) + 3ac + 3c(a-c)\frac{x+a}{x+c}\right\} \frac{2x\,dx}{(x+a)^3(x+c)^{\frac{3}{2}}},$$

which is evidently positive.

Thus, putting $x = y$, we infer that *the spheroids are stable for spheroidal displacements for all values of* α up to $\frac{\pi}{2}$.

We can prove that

$$\sigma = 6[p(1+t^2)]^{\frac{1}{6}}\frac{(1+t^2)^{\frac{1}{2}}}{t^2}\left[\frac{\alpha}{t}(1 + \tfrac{1}{3}t^2) - 1\right], = 6[p(1+t^2)]^{\frac{1}{6}}\,U,$$

say, where $(1 + t^2)^{\frac{1}{6}}$ increases with t, while

$$\frac{dU}{dt} = \frac{d}{dt}\left\{\frac{(1+t^2)^{\frac{1}{2}}}{t^2}\left[\frac{\alpha}{t}(1+\tfrac{1}{3}t^2)-1\right]\right\} = \frac{9+7t^2}{3t^4(1+t^2)^{\frac{1}{2}}}\eta,$$

wherein

$$\eta = t\frac{9+4t^2}{9+7t^2} - \alpha,$$

leading to

$$\frac{d\eta}{dt} = \frac{4t^4}{1+t^2}\cdot\frac{6+7t^2}{(9+7t^2)^2}.$$

Hence σ, and therefore the angular momentum, constantly increases with the angle α, along the series of spheroids.

But, we may remark,

$$\frac{\omega^2}{2\pi\rho}, = 3\cos\alpha\frac{(1+t^2)^{\frac{1}{2}}}{t^2}\left[\frac{\alpha}{t}(1+\tfrac{1}{3}t^2)-1\right], = 3\cos\alpha U,$$

increases with α only until

$$\frac{\alpha}{t}(1+t^2) = \frac{9+7t^2}{9+t^2}$$

(approximately $\alpha = 68^\circ\ 26'$, $e = \cdot 93$, giving $\frac{\omega^2}{2\pi\rho} = \cdot 2247$), and afterwards diminishes.

On the general theory of the stability of rotating masses of liquid. By Professor H. F. Baker.

[*Read* 9 February 1920.]

I venture to make some brief provisional remarks, to which I have hoped now for some years to give a more detailed examination, relating to the question why Sir George Darwin on the one hand, and Liapounoff and Mr Jeans on the other, arrive at different conclusions in regard to the stability of the so-called pear-shaped figure of equilibrium of a rotating liquid mass. These point to the conclusion that this is a case in which the empirical treatment of the convergence of an infinite series may lead to erroneous results in a concrete practical matter.

§ 1. For a mass of homogeneous incompressible liquid rotating as if solid about an axis, with angular momentum μ, moment of inertia I, and potential energy of gravitation $-W$ (where W is the volume integral of the product $dm.dm'$ of two elements of mass divided by their mutual distance), we consider the Hamiltonian function

$$H = -W + \tfrac{1}{2}\mu^2/I.$$

Let H' be the corresponding function belonging to another such mass, "sufficiently" near to the former, rotating about the same axis with the same angular momentum (and, for brevity, of the same mass, and the same centre of mass lying on the axis), but with different I and different W. We conceive that the form of this second mass can be specified, relatively to that of the former, by a certain number of parameters. In the actual problem the tale of these parameters must be unlimited; but the methods applicable when this tale is finite cannot be extended to the actual case without careful examination, and in what follows we think only of a limited tale. The difference $H' - H$ is then a function of these parameters. In the case in which a change of form of the rotating mass involves a dissipation of energy, a necessary and sufficient condition that the form of the mass first considered should be one of stable rotary equilibrium, under its own gravitation, is that $H' - H$ should be positive for all small values of the parameters. We adopt this as the condition. In the problem now being considered the first form of the mass is itself regarded as arising from another (of the same mass and centre of mass), with a different angular momentum, μ_0, and different I and W, say I_0 and W_0, of which the relative rotary equilibrium and stability have already been investigated. We have then a known form of equilibrium, to

which there belongs a function H_0, then a contiguous form whose equilibrium is under examination, to which belongs the function H, and then, further, a "virtual" form, which has the same angular momentum as arises in H, to which the function H' applies. We may then suppose the parameters, above spoken of as identifying the form, to vanish for the form (H_0), may denote their values for (H) by $x, y, \ldots$, and their values for (H') by $x+\xi, y+\eta, \ldots$. We put $\frac{1}{2}\mu^2 - \frac{1}{2}\mu_0^2 = k$. Presuming certain conditions of continuity for the functions involved, the equations of equilibrium of the form (H), which are such as $\partial H/\partial x = 0$, $\partial H/\partial y = 0$, ..., must, for $x=0$, $y=0$, ..., together with $k=0$, be satisfied for (H_0). In general (certain conditions being introduced in the choice of the parameters) these equations determine a form (H) corresponding to every arbitrary small k; the necessary and sufficient condition however that (H_0) should be a so-called form of bifurcation, or branch form, is that these equations should lead to more than one form (H) for any given small k. A sufficient condition for the stability of the form (H), is that the quadratic form, in the arbitrary variables $\xi, \eta, \ldots$, consisting of such terms as

$$\tfrac{1}{2}\xi^2\partial^2 H/\partial x^2 + \xi\eta\partial^2 H/\partial x\partial y + \ldots,$$

should be definite, and be positive, when, therein, the coefficients $\partial^2 H/\partial x^2$, $\partial^2 H/\partial x\partial y$, ..., are those functions of k arising by substituting the value of $x, y, \ldots$, just found from $\partial H/\partial x = 0$, $\partial H/\partial y = 0$, etc. And, in particular, if (H_0) be stable, this quadratic form must be definite and positive for $k=0$, $x=0$, $y=0$, ... Conversely this last fact, when (H_0) is known to be stable, considerably reduces the labour of considering the stability of (H).

§ 2. In our case, (H_0) is an ellipsoid, and (H) a contiguous, so-called pear-shaped, form. Sir George Darwin calculates (*Papers*, III, 349), a form for the increment of the Lagrangian function $W + \frac{1}{2}\mu^2/I$, in passing from the form (H) to the form (H'), which leads, to the same approximation, to a form for δH, or $H' - H$, which may be written thus:

$$\begin{aligned}\frac{\delta H}{3M^2/2k_0} &= k'\,(1 - lx^2 - my - nz)\\ &\quad + \tfrac{1}{2}ax^4 + \tfrac{1}{2}by^2 + \tfrac{1}{2}cz^2 + gx^2z + hx^2y + fyz\\ &\quad + \Sigma\,[h'x^2y' + \tfrac{1}{2}b'y'^2];\end{aligned}$$

for facility of comparison we may give the equations which connect the notation here employed with that used by Sir George Darwin:

$$x, \quad y, \quad z, \quad y'; \qquad\qquad k', \quad l, \quad m, \quad n,$$

are used respectively in place of

$$e, \quad f_2, \quad f_2^{(2)}, \quad f_2^{(s)}; \qquad\qquad \frac{k}{a}, \quad \frac{b}{a}, \quad \frac{c}{a}, \quad -\frac{d}{a};$$

also the coefficients a, b, ..., b', h', expressed in Sir George Darwin's notation are, respectively, given by

$$a = \frac{\mathfrak{b}^2\omega^2}{\mathfrak{a}} - 2A_0, \qquad b = \frac{\mathfrak{c}^2\omega^2}{\mathfrak{a}} + 2C_2, \qquad c = \frac{\mathfrak{d}^2\omega^2}{\mathfrak{a}} + 2C_2^{(2)},$$

$$f = -\frac{\mathfrak{c}\mathfrak{d}\omega^2}{\mathfrak{a}}, \qquad g = -\frac{\mathfrak{b}\mathfrak{d}\omega^2}{\mathfrak{a}} - 2B_2^{(2)}, \quad h = \frac{\mathfrak{b}\mathfrak{c}\omega^2}{\mathfrak{a}} - 2B_2,$$

$$h' = -2B_i^{(s)}, \qquad b' = 2C_i^{(s)}.$$

§ 3. For a form of possible rotary equilibrium to which such an expression for δH is appropriate, the equations for δH to be stationary are

$$\begin{aligned} 2x\,[ax^2 + hy + gz + \Sigma h'y' - k'l\,] &= 0, \\ hx^2 + by + fz \qquad\quad - k'm &= 0, \\ gx^2 + fy + cz \qquad\quad - k'n &= 0, \\ h'x^2 + b'y' \qquad\quad &= 0. \end{aligned}$$

The solution $x = 0$ belongs to the series of ellipsoids; omitting this for the moment, the equations give

$$\Sigma h'y' = -\Sigma \frac{h'^2}{b'}\,x^2,$$

$$x^2 \left| \begin{matrix} a - \Sigma\,(h'^2/b'), & h, & g \\ h & , & b, & f \\ g & , & f, & c \end{matrix} \right| = k' \left| \begin{matrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{matrix} \right|, \quad y/k' = \text{etc.}, \quad z/k' = \text{etc.},$$

corresponding to two possible forms other than ellipsoids.

The stability, for displacements in which only x, y, z, y' vary, depends on the quadratic form

$$\tfrac{1}{2}\xi^2 \frac{\partial^2(\delta H)}{\partial x^2} + \xi\eta \frac{\partial^2(\delta H)}{\partial x \partial y} + \ldots + \tfrac{1}{2}\Sigma\eta'^2 \frac{\partial^2(\delta H)}{\partial y'^2} + \Sigma\eta'\eta'' \frac{\partial^2(\delta H)}{\partial y' \partial y''},$$

namely

$$2ax^2\xi^2 + 2hx\xi\eta + 2gx\xi\zeta + 2\Sigma h'x\xi\eta' + \tfrac{1}{2}b\eta^2 + f\eta\zeta + \tfrac{1}{2}c\zeta^2 + \tfrac{1}{2}\Sigma b'\eta'^2;$$

this is, however, the same as

$$\begin{aligned} &\tfrac{1}{2}\Sigma b' \left(\eta' + \frac{2h'}{b'}\,\xi x\right)^2 + \tfrac{1}{2}b \left(\eta + \frac{f}{b}\,\zeta + \frac{2h}{b}\,\xi x\right)^2 \\ &\quad + \tfrac{1}{2}\left(c - \frac{f^2}{b}\right)\left(\zeta + 2\xi x \frac{gb - hf}{bc - f^2}\right)^2 \\ &\quad + \frac{2\xi^2 x^2}{bc - f^2} \left| \begin{matrix} a - \Sigma\,(h'^2/b'), & h, & g \\ h & , & b, & f \\ g & , & f, & c \end{matrix} \right|, \end{aligned}$$

wherein, by what has appeared, the last term is the same as

$$\frac{2\xi^2 k'}{bc-f^2}\begin{vmatrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{vmatrix}.$$

The solution, above omitted, arising by taking $x=0$, requires

$$by+fz-k'm=0, \quad fy+cz-k'n=0, \quad y'=0,$$

so that, if we introduce σ, given by $\sigma = hy+gz-k'l$, we obtain

$$\sigma = -\frac{k'}{bc-f^2}\begin{vmatrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{vmatrix},$$

and the quadratic form arising in considering the stability is

$$\sigma\xi^2 + \tfrac{1}{2}b\eta^2 + f\eta\zeta + \tfrac{1}{2}c\zeta^2 + \tfrac{1}{2}\Sigma b'\eta'^2.$$

Assuming then that the ellipsoids up to the form of bifurcation, that is for $k'<0$, are stable (as is well known), we can infer that b, $bc-f^2$, b' and σ are all positive, and hence that

$$\begin{vmatrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{vmatrix}$$

is positive.

Hence, returning to the quadratic form above wherein x is not zero, we infer (1) that *the pear-shaped figure*, so far as the increment δH is appropriately represented by the form above, *is stable if k' is positive*. By Darwin and Jeans this conclusion is made to depend on general reasoning, due in the first place to (Liapounoff and) Schwarzschild, writing in correction of Poincaré (*Inaugural Dissertation, München*, 1896, or *Neue Annalen d. Sternwarte München*, III, 1898, 275); and (2) that a necessary and sufficient condition for this is that the determinant

$$\begin{vmatrix} a-\Sigma\,(h'^2/b'), & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

should be positive. It is however easy to verify that this determinant is, in the notation of Sir George Darwin, equal to

$$-(bc-f^2)\,N + \frac{4C_2C_2^{(2)}\omega^2}{\mathrm{a}}\left(\mathrm{b}+\mathrm{c}\,\frac{B_2}{C_2}-\mathrm{d}\,\frac{B_2^{(2)}}{C_2^{(2)}}\right)^2,$$

where

$$N = 2\left[A_0+\frac{(B_2)^2}{C_2}+\frac{(B_2^{(2)})^2}{C_2^{(2)}}+\Sigma\,\frac{(B_i^{(s)})^2}{C_i^{(s)}}\right]^2;$$

from the expression of the increment of the function H, for variation to the second order, in the case of the ellipsoids, it is known that C_2 is negative, and $C_2^{(2)}$ is positive; (these are in fact the "coefficients of stability" corresponding to Lamé functions of one root, in the former case lying between the negative squares of the greatest and mean axes of the ellipsoid, in the latter case between the negative squares of the mean and least axis). Thus (3), we infer, by *algebraic methods only*, that a sufficient condition for the *instability* of the pear-shaped figure, if we assume the preceding ellipsoids stable, is the single condition

$$\tfrac{1}{2}N, \text{ or } A_0 + \Sigma \frac{(B_m^{(n)})^2}{C_m^{(n)}} > 0, \qquad \begin{pmatrix} m = 2, 2, i \\ n = 0, 2, s \end{pmatrix}.$$

Darwin (*Works*, III, 378) computes

$$N = -\cdot 000235513.$$

It is easy to show, from what precedes, that, if $\delta\omega^2$ denote the increment of ω^2 in passing from the ellipsoid to the pear, we have

$$Nx^2 + \tfrac{1}{8}\frac{\mathrm{a}\delta\omega^2}{C_2 C_2^{(2)}} \begin{vmatrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{vmatrix} = 0,$$

or

$$Nx^2 + \tfrac{1}{2}\delta\omega^2 \left(\mathrm{b} + \mathrm{c}\frac{B_2}{C_2} - \mathrm{d}\frac{B_2^{(2)}}{C_2^{(2)}}\right) = 0.$$

Thus a sufficient condition for instability is $\delta\omega^2 > 0$.

This again, we see algebraically, involves that the increment δI of the moment of inertia, is negative; for, if we put D for the coefficient of $\frac{1}{2}\delta\omega^2$ in the last written equation, we easily find

$$\delta I = \frac{x^2}{D}\left[D^2 + N\frac{\mathrm{a}}{\omega^2}\left(1 - \frac{bc - f^2}{4C_2 C_2^{(2)}}\right)\right].$$

As $\delta\omega^2$, δI are then of opposite sign it is evidently desirable, if possible, to calculate N independently and not from these, as do Darwin (*loc. cit.*, p. 379) and Jeans (*Cosmogony*, 1919, p. 101).

Mr Jeans (*Phil. Trans.* A, CCXV, 1915, 76, 77; *Cosmogony*, 1919, p. 92, § 94) appears, if I understand him aright, to hold the view that an expression such as that above taken for δH, does not suffice to enable us to draw inferences in regard to the stability, and in particular that there should be present therein a term in x^2. And he seeks to find in this way the explanation of the difference in conclusion of Sir George Darwin and himself. I cannot agree with this view; if the expression for δH is accurate as far as it goes, it appears to suffice for forming a judgment, though hypotheses as to the relative order of smallness of x, y, ..., should clearly not form part of the process for calculating δH.

§ 4. I believe that the discrepancy of conclusion arises in another way. Sir George Darwin computes his form N only to a limited number of terms, and is satisfied with the verification (*loc. cit.*, p. 380) that the terms he would next calculate are very small in comparison. It is easy to show however that the terms of the first few orders, in Mr Jeans' expression for the normal variation from the ellipsoid to the contiguous pear, involve terms extending to infinity in Sir George Darwin's expression for this normal variation.

Taking a point (x, y, z) near to a point (x_0, y_0, z_0) of the ellipsoid

$$x^2/a + y^2/b + z^2/c = 1,$$

given by

$$x\,(a+\lambda)^{-\frac{1}{2}} = x_0 a^{-\frac{1}{2}}, \quad y\,(b+\lambda)^{-\frac{1}{2}} = y_0 b^{-\frac{1}{2}}, \quad z\,(c+\lambda)^{-\frac{1}{2}} = z_0 c^{-\frac{1}{2}},$$

Darwin (*loc. cit.*, p. 320) has an expansion in Lamé products

$$-\tfrac{1}{2}\lambda/{p_0}^2 = \text{const.} - e\Omega_{10}{}^a - f_2\Omega_{10} - f_2{}^{(2)}\Omega_{01} - \Sigma f_i{}^{(s)}\Omega_i{}^s,$$

where p_0 is the central perpendicular on the tangent plane at (x_0, y_0, z_0), so that, in terms of elliptic coordinates, ${p_0}^2 = abc/\mu\nu$.

Jeans' form of contiguous surface (*Phil. Trans.* A, CCXV, CCXVII, 1915, 1916, or *Cosmogony*, p. 88) is

$$0 = -1 + \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} + exu_1 + e^2u_2 + e^3xv_2 + \ldots,$$

where u_1, u_2, v_2 are integral polynomials in $x^2, y^2, z^2, 1$, of respective orders 1, 2, 2. If herein we substitute

$$x = x_0\left(\frac{a+\lambda}{a}\right)^{\frac{1}{2}} = x_0\left(1 + \frac{\lambda}{2a} - \frac{\lambda^2}{8a^2} + \ldots\right),$$

with similar expressions for y and z, and solve for λ, we shall find a series of the form

$$\frac{\lambda}{{p_0}^2} = -ex_0u + e^2{p_0}^2v + e^3x_0{p_0}^4w + \ldots,$$

where u, v, w, ..., are polynomials in ${x_0}^2$, ${y_0}^2$, ${z_0}^2$, which, on examination, prove not divisible by $\dfrac{1}{{p_0}^2}$ or $x_0{}^2/a^2 + y_0{}^2/b^2 + z_0{}^2/c^2$. These are then expressible by finite series in Lamé functions. But ${p_0}^2 = \dfrac{abc}{\mu\nu}$, ${p_0}^4 = \dfrac{a^2b^2c^2}{\mu^2\nu^2}$, ..., can only be expressed by infinite series of this form.

Thus even the terms of the second order in Mr Jeans' form for $\lambda/{p_0}^2$ cannot be put down exactly from Darwin's results without taking account of the whole aggregate of terms in Darwin's series; and the same for terms of higher order.

§ 5. The conclusion so reached is in accordance with a note appearing in the *Compt. Rendus* (CLXX, 5 Jan. 1920, 38; "Calculs de G. H. Darwin sur la stabilité de la figure piriforme," Note de M. Pierre Humbert) long after the above was written, to which my attention was called by Mr F. P. White of St John's College. The author has calculated Darwin's series to a further approximation, and finds a result not strengthening Darwin's prevision.

But the conclusion is made almost certain by a comparison with Liapounoff's paper, "Sur un problème de Tchebychef," *St Pétersbourg Mémoires*, XVII. For the statements there given reference is made to another memoir, of which the first, the theoretical, part, appears not to be obtainable in England; indeed, were it otherwise, there might be little justification for the preceding summary remarks, save perhaps on account of the total difference of method, in view of the rigour with which Liapounoff's results in these problems are developed. As was pointed out to me by Mr S. R. U. Savoor, of Trinity College, Liapounoff makes the remark (*loc. cit.*, p. 27) that the terms of various orders in his development of what is here called λ, though presenting themselves in the first place as infinite series, can be summed, and then take the form above remarked (§ 4) as belonging to the expansion which can be deduced from Mr Jeans' work.

Liapounoff however also remarks (*loc. cit.*, p. 30) that the instability of the pear follows, when the ellipsoids have been examined, from the sign of one term only, which, in a footnote, he identifies with that above denoted by $\frac{1}{2}N$—which he states he has expressed in finite terms as an algebraic function of the axes of the ellipsoid; and gives further, also without proof, a general expression for the increase of angular momentum in passing from the ellipsoid to the pear, with the remark that the (positive) sign of this also follows from the sign of N^*. (See *Péters. Mém.* XXII, 1908, 126–131.)

* The following references to Liapounoff's papers may be useful to the reader: (1) 1884, "Sur la stabilité des figures ellipsoïdales," *Toulouse Annales*, VI, 1904 (translated from the Russian); (2) 1903, "Recherches dans la théorie de la figure...," *St Pétersbourg Mémoires*, XIV; (3) 1904, "Sur l'équation de Clairaut," *ibid.* XV; (4) 1905, "Sur un problème de Tchebychef," *ibid.* XVII; (5) 1906, *Sur les figures d'équilibre peu différentes des ellipsoïdes*, Part I—published separately, unobtainable in England; (6) 1908, "Problème de minimum...," *St Pétersbourg Mémoires*, XXII; cf. *Toulouse Annales*, IX, "Problème général de la Stabilité du mouvement"; (7) 1909, second part of the memoir (5); (8) 1912, third part of the same. The second and third parts are in the British Museum (as was first discovered for me by Mr F. P. White); (9) 1916, "Sur les équations qui appartiennent aux surfaces des figures d'équilibre dérivées des ellipsoïdes d'un liquide homogène en rotation"; and "Nouvelles considérations relatives à la théorie des figures d'équilibre dérivées des ellipsoïdes dans le cas d'un liquide homogène," *St Pétersbourg Bulletin*, 1916. These last two references I take from an article by L. Lichtenstein, "Gleichgewichtsfiguren rotierender Flüssigkeiten," *Math. Zeitschrift*, VII, Berlin, 1920, 132. The same writer also gives (*ibid.* I, 1918, 232) reference to a fourth part of the memoir (8), under date 1914, and to a memoir, *Ann. de l'Éc. Norm.* XXVI, 1909, 473.

Mr S. R. U. Savoor has made a detailed application of the method of Liapounoff to the case of the rotating cylinders, of which it is hoped that a summary may be published in the *Trans. Camb. Phil. Soc.*

Sur le principe de Phragmén-Lindelöf. Par MARCEL RIESZ. (Extrait d'une lettre adressée à M. G. H. Hardy.)

[*Received* 29 July 1920.]

1. La lecture attentive de votre travail récemment paru dans les *Acta Mathematica** et de ceux de MM. Phragmén et Lindelöf†, P. Persson‡, F. Carlson ¶, etc., me fait croire que le théorème suivant n'a jamais été observé.

Si la fonction $g(z)$ holomorphe dans le demi-plan $R(z) \geqq 0$ (sauf peut-être à l'infini) y satisfait à l'inégalité

$$|g(z)| < C_1 e^{lr} \qquad (r = |z|) \tag{1}$$

et sur l'axe imaginaire à celle-ci

$$|g(z)| < C_2 e^{-kr}, \tag{2}$$

C_1, C_2, k et l désignant des constantes positives, cette fonction s'annulera identiquement.

Ce théorème rend la première démonstration dans votre travail beaucoup plus simple qu'elle ne l'était. Vous le verrez dans un instant. Mais démontrons d'abord le théorème lui-même. Il découlera immédiatement des principes généraux de MM. Phragmén et Lindelöf.

Formons la fonction

$$\psi(z) = e^{\omega z} g(z),$$

ω désignant un nombre positif. Désignons par C le plus grand des nombres C_1 et C_2. Sur l'axe réel positif on aura d'après (1)

$$|\psi(z)| < C e^{(\omega+l)r}$$

et sur l'axe imaginaire

$$|\psi(z)| < C e^{-kr}.$$

Il s'ensuit d'après un théorème connu de Phragmén-Lindelöf||,

* G. H. Hardy, "On two theorems of F. Carlson and S. Wigert," *Acta Mathematica*, t. 42, 1920, pp. 327–339.

† E. Phragmén et Ernst Lindelöf, "Sur une extension d'un principe classique de l'Analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier," *Acta Mathematica*, t. 31, 1908, pp. 381–406.

‡ Paul Persson, "Recherches sur une classe de fonctions entières," *Thèse pour le doctorat*, Upsal, 1908.

¶ F. Carlson, "Sur une classe de séries de Taylor," *Thèse pour le doctorat*, Upsal, 1914.

|| *Loc. cit.*, pp. 393–394; an explicit proof of the particular result required is given by H. Cramér, "Sur une classe de séries de Dirichlet," *Thèse pour le doctorat* (Stockholm), Upsal, 1917, pp. 34–36.

ϕ désignant un nombre arbitraire situé entre $-\frac{\pi}{2}$ et $\frac{\pi}{2}$,

$$|\psi(z)| < Ce^{(-k|\sin\phi| + (\omega+l)\cos\phi)r}.$$

On aura donc, δ étant choisi assez petit, pour $\phi = \pm\left(\frac{\pi}{2} - \delta\right)$,

$$\psi(z) < C, \tag{3}$$

inégalité valable aussi sur l'axe imaginaire. En appliquant encore une fois le théorème en question de Phragmén-Lindelöf aux angles

$$\left(-\frac{\pi}{2}, -\frac{\pi}{2}+\delta\right), \quad \left(-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right) \text{ et } \left(\frac{\pi}{2}-\delta, \frac{\pi}{2}\right)$$

tous moindres que π, on voit que (3) subsiste dans tout le demi-plan. C'est à dire, dans tout ce demi-plan,

$$|g(z)\, e^{\omega z}| < C. \tag{4}$$

En tout point intérieur au demi-plan, on a $|e^z| > 1$. On conclut donc de (4), en faisant tendre ω vers l'infini,

$$g(z) \equiv 0.$$

2. Appliquons ce résultat à démontrer le théorème de M. Carlson.

Si $f(z) = f(re^{i\theta})$ *est holomorphe dans l'angle* $-\alpha \leqq \theta \leqq \alpha$, *où* $\alpha \geqq \frac{\pi}{2}$, *et y satisfait à l'inégalité*

$$|f(z)| < Ae^{pr} \qquad (p < \pi),$$

et

$$f(n) = 0 \qquad (n = 0, 1, 2, \ldots),$$

alors $f(z)$ *s'annule identiquement.*

En effet, la fonction $g(z) = \frac{f(z)}{\sin \pi z}$ satisfait aux hypothèses du numéro précédent, par suite elle est identiquement nulle. Quant à la condition (2), cela est évident. En ce qui concerne (1), on trouve les éléments d'une démonstration rigoureuse dans votre travail ci-mentionné (p. 329). (En réalité, votre démonstration en question du théorème de M. Carlson présente une petite lacune. Pour pouvoir appliquer le théorème de Phragmén-Lindelöf, vous auriez dû démontrer que pour tout angle $-\alpha + \delta \leqq \theta \leqq \alpha - \delta$, il y a une majorante (une certaine fonction exponentielle) qui dépend seulement de r. Vous démontrez seulement que $+1(\theta) \leqq k$ sur chaque vecteur issu de l'origine, sans vous préoccuper de l'uniformité.)

Voilà une autre application de ce théorème. M. Cramér a déduit d'un beau théorème qui lui est dû, le corollaire qui suit*.

* H. Cramér, "Un théorème sur les séries de Dirichlet et son application," *Arkiv för Matematik, Astronomi och Fysik*, t. 13, No. 22, 1918, pp. 1–14 (p. 12).

Si le module de la fonction entière $\phi(z)$ finit par rester inférieur à $e^{k|z|}$, où $k > 0$, il y a sur chaque vecteur issu de l'origine une infinité de points aux modules indéfiniment croissants pour lesquels

$$|\phi(z)| > e^{-(k+\epsilon)|z|}$$

et cela a lieu pour tout $\epsilon > 0$.

L'application du théorème démontré au début à la fonction $g(z) = \phi(z)\phi(-z)$ donne de suite le résultat de M. Cramér. D'ailleurs il est facile de déduire ce résultat directement du théorème fondamental de Phragmén-Lindelöf que nous venons d'appliquer.

3. Un corollaire immédiat de notre théorème est encore le suivant.

Une fonction $g(z)$ qui est holomorphe dans un angle d'étendue $\geqq \pi$ et y satisfait à l'inégalité

$$|g(z)| < Ce^{-kr} \qquad (k > 0)$$

s'annule identiquement.

Par une substitution de variable on en obtient immédiatement:

Soit $\Phi(x)$ une fonction analytique de la variable complexe $x = re^{i\phi}$ qui jouit des propriétés suivantes:

1°. *Elle est holomorphe à l'intérieur et sur le contour d'un domaine T renfermant l'angle $-\frac{\pi}{2\alpha} \leqq \phi - \phi_0 \leqq \frac{\pi}{2\alpha}$, sauf peut-être certaines parties de cet angle situées à l'intérieur d'un cercle autour de l'origine.*

2°. *A l'intérieur et par suite au contour de ce domaine, on a*

$$|\Phi(x)| < Ce^{-kr^\alpha}, \tag{5}$$

C et k étant des constantes positives.

Dans ces conditions, on a identiquement $\Phi(x) = 0$.

Le meilleur théorème de ce genre fut jusqu'ici, à ce qu'il semble, un théorème de M. Paul Persson démontré dans sa thèse (*l.c.* p. 8) et cité *in extenso* dans la thèse de M. Carlson (*l.c.* p. 36). Chez M. Persson, la condition

$$|\Phi(x)| < e^{-v(r)r^\alpha}, \quad \lim_{r=\infty} v(r) = \infty$$

figure au lieu de (5).

Note by G. H. Hardy.

I know from my own experience that it is sometimes a little difficult to pick out from Phragmén and Lindelöf's classical memoir the precise proposition of which one may have need; and Dr Cramér's dissertation, referred to by Dr Riesz, is not easily accessible to English readers. It may therefore be worth while to give an explicit statement and proof of the particular theorems used by Dr Riesz.

1. *Suppose that T is an angle of magnitude less than π, whose vertex is at the origin, that C, C_1, and K are constants, and that*

(1) $$|f(z)| \leqq C$$

on the boundary of T,

(2) $$|f(z)| \leqq C_1 e^{Kr}$$

throughout T. Then (1) *holds throughout T.*

We may plainly suppose, without loss of generality, that the boundaries of T are $\phi = -\alpha$ and $\phi = \alpha$, where $0 < \alpha < \frac{1}{2}\pi$. Let δ be positive, $1 < k < \dfrac{\pi}{2\alpha}$, and

$$g(z) = e^{-\delta z^k},$$

so that

$$|g(z)| = e^{-\delta r^k \cos k\phi} \leqq e^{-\delta r^k \cos k\alpha} < 1$$

throughout T. Finally, let

$$f(z)\, g(z) = h(z).$$

Since $|g(z)| \leqq 1$, we have $|h(z)| \leqq C$ on the boundary of T. Also, since $k > 1$ and $\cos k\alpha > 0$,

$$|h(z)| \leqq C_1 e^{Kr - \delta r^k \cos k\alpha}$$

tends uniformly to zero when z tends to infinity in T. Hence $|h(z)| \leqq -C$ if $-\alpha \leqq \phi \leqq \alpha$, $r = R$, and R is sufficiently large; and therefore at all points of the boundary of the region $T(R)$ defined by the inequalities just written; and therefore throughout $T(R)$.

As R is arbitrarily large:

$$|h(z)| = |e^{-\delta z^k} f(z)| \leqq C$$

throughout T; and therefore, making δ tend to zero, $|f(z)| \leqq C$ throughout T.

2. *Suppose that T is the angle $\phi_1 \leqq \phi \leqq \phi_2$, where $\phi_2 - \phi_1 < \pi$; that*

(3) $$|f(re^{i\phi_1})| \leqq Ce^{a_1 r},$$

(4) $$|f(re^{i\phi_2})| \leqq Ce^{a_2 r},$$

and that (2) *is satisfied throughout T. Then*

(5) $$|f(re^{i\phi})| \leqq Ce^{h(\phi)r}$$

where $h(\phi)$ *is the function* $A\cos\phi + B\sin\phi$ *which assumes the values* a_1 *and* a_2 *for* $\phi = \phi_1$ *and* $\phi = \phi_2$.

We may plainly suppose, without loss of generality, that $-\phi_1 = \phi_2 = \psi$, where $0 < \psi < \frac{1}{2}\pi$.

Let $$g(z) = e^{-(A-iB)z}$$

so that $$|g(z)| = e^{-h(\phi)r}$$

and $$f(z)\,g(z) = h(z).$$

Plainly $h(z)$ satisfies a condition of the type (2); and $|h(z)| \leqq C$ for $\phi = \phi_1$ and for $\phi = \phi_2$. Hence $|h(z)| \leqq C$ throughout T, which proves the theorem.

It seems worth while also to fill up explicitly the "petite lacune" in my proof in the *Acta Mathematica* signalised by Dr Riesz. It is a question simply of proving that

$$g(z) = \frac{f(z)}{\sin \pi z}$$

satisfies an inequality

(6) $$|g(z)| < Ce^{Kr}$$

for $0 \leqq \phi \leqq \frac{1}{2}\pi$. Suppose that λ is a positive constant, U the part of the positive quadrant above, and V the part below, the line $y = \lambda$. Since $|\sin \pi z|$ is greater than a constant throughout U, (6) is satisfied in U, and we need only consider V. Also

$$f'(z) = \frac{1}{2\pi i}\int \frac{f(u)}{(u-z)^2}\,du,$$

z lying in V and the contour of integration being the circle $|u - z| = 1$. Hence $f'(z)$ satisfies an inequality

$$|f'(z)| \leqq C_1 e^{Kr}$$

throughout V.

Suppose now that z lies in V and that n is the integer nearest to z (either, if there are two equidistant from z). Then

$$f(z) = f(z) - f(n) = \int_n^z f'(u)\,du,$$

the path of integration being rectilinear; and so

$$|f(z)| \leqq M|z - n|,$$

where M is the maximum of $|f'(u)|$ on the path of integration. Hence

$$|F(z)| = \left|\frac{f(z)}{\sin \pi z}\right| \leqq C_1 e^{K(n+\frac{1}{2}+\lambda)}\left|\frac{z-n}{\sin \pi z}\right| \leqq C_2 e^{Kn} \leqq Ce^{Kr},$$

which is the inequality required. This is substantially the argument I used before, extended however to complex values of z.

A note on the nature of the carriers of the Anode Rays. By G. P. Thomson, M.A., Fellow of Corpus Christi College.

[*Received* 3 August 1920.]

The importance of positive rays as an instrument of research is now thoroughly established, and the success of the recent investigations of Dr Aston on isotopes makes it the more to be regretted that so far, with the exception of mercury, no metal has with certainty been identified in the photograph of the mass spectra of positive rays; and this in spite of a considerable number of attempts. In these circumstances the work of Gehrcke and Reichenheim* on anode rays suggested the possibility that these might fill the gap. From a rough measurement of e/m Gehrcke and Reichenheim showed that the anode rays probably consist of positively charged atoms of metal. They did not however photograph the particles and their method of determining e/m made use of magnetic deflection only, and so involved the assumption that the velocity was constant and known. It was therefore not capable of giving more than a rough approximation and could not lead to separation of any isotopes which may exist. In the experiments described below, which are of a preliminary nature, I have tried to improve on their method in these points.

Anode rays are formed when a mixture of various metallic chlorides and iodides with graphite is used as the anode in a discharge tube under reduced pressure. They are visible as a slightly diverging pencil of light starting from the surface of the anode and very roughly normal to it. In order to investigate the value of e/m for the particles which cause this light I placed in the path of the beam, and 2 or 3 cm. from the anode, an insulated aluminium plate pierced with a hole in which was fastened the end of a fine tube of ·3 mm. bore. The parallel beam of rays thus produced was analysed by coincident electrostatic and magnetic fields in the usual manner for positive rays†, and the deflected particles allowed to strike on a photographic plate. With exposures of about 20 minutes well-marked results have been obtained, the particles affecting the photographic plate in the same manner as positive rays.

Using a paste composed of a mixture of KI, LiCl and graphite, short parabolic arcs appeared whose magnetic displacements were

* *Verh. d. Phys. Gesell.*, 8, p. 559; 9, pp. 76, 200, 376; 10, p. 217.
† *Rays of Positive Electricity*, J. J. Thomson, p. 20.

in the ratio $\sqrt{7}$ to $\sqrt{39}$, indicating that they were due to Li and K respectively. On one plate a faint line appeared corresponding to about 140 atomic weight which is probably due to iodine, the accuracy being poor for large atomic weights.

In order to confirm the interpretation of these lines an absolute determination of e/m was made from measurements of the electric and magnetic forces. Taking e/m for the hydrogen atom as 9571 the measurements gave an atomic weight for the more deflected lines of 6·9, assuming a single charge, thus confirming the theory that they are due to single charged atoms of lithium. For accurate work a better method would be to use the apparatus, with an ordinary anode and the aluminium plate as cathode, to give positive rays, and use these to calibrate the plate; it may even be possible to get both anode and positive rays simultaneously.

Owing probably to an irregularity in the electric field the lines so far obtained are not very sharp, but one plate has been obtained where the width of the line due to lithium is no larger than the separation due to a difference of a unit in atomic weight. There is no trace of doubling, and it seems clear that if lithium consists of a mixture of isotopes with atomic weights differing by whole numbers, only that corresponding to an atomic weight of 7 occurs in any considerable quantity. Further experiments are in progress with the object of improving the definition and detecting small proportions of isotopes if these exist.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1919—1920.

ANNUAL GENERAL MEETING.

October 27, 1919.

In the Comparative Anatomy Lecture Room.

Mr C. T. R. Wilson, President, in the Chair.

The following were elected Officers for the ensuing year:

President:

Mr C. T. R. Wilson.

Vice-Presidents:

Prof. Marr.
Prof. Sir W. J. Pope.
Prof. Sir E. Rutherford.

Treasurer:

Prof. Hobson.

Secretaries:

Mr Alex. Wood.
Mr G. H. Hardy.
Mr H. H. Brindley.

Other Members of Council:

Mr F. F. Blackman.
Prof. Sir J. Larmor.
Prof. Eddington.
Dr Marshall.
Prof. Baker.
Prof. Newall.
Dr Fenton.
Prof. Inglis.
Prof. Seward.
Dr Rivers.
Dr E. H. Griffiths.
Mr F. A. Potts.

The following were elected Fellows of the Society:

A. C. Banerji, B.A., Clare College.
W. E. H. Berwick, M.A., Clare College.
B. M. Jones, M.A., Emmanuel College, Professor of Aeronautical Engineering.
K. Prosad, B.A., St John's College.

The following were elected Associates:

T. A. Browne, Trinity College.
J. Chadwick, Gonville and Caius College.
Miss L. V. Craies, Newnham College.
H. Henderson, Gonville and Caius College.
F. P. Slater, Gonville and Caius College.
E. F. Vacin, Trinity College.

The future policy of the Society, with special reference to the form and character of its publications, and the conduct of its meetings, was discussed.

November 10, 1919.

In the Comparative Anatomy Lecture Room.

Mr C. T. R. Wilson, President, in the Chair.

The following was elected an Associate:

Donald A. MacAlister, King's College.

The following Communications were made to the Society:

1. Colourimeter Design. By H. Hartridge, M.D., King's College.

2. A note on photosynthesis and hydrogen ion concentration. By J. T. Saunders, M.A., Christ's College.

3. (1) The effects of some electrolytes upon spermatozoa.
(2) The effects of ions upon ciliary movement.
By J. Gray, M.A., King's College.

4. Note on the solitary wasp, *Crabro cephalotes*. By C. Warburton, M.A., Christ's College.

5. Preliminary note on the life-history of a Proctotrypid (*Lygocerus sp.*) hyperparasite of *Aphidius*. By Miss M. D. Haviland. (Communicated by Mr H. H. Brindley.)

6. The Natural History of Rodriguez, *with Exhibits*. By H. J. Snell and W. H. Tams. (Communicated by Professor Stanley Gardiner.)

November 24, 1919.

In the Cavendish Laboratory.

Mr C. T. R. Wilson, President, in the Chair.

The following were elected Fellows of the Society:

J. Line, M.A., Emmanuel College.
L. F. Newman, M.A., Downing College.
A. D. Ritchie, M.A., Trinity College.
J. B. Seth, B.A., Trinity Hall.
G. I. Taylor, M.A., Trinity College.
G. P. Thomson, M.A., Corpus Christi College.
S. M. Wadham, M.A., Christ's College.

The following were elected Associates:

Miss M. D. Haviland, Newnham College.
L. W. G. Malcolm, Christ's College.

The following Communications were made to the Society:

1. (1) Photographs of a Solar Prominence taken during the eclipse of 1919 May 29.

(2) The theory of relativity and recent eclipse observations.

By Professor Eddington and E. T. Cottingham.

2. (1) The Hydrodynamical theory of the Lubrication of a Cylindrical Bearing under Variable Load, and of a Pivot Bearing.

(2) The pressure in a viscous liquid moving through a channel with diverging boundaries.

By W. J. Harrison, M.A., Clare College.

January 26, 1920.

In the Comparative Anatomy Lecture Room.

Prof. Sir Ernest Rutherford, Vice-President, in the Chair.

The following was elected a Fellow of the Society:

E. A. Milne, B.A., Trinity College.

The following were elected Associates:

A. H. Compton.
T. Kikuchi, St John's College.
D. F. Scanlan, Jesus College.
G. Shearer, Emmanuel College.

The following Communications were made to the Society:

1. Gravitation and Light. By Prof. Sir Joseph Larmor.

2. Note on Mr Hardy's extension of a theorem of Mr Pólya. By E. Landau. (Communicated by Mr G. H. Hardy.)

3. On a Gaussian series of six elements. By L. J. Rogers. (Communicated by Mr G. H. Hardy.)

February 9, 1920.

In the Cavendish Laboratory.

Mr C. T. R. Wilson, President, in the Chair.

The following were elected Fellows of the Society:

M. D. Bhansali, B.A., St John's College.
D. C. Henry, B.A., Trinity College.

The following Communications were made to the Society:

1. The Mass Spectra of the Chemical Elements. By F. W. Aston, M.A., Trinity College.

2. An examination of Searle's method for determining the viscosity of very viscous liquids. By Kurt Molin. (Communicated by Dr G. F. C. Searle.)

3. Note on the Diophantine equation $t^3 + x^3 + y^3 + z^3 = 0$. By H. W. Richmond, M.A., King's College.

4. Mathematical notes: (1) On the stability of rotating liquid ellipsoids; (2) on the general theory of the stability of rotating masses of liquid; (3) on the stability of periodic motions in general dynamics; (4) on the invariance of the equations of electrodynamics in the Maxwell and in the Einstein forms; (5) on a property of focal conics and of bicircular quartics; (6) on the Hart circle of a spherical triangle; (7) on a proof of the theorem of a double six of lines by projection from four dimensions; (8) on a set of transformations of rectangular axes; (9) on transformations with an absolute quadric; (10) on the reduction of homography to movement in three dimensions. By Professor H. F. Baker.

February 23, 1920.

In the Botany School.

Mr C. T. R. Wilson, President, in the Chair.

The following were elected Fellows of the Society:

E. Cunningham, M.A., St John's College.
M. C. Vyvyan, B.A., Gonville and Caius College.

The following Communication was made:

The Origin of the Vegetation of the Land. By Professor Seward.

Mr Lister exhibited and described a large collection of butterflies.

Many exhibits of botanical interest were shewn.

March 8, 1920.

In the Cavendish Laboratory.

Mr C. T. R. Wilson, President, in the Chair.

The following was elected a Fellow of the Society:

Terry Thomas, M.A., St John's College.

The following Communications were made to the Society:

1. Further notes on the food plants of the Common Earwig (*Forficula auricularia*). By H. H. Brindley, M.A., St John's College.

2. Preliminary note on antennal variation in an aphis (*Myzus ribis*, Linn.). By Miss Maud D. Haviland. (Communicated by Mr H. H. Brindley.)

3. Studies on Cellulose Acetate. By Dr Fenton and A. J. Berry, M.A., Downing College.

4. The rotation of a non-spinning gyrostat, and its effect on the aeroplane compass. By G. T. Bennett, M.A., Emmanuel College.

5. Lagrangian methods for high-speed motion. By C. G. Darwin, M.A., Christ's College.

6. The effect of a magnetic field on the intensity of spectrum lines. By H. P. Waran. (Communicated by Professor Sir Ernest Rutherford.)

7. Generation of sets of four tetrahedra mutually inscribed and circumscribed. By C. V. Hanumanta and Professor H. F. Baker.

8. On the term by term integration of an infinite series over an infinite range and the inversion of the order of integration in repeated infinite integrals. By S. Pollard, M.A., Trinity College. (Communicated by Professor G. H. Hardy.)

9. On rotating liquid cylinders. By S. R. U. Savoor, B.A., Trinity College.

May 3, 1920.

In the Cavendish Laboratory.

Professor Sir Ernest Rutherford, Vice-President, in the Chair.

The following were elected Fellows of the Society:

N. K. Adam, M.A., Trinity College.
P. A. Buxton, M.A., Trinity College.
N. M. Shah, B.A., Trinity College.
F. P. White, M.A., St John's College.

The following was elected an Associate:

K. W. Braid, Fitzwilliam Hall.

The following Communications were made to the Society:

1. Notes on the Theory of Vibrations. By W. J. Harrison, M.A., Clare College.

2. On cylical octosection. By W. Burnside, M.A., Pembroke College.

3. (1) A bifilar method of measuring the rigidity of wires.
(2) An experiment on a piece of common string.
(3) Experiments with a plane diffraction grating, using convergent light.

By Dr G. F. C. Searle, Peterhouse.

4. Congruences with respect to composite moduli. By Major P. A. MacMahon.

5. Equivalence of different mean values. By Alfred Kienast. (Communicated by Professor G. H. Hardy.)

6. Construction of the ninth intersection of two cubic curves passing through eight given coplanar points. By Professor H. F. Baker.

7. Quintic transformations and singular invariants. By W. E. H. Berwick, M.A., Clare College.

May 17, 1920.

In the Cavendish Laboratory.

Mr C. T. R. Wilson, President, in the Chair.

The following were elected Fellows of the Society:

W. M. H. Greaves, B.A., St John's College.
W. M. Smart, M.A., Trinity College.

The following was elected an Associate:

T. Shimizu.

It was announced that the adjudicators for the Hopkins Prize had made the following awards:

For the period 1903–06 to W. Burnside, M.A., F.R.S., of Pembroke College, for investigations in Mathematical Science. For the period 1906–09 to G. H. Bryan, Sc.D., F.R.S., of Peterhouse, for investigations in Mathematical Physics, including aerodynamic stability. For the period 1909–12 to C. T. R. Wilson, M.A., F.R.S., of Sidney Sussex College, for investigations in Physics, including the paths of radioactive particles.

The following Communication was made to the Society:

The atomic nature of matter in the light of modern physics. By F. W. Aston, M.A., Trinity College. With Experiments.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

The Problem of Soaring Flight. By E. H. Hankin, M.A., Sc.D., late Fellow of St John's College, Cambridge, Chemical Examiner to Government, Agra, India. (Communicated by Mr H. H. Brindley.)

[*Read* 22 November 1920.]

With an introduction by

F. Handley Page, C.B.E., F.R.Aer.S.

INTRODUCTION.

By F. Handley Page, C.B.E.

The study of bird flight has always fascinated those who were interested in the early development of aviation, and all the original attempts at heavier-than-air flight were based on imitation of birds, both in constructive and in propelling mechanism.

Progress from such study was, however, well nigh impossible. The bird represents the finished article of millions of years of slow development to suit the difficult condition of taking the best advantage of the air structure in which it had to fly. Observers therefore had not only to study a complicated mechanism to find its basic principle of operation, but also to do so in a medium whose movements were but imperfectly understood.

The methods of observation were crude, and the observers not trained for the work of exact measurements and recording of the results. A great many of the early observers, from Leonardo da Vinci down to Weiss and others of the present day, were artists accustomed to observe with the trained eye the picture presented to their mind by bird flight, and from that they endeavoured to reproduce empirically in a mechanical form the design which they deemed the most successful in bird flight.

Directly engineering and scientific thought was directed to the study of flight problems the component parts were reduced to

simple and elementary forms. Instead of the complicated supporting *plus* propelling mechanism of the wing of varying plan form and cross section the aeroplane was developed with a wing of simple geometrical shape which was used for support only and with a screw propelling mechanism quite distinct from its wings.

Great progress has been made in the design of aircraft based on mechanical methods of investigation in wind channels, where the air is driven at uniform speed with little or no turbulence against the model to be tested. There still remains a vast amount to be discovered in the actual air structure in which aircraft fly and in the aerodynamical design best suited to fit the conditions discovered. Here, without a doubt, lies the value of a careful and considered study of soaring flight. In this country, owing to the complicated and ever-changing meteorological conditions, research is difficult. In continental and particularly tropical climates conditions are more stable and investigation easier. In India Dr Hankin has been particularly fortunate in having most favourable conditions for observation and in bringing to bear on the problem observing powers trained to a high degree of accuracy by long years of practice.

The measurements made and the facts discovered relating to soaring flight are so extraordinary as to awaken suspicions at once as to their accuracy. Were they but chance observations of an observer made at odd intervals such criticisms would be just, but the careful system of investigation and re-investigation continued by Dr Hankin over a long period of years makes his results worthy of consideration by all interested in flight phenomena and open up an entirely new field for aviation and meteorological research.

The high soaring speed of 50 miles per hour over the plains of India without any discoverable wing movement brings up visions of possible motorless flight in tropical climates, where conditions of visibility and weather are ideal for this form of transport. To all those who are interested in aviation development in India and other tropical countries, Dr Hankin's discoveries must be of the greatest interest and his published results well worthy of most careful study.

The results are by no means on all fours with present aircraft design. The wing sections of the best soaring birds and of the soaring dragon-flies are characterised by ridges projecting on the under side of the wing, forming to the eye of the aeroplane designer a source of resistance entirely uncalled for. On the other hand, birds of more streamline wing cross section, where these ridges are not present to anything like the same extent, invariably fly by flapping their wings, and have little or no power to soar. Can it be that the more crude cross sections fit in with, and thus take advantage more readily of, some form of air turbulence or movement and so allow

the soaring bird to draw on some at present unknown source of energy—kinetic or other—in the air?

Another observation is of interest. Birds of the smaller size are more lightly loaded than the larger birds. Thus the cheel, with a span of 51 inches, is loaded ·5 lb. per sq. ft. as compared with the vulture, with a span of 85 inches and loading of 1·5 lbs. per sq. ft. Each, however, in similar conditions of soarable air glides at approximately the same speed. According to aerodynamical theory, the speed of the vulture should be approximately $\sqrt{3}$ times as great, the loading being three times as great—if we neglect bodily resistance. Even if this is taken into account, their speeds should not be anything like the same. The explanation lies probably in "scale effect." The larger bird is, owing to its larger dimensions, able to lift more in proportion to its area, just as the full size aeroplane can compared with the model. Here nature, faced with the problem of making large birds, has avoided the increased percentage of wing weight due to increased span by concurrently increasing the lifting power of the wing with increase in area.

Apart from the study of natural soaring flight its mechanical equivalent has aroused interest and a good deal of research has been carried out on the Continent. In Germany, Gustav Lillienthal, brother of the famous Otto Lillienthal, has, according to a paper published by him in 1917*, designed a wing section of alleged similarity to that of a soaring bird, and with a glider of this wing cross section a German engineer, Friedrich Harth, claims to have flown 500 metres against a wind of 12 metres per second at a height of 40 metres.

Such attempts are, however, premature, and the results unauthenticated. Firstly, we want to know whether soaring birds can soar—as apparently they can—for indefinite distances, provided weather conditions are suitable. Secondly, if such flight is as effortless as it appears to be, from what source is the energy obtained and how the bird is able to take advantage of it. When this is discovered, then will be the time for practical application. Its importance for aviation work in tropical climates has already been referred to.

Dr Hankin's discovery of the soaring flight of dragon-flies and flying-fishes and its similarity in speed and other respects to that of the soaring bird affords a means of more closely investigating the phenomenon of soaring. Observation of birds soaring at 2000 or 3000 feet is difficult compared with the observation of insect flight at a few yards' distance. The whole subject demands most careful investigation and merits the attention of all those interested in the scientific development of the world's latest form of locomotion.

* "Der Einfluss der Flugelform auf die Flugart der Vögel" (*Sitzungsberichte der Gesellschaft naturforschender Freunde*, Berlin, 1917, No. 4).

THE PROBLEM OF SOARING FLIGHT.

By E. H. Hankin, M.A., Sc.D.

This paper contains a short summary of existing evidence as to the nature of soaring flight.

1. *Comparison of soaring flight in birds, dragon-flies and flying-fishes.*

Birds, dragon-flies, and flying-fishes can exhibit two kinds of soaring flight—slow and fast—characterised by different wing dispositions.

As a rule, both with birds and flying-fishes (XI)*, slow soaring flight is carried out with wings dihedrally-up (*i.e.* with wing-tips at a higher level than the body). In fast soaring flight the wings are either flat or occasionally in the case of flying-fishes dihedrally-down. In dragon-flies the wings are strongly dihedrally-up in slow flight and either flat or less dihedrally-up in fast flight (XII).

In each of the three classes of soaring animals slow soaring flight is dependent, as a rule, on the presence of sunshine and fast flight is always dependent on the presence of wind (IV, pp. 52, 98, 251, 299).

With birds and flying-fishes lateral instability occurs more often late in the afternoon, when the air is becoming unsuited for soaring flight, than at other times of the day (IV, p. 294 and XI). A rare form of lateral instability in which the oscillations are just too rapid to count has been observed in each of the three classes of flying animals.

The speeds attained in soaring flight seem to be remarkably similar in the three classes. The slow speed flight is between 5 and 10 metres per second. The high speed flight has been estimated for dragon-flies at above rather than below 15 metres per second (XII). For flying-fishes, in strong winds, it may exceed 20 metres per second (XI). The air speed of vultures has been measured and found to reach a mean speed of 20 metres per second in winds of medium strength (XI).

In view of the above resemblances between the soaring flight of birds, flying-fishes and dragon-flies there seems to be no room for doubt that we are dealing with the same phenomenon in each case.

2. *The regularity of soaring flight.*

A remarkable, though familiar, phenomenon is shown by a flock of cranes in soaring flight. The birds may be seen to keep their distances from each other with marvellous exactitude and this not only when they are gliding in a straight line but also when they are

* Roman numerals in brackets refer to the appended list of my publications.

on a curved course. When watched through a binocular they resemble a number of dead birds pinned on a blue wall (IV, pp. 60 and 61).

The regularity of soaring flight thus shown conspicuously by cranes, and less strikingly by all soaring animals that fly in groups, furnishes a clear proof that the energy involved does not come from any chance or irregular currents of air.

Two alternatives appear to be suggested. Either such regular flight is due to undiscovered wing movements or its cause must be some condition widely and uniformly distributed in the atmosphere.

3. *The flight of the puttung.*

The puttung is a kite made of paper and bamboo in common use by Indian boys.

The chief peculiarity of its flight is that, in a suitable wind, it flies vertically over its string and when so flying its flight is particularly stable. To achieve this result the front limb of the bridle that attaches it to the string must be a little shorter than the hind limb.

This mode of flight can only be explained by the supposition that this kite takes energy from the air after the manner of the soaring bird.

Its flight resembles that of the soaring bird in that it is more liable to show lateral instability late in the afternoon than at other times of the day.

When struck by a gust it may, for a few seconds, fly up wind in advance of the vertical. Similarly, birds and dragon-flies soaring up wind when struck by a gust often show an increase of speed.

The structure of this kite is such that when it is exposed to wind pressure there must be a ridge on its under surface that lies transverse to the line of flight. In all the more efficient soaring animals ridges transverse to the line of flight are present on the under surfaces of the wings (IV, pp. 242 and 341).

If it is admitted that the flight of this kite is an instance of soaring then obviously the idea of undiscovered wing movements must be given up and, also, a means is indicated by which the phenomena of soaring may be submitted to an experimental investigation.

4. *Soaring flight not due to undiscovered wing movements.*

(*a*) Flying-fishes can check their speed by hanging their hind wings downwards. They habitually do so towards the end of a flight if it has been carried out at high speed. But in the highly soarable monsoon winds they often use this air brake during the whole of a flight and, as may be seen, with this brake in action

they attain less speed than other fishes near them that are not using the adjustment (XI).

If their flight was due to undiscovered wing movements why should they not be able to check it by decreasing these movements?

(*b*) Dragon-flies can check speed by hanging down the abdomen and hind legs (IX and XII). Those kinds of soaring dragon-flies that habitually soar in a group over a restricted area commonly use this air brake, when in continued flight, between about 11 a.m. and 3 p.m., if there is strong sunshine. If small clouds pass over the sun the brake is taken out of use to be applied again when the sun is clear. This happens even when the clouds are too thin to throw a shadow or to cause any appreciable decrease in the intensity of the sunshine. After the sun comes out there is a latent period of about 23 seconds before the brake is reapplied (XII). These facts give a striking proof that, in fine weather, the energy for their flight is derived from sunshine and are quite inconsistent with the idea that their soaring is due to undiscovered wing movements.

(*c*) Among birds the albatross furnishes a proof that soaring flight is not due to undiscovered wing movements. Observers are agreed that this bird cannot soar in a calm when near sea level (VII). If its soaring flight was due to wing movements, why, it may be asked, should it be unable to execute these movements in the absence of wind?

5. *Soaring flight is not due to the effect of lateral gusts of wind.*

If soaring animals habitually carried their wings inclined so that the wing-tips were at a higher level than the body, then it is conceivable that soaring flight might be due to the effect of lateral gusts which, striking the underside of the wings from one side or the other, would give a succession of lifting impulses and hence keep the bird aloft.

This idea is negatived by the fact that in high speed flight the wings are placed in the flat position. When this is the case lateral gusts can have no lifting effect. Flying-fishes sometimes place their wings dihedrally-down, *i.e.* the wing-tips are at a lower level than the bases of the wings. This disposition is probably that used in flight at highest speed (XI). If lateral gusts existed and were operative their only effect in this case would be to drive the fish under water.

Though wind is favourable to the high speed flight of dragon-flies, the clearest proof exists that, in calm air in the early morning, when soarability is developing under the influence of sunshine, the coming of the lightest draught of wind causes soaring to be replaced by flapping (XII). This observation is entirely inconsistent with the idea that soaring flight is due to the effect of lateral pulsations.

6. *Soaring flight is not due to the effect of ascending currents.*

(*a*) If isolated clouds are passing over the sky soaring dragon-flies are apt to collect in the neighbourhood of a convenient ascending current and glide into it whenever the sun is observed. They glide out of it and keep out of its range as soon as sunshine returns. The behaviour of dragon-flies is such as to suggest that ascending currents are in some way inimical to their soaring flight (XII).

(*b*) Inland birds similarly avoid ascending currents so long as the sun is shining during the day time. They use them when the sun is obscured and also both in the early morning and late in the afternoon when the air is not soarable. Under a cloudy sky in disturbed weather some winds are soarable and some are un-soarable. In the latter case only, do birds make use of ascending currents (IV, pp. 20 and 283).

7. *Convection currents and soarability.*

Ample evidence exists that convection currents in the air caused by the heat of the sun, whether at ground level (IV, p. 263) or at a height (IV, p. 23) have nothing to do with soarability.

8. *The theory that turbulent motion in the air may be the source of the energy of soaring flight.*

In favour of this idea the following facts may be noticed:

(*a*) Atmospheric turbulence, in fine weather, decreases towards sunset. At this time soarability for inland birds and dragon-flies comes to an end at ground level. But soarability at this level may persist after sunset in the presence of stormy winds when turbulence must obviously continue (IV, pp. 80, 281 and 375). In the case of flying-fishes a loss of soarability has been observed shortly after sunset in the absence of appreciable wind (XI).

(*b*) The stronger the wind the more turbulence is likely to be present. The speeds attained by vultures in horizontal soaring flight have been measured and found to be greater the stronger the wind. In winds of 10 to 20 metres per second the mean air speed of the vulture has been found to be about three times the speed the same bird reaches in calm after sunset when in flapping flight. As above stated, both cheels and dragon-flies often show an increase of speed when struck by a gust (IV, pp. 250 and 377, XII).

9. *Difficulties in accepting turbulencies whose effects can be seen as an explanation.*

(*a*) Cheels have been observed catching locusts. Each locust flew in a straight line whether it was flapping, as was usually the case, or gliding downwards as sometimes occurred. The cheels were

in gliding flight and always glided at a faster rate than the locusts. This was the case whether they were travelling horizontally or gliding upwards at an angle of 10 or more degrees with the horizon. Such upward gliding was often a continuation of a horizontal course. It was not due to any momentum obtained by a preceding glide in a downward direction (X).

How, it may be asked, could turbulent motion be present of sufficient force to propel the cheels and yet to have no visible effect on the course of the locusts? If, as is the fact, in virtue of its larger size, the cheel is a more efficient flying machine than the locust, one would expect the latter to be the more readily deflected or influenced by turbulence. Had dragon-flies been present, animals smaller than locusts, there is no doubt that they would have been seen flying faster than the latter.

(*b*) A dragon-fly whose flight is mostly by flapping (*Rhyothemis variegata*) is common near Calcutta. A soaring dragon-fly may sometimes be seen gliding to and fro in a group of these flapping dragon-flies. Its speed is distinctly faster than that of the latter (IV, p. 388). What form of pre-existing air movement can be imagined that propels the soaring dragon-fly and yet has no effect on the flapping dragon-fly though the latter is of lighter weight and loading?

(*c*) Cheels have been seen soaring in air containing small masses of discrete cloud material which were so numerous that the movement of any one cubic foot of air relatively to that of adjacent cubic feet could be seen if it existed. No such relative movement was visible. It is difficult to see how such an observation is consistent with the idea that turbulent movement is the cause of soaring flight (IV, p. 104).

For soaring flight to be possible under such conditions a strong glare from the sun shining through cloud is necessary. Glare due to light reflected from a cloud seems unable to furnish the energy necessary for soaring flight (IV, pp. 102 and 105, II, p. 24).

(*d*) Vultures have been seen gliding in air that was so full of aerial seeds that it looked like a snowstorm. These floating seeds were in slow equable movement that showed no turbulent motion such as one might expect to be necessary to propel a bird weighing ten or twelve pounds (IV, p. 102).

(*e*) Small feathers have often been seen floating in the air in the midst of soaring birds. These feathers in their course showed only the smallest deviations from a straight line. Sometimes a slow partial rotation of a feather may be seen but any evidence of energetic turbulent movement is conspicuously absent (IV, p. 57).

On one occasion a floating feather was seen to pass directly under the wing of a cheel. It was instantly shot out sideways to a distance of several metres from the bird, thus yielding evidence of

active air movement which, so far as is yet known, must have been the effect rather than the cause of the bird's progress.

10. *Conclusions.*

This brief summary of the facts observed by me during the last ten years amply supports the view that soaring flight is inexplicable in the light of existing knowledge.

In the case of soaring flight at slow speed a proof exists that the energy involved is derived from the sun's rays. But the mode by which it becomes available to the soaring animal is, as yet, a complete mystery. Direct observation having failed to point the way to a solution, it is to be hoped that the subject will be attacked with the aid of an experimental investigation. It is only in this way that an explanation of the problem is likely to be attained.

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Preliminary Note on the Superior Vena Cava of the Cat. By W. F. Lanchester, M.A., King's College, and A. G. Thacker.

[*Read* 22 November, 1920.]

The present series of observations arose out of the more or less accidental discovery of a discrepancy between the factors of the Superior Vena Cava in a cat that we were dissecting and the description of these veins given in St George Mivart's well-known text-book on the Cat. It will be remembered that in the cat the two innominate veins (or, as they are otherwise called, the brachiocephalic veins) are formed by the union, on either side, of the External Jugulars and Subclavians, the two Innominates then uniting with each other slightly to the right of the trachea to form the single Superior Vena Cava. The right Innominate is of course the anterior portion of the surviving Superior Vena Cava. Now, according to Mivart's description, the Internal Jugular (of either side) runs into the corresponding Innominate. In our first cat we found on the contrary that the Internal Jugular ran into the External Jugular, or in other words, that the vein which unites with the Subclavian is not really an External Jugular but a Common Jugular. There are several American text-books on the cat, and we found that they agreed with our own first observations, but none of them refer to the different description given by Mivart, neither do any of them mention another vein, which does in fact usually join the Innominate (on one side only) at the point Mivart indicates for the Internal Jugular and which is usually much more conspicuous in dissection than the Internal Jugular itself. This is the vein coming from the thyroid glands, which, though it is of course a paired structure in the anterior, is usually single in the posterior part of its course. We decided to clear up this small point regarding the position of the Internal Jugular. The American text-books are correct. Out of 30 cats dissected, in 29 the Internal Jugular entered the External Jugular before the union of the latter with the Subclavian. In one small kitten the three veins—the Subclavian, the External Jugular, and the Internal Jugular—appeared to meet one another at the same point. Our observations show, therefore, that there is normally a Common Jugular in the cat, and this vein is often of considerable length. It is often longer, sometimes much longer, than the corresponding Innominate. The Thyroidean, as stated, usually falls into one or other of the Innominates; but it sometimes falls into the Common Jugular or even into the Internal Jugular. In two cases the Thyroideans were separate throughout their course.

Whilst investigating this point we began another series of observations on the Superior Vena Cava itself, which will perhaps be of more general interest than the point of anatomical detail just described. This work is still proceeding, but the results already obtained are perhaps of sufficient interest to justify a brief summary of them.

Of the large changes in the circulatory system by far the most recent is that from the condition of two Superior Venae Cavae to that of one Superior Vena Cava, and the latter condition has apparently been evolved independently in more than one order of the Mammalia. The nature of the variation in the single Vena Cava would therefore appear to have considerable interest. Cases of two Superior Venae Cavae have, we believe, been recorded for the human subject, and at least one such case is recorded for the dog. The Superior Vena Cava of the cat is not only large but long, being normally, as we shall see in a minute, almost half as long as the trachea. It is normally about four times as long as the Innominate Veins, which are, therefore, very short vessels running almost at right angles to it. We decided to measure the variation in the length of the Vena Cava, taking as our standard the length of the trachea from the posterior edge of the cricoid cartilage to its bifurcation into the two main bronchi. The trachea proved to bear a fairly constant relation to the total length of the animal, exclusive of the tail, but we checked our standard by making other measurements. The length of the trachea varied from 21% of the total length to 26·7%. To summarise the results, we found it necessary to exclude the kittens from the scheme as they showed in many respects very anomalous relations between the sizes of their different organs. This left us with only twenty-one specimens, all these being over 40 cms. in length. Taking the length of the trachea as 100, the extreme limits of variation in the length of the Vena Cava were 19 and 47·9. But the majority of the cats are by no means massed about the mean between these extremes. On the contrary, the great majority, 16 out of 21, are massed close to the upper limit. These 16 varied from 38 to 47·9. This is obviously the normal type and in it the Vena Cava is approximately four times the mean length of the Innominates. And it may be said that in this type the blood coming from the fore-limbs and head runs into a single channel at the earliest possible moment. There was one case in which the Vena Cava sank to just under 35 %, but which, nevertheless, probably belongs, we think, to the normal category; because this was the cat which had the longest trachea (26·7 %), and if the Vena Cava be judged either by the total length of the animal or by the Innominates, it comes up to the normal length. Then there are three other cases which varied from 29 % to 33 %. These are very possibly a different type, but the numbers

are at present too small for us to speak with confidence on this point. In this second type the Vena Cava was approximately twice the length of the Innominates.

Finally, we had one animal which was clearly in quite a different class. In this specimen the Vena Cava was only 19 % the length of the trachea and the two Innominates were themselves swollen almost to the proportions of Venae Cavae. The mean length of the Innominates was, in fact, slightly greater than that of the Vena Cava. It is owing to the existence of this specimen that, notwithstanding the small total number, we think the curve of variation cannot be that of normal continuous variation. In view of the evolution of the single Vena Cava we think that even these preliminary results may be of some interest and, at present, they would appear to be congruous with a discontinuous method of evolution of the single Vena Cava condition.

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A Note on Vital Staining. By F. A. Potts, M.A., Trinity Hall.

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Great confusion exists as to the phenomena which are classed under the term 'vital staining' but it is established that certain stains can penetrate the living cell and enter into combination with bodies in the cell without apparently affecting the normal course of cell life. These bodies are principally granules of various kinds and some of them are undoubtedly concerned with the process of secretion. How far they are actually part of the living protoplasm is a point at issue. Very little work has been done, in the Metazoa at least, on the further history of stained granules of this kind, and it seems to offer a promising field for students of vital processes. Exceedingly interesting in this connection are the observations of Oxner* on Nemertines, the mucus cells of which contain numerous granules staining with neutral red and methylene blue but produce colourless mucus. The earlier work of Apathy† had brought to light similar effects and is more satisfactory in that the author states that he has seen in the mucus cells of Hirudinea granules stained with methylene blue actually forming blue mucus. It is not obvious that Oxner had definitely traced the genesis of his colourless mucus from stained granules.

The Nematoda are a group which offers wonderful facilities for researches of this kind, for the smaller members of the phylum are of an almost perfect transparency, are exceedingly hardy under experimental conditions, and contain many kinds of granules which take up vital stains. Cobb‡ has already indicated this extraordinary suitability of the free-living forms in particular and appealed to scientific workers to give their attention to these and other related problems. I wish in this paper to briefly summarise some observations made on a species of *Diplogaster* which was found in garden soil in Cambridge during 1919.

Neutral red was found to be by far the most effective stain. In fact no others of those used showed up the granules which I go on to describe here. The solutions employed were very concentrated especially if compared with the exceedingly dilute ones which give the best result for freshwater Crustacea (Fischel) and marine organisms. A solution made up more than ten years ago of neutral red in distilled water was especially good, its staining properties had evidently improved with age and it was almost non-toxic.

* Oxner, *Bull. de l'Institut. Oceanograph*, Monaco, 1908.

† Apathy, *Zeitschr. wiss. Mikr.*, Bd. 9, 1892. *Cf.* also Hardy, *Journ. Physiol.*, vol. XIII, 1892.

‡ Cobb, *Natural Science*, vol. 46, 1917, p. 167.

The nematode is surrounded by an elastic cuticle which is continued into the gut at the mouth and anus. Anteriorly it lines the buccal cavity and oesophagus and posteriorly the short rectum. When the animal moults this lining is detached with the external cuticle and retains its shape after detachment. The midgut or intestine consists of a single layer of cells which according to some authorities (*Cambridge Natural History*, Vol. II. p. 131) is 'coated internally and externally by a layer of cuticle.' In my experience, however, of free-living forms the cells of the midgut do not possess a firm and definite cuticle like that of the oesophagus and rectum but the cell membrane is thin and without any apparent structure.

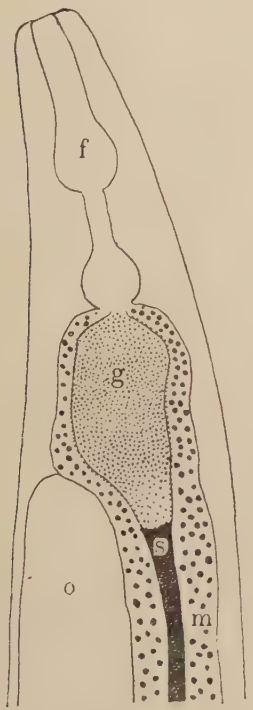
The frequent pulsation of the second oesophageal bulb is the means by which the liquid culture medium is pumped into the gut of the nematode. Hence, when the animal is put into a solution of stain this rapidly penetrates into the cavity of the oesophagus and the anterior part of the midgut. It is some time, however, before the whole lumen of the gut is filled with the stain. But none of the neutral red ever penetrates through the external cuticle. The cuticle of the buccal cavity and oesophagus is apparently porous enough to allow the penetration of a certain amount, but the greater part passes through the cell membranes of the midgut. When a nematode has been in stain for about six hours it presents a remarkable appearance, the gut being stained pink or purple in its various parts while the remaining tissues are still perfectly transparent. The stain is taken up by the various scattered granules of the gut-cells and when their capacity for stain is exhausted, which does not occur till the experiment has lasted about twelve hours, the neutral red begins to pass through the external membrane of the gut and irregular deposits appear in the genital organs and the lateral fields of the hypodermis. Lastly, the muscle fibres take up the stain, showing a transverse banding of the individual fibres. It does not appear that the slow penetration is due to the resistance of the external membrane of the gut, but merely to the avidity with which the gut granules take up the stain.

In the midgut cells of *Diplogaster* these granules fall into two categories: (1) brownish highly refractive granules of various sizes, resistant to most reagents and scattered irregularly through the cell. Their composition is unknown, but they take up a good deal of stain. It is not proposed to deal with them in the present paper. (2) Smaller secretory granules of uniform size forming a peripheral zone round the lumen of the gut, normally colourless, but purple with neutral red. They are especially thickly developed in the first four cells between which the gut cavity is enlarged and often occupied by a bolus of living or dead bacteria.

There are, however, individuals which do not show the granular zone, but instead the gut-lumen appears to be lined with a structure-

less highly refractive layer which stains a brownish-red. Some Diplogasters show the granules anteriorly in the gut and the structureless layer posteriorly, but never both together. The explanation of this distribution is given by observing that the granules break down and form the substance of the structureless layer. This is a change which can be watched, beginning posteriorly and working slowly forward until the whole of the midgut is lined by the substance.

The structureless layer is not a firm cuticle as is shown by the fact that when a stained *Diplogaster* is compressed under a cover-



Diagrammatic figure of *Diplogaster* sp. to show staining of midgut with neutral red after about six hours. In the specimen figured the individual is in an intermediate state, the granules being well developed anteriorly but have broken down posteriorly to form the secretion. The cell boundaries and nuclei of the midgut are omitted.

f. oesophagus.
g. granules surrounding lumen of midgut.
s. secretion formed by breaking down of these.
m. midgut with larger resistant granules.
o. ovary.

slip to such an extent that the contents of the body are squeezed out through the anus, as the gut lining passes through the narrow aperture it changes its form and flows like a plastic material. Moreover the lining of the midgut does not appear as a definite detached layer in moulting or after the death and disintegration of the animal, as is the case in the cuticle of the oesophagus and rectum.

That the granules and the substance they produce are not artifacts or products of degeneration is shown by the fact that they

are visible in unstained nematodes, taken fresh from cultures where they were growing under the most favourable conditions. Neutral red in this case only serves to differentiate pre-existing structures which are otherwise very difficult to make out.

The function of the layer is very doubtful. It might be supposed that it is digestive and the great development of the granules in the anterior part of the midgut where the lumen is enlarged and often contains an accumulation of bacteria and organic fragments would seem to support this view. But living bacteria exist embedded in or attached to the lining and appear to grow and divide freely there as in a culture medium. From this observation it seems possible that a symbiotic relationship exists between the bacteria and the nematode which is in some way furthered by the secretion of a soft lining to the gut.

While the granular phase is often seen passing into the plastic phase I have not succeeded in observing a natural evacuation of this lining preparatory to the formation of fresh granules in the gut-cells. I hope, however, that further study will reveal the relation of this phenomenon to the life of the nematode. To this end it will be necessary to find out whether it is a regularly, or at least frequently, repeated event.

Preliminary note on a Cynipid hyperparasite of Aphides. By Maud D. Haviland, Fellow of Newnham College. (Communicated by Mr H. H. Brindley.)

[*Read* 22 November 1920.]

Aphides are liable to parasitisation by certain Braconidae of the sub-family Aphidiidae. The larva develops in the haemocoele of the host, which dies just before the metamorphosis of the parasite, and the latter lines the empty skin with silk and pupates within it. Throughout its larval life the *Aphidius* is in its turn liable to parasitisation by certain Chalcids, Cynipids, and Proctotrypids, which are thus hyperparasites of the aphid. I described the development of one of these hyperparasites, a Proctotrypid, *Lygocerus cameroni*, Kieff., in a paper read before the Society last February. The following is a summary of some observations made on the development of certain Cynipid endo-hyperparasites of the genus *Charips*, formerly known as *Allotria*. The aphid used in the breeding experiments was *Macrosiphum urticae*, Kalt. from the nettle, and the primary, or host, parasite was *Aphidius ervi*, Hal.

These Cynipids have long been known to be hyperparasites, but at first I could not induce them to oviposit in captivity. The cause of this failure was that the material offered them was in too advanced a stage; for, unlike most of the hyperparasites of this group, which do not oviposit until the aphid is dead and the primary parasite is in metamorphosis, these Cynipidae seek out an *Aphidius* preferably in the third, or early in the fourth, instar, though a second instar larva may also be chosen. At this time the parasite is lying in the host's body cavity, and the aphis feeds as usual. Until twelve hours or so before its death there is no external sign that it contains a parasite, and yet the Cynipid unerringly recognises the presence of the latter and ignores unparasitised aphides when they are offered to it. A necessary condition for oviposition appears to be that the aphid host should be alive, and the *Aphidius* be still bathed in its body fluids. Larvae which had emptied the Aphid's skin of its contents, and had already begun to spin the cocoon, were never selected.

In captivity, ovipositions were sometimes, but not always, parthenogenetic. The female *Charips* ran among the aphides, tapping them excitedly with her antennae. When a victim was chosen, she leaped upon its back, facing the head, and clung there in spite of its struggles. Sometimes she was thrown off, but in such cases always returned to the attack, until the aphid became passive from

exhaustion. The oviposition took from three to six minutes to complete, which is not surprising when it is remembered that the ovipositor must be thrust through the chitin and body wall of the aphid before probing for the *Aphidius*. Even then the organ must possess an exquisite sense of touch, for the gut of the host larva is so distended with food that the haemocoele is correspondingly reduced; and if the ovipositor were to be driven in a fraction too far the egg might be deposited in the gut cavity, and be lost at evacuation of the meconium.

The egg is oval, with a pedicel or stalk at one end and a smooth chorion. As development proceeds it becomes more spherical and the stalk disappears. It hatches about two to three days after oviposition. As in certain other internal Hymenoptera Parasitica, a 'pseudo-serosa,' or envelope of large deeply staining cells, is developed round the embryo, and presumably fulfils a trophic function. The remains of this embryonic membrane may be found in the host when the larva has ruptured and emerged from it.

The newly hatched larva is a remarkable form, armoured with dark segmental plates of chitin, which render it easily visible through the tissues of the host. There are a distinct head and thirteen body segments, the last terminating in a long tail. The mouth is produced into a proboscis, within which lie two simple mandibles. The head is furnished with four pairs of chitinous nodules, three on the ventral and one on the dorsal side. Each bears a transparent spot at the summit, possibly sensory in function. The anus, which is dorsal to the cauda, is a large conspicuous structure surrounded by a chitinous ring, and striae of chitin may be seen radiating into the lumen. At this stage, as in other Hymenoptera, there is no passage from the mid- to the hind-gut, but the bulb-shaped cavity of the latter and the wide anus, suggest that in this form it may serve some especial function in early larval life. The duration of this stage is variable. In one observed case, the chitin had been cast when the larva hatched, and was left behind in the pseudo-serosa. In other instances, it lasted from two to four days. Three or four of these larvae may be found in the same host, but, so far as is known, only one reaches maturity. In ecdysis the chitinous skin either splits down the ventral median line or else transversely across the thorax.

The second stage larva differs from the first chiefly in the absence of the chitinous plates. It is transparent, and the gut contents tinge it pale yellow. The mouth parts are less produced, but the ventral papillae on the head are more conspicuous and the first three segments bear ventral processes.

In the third stage, the tail is greatly reduced, the thoracic processes disappear, and the cephalic papillae are hardly visible. It was not ascertained whether there was actually a moult between

these two stages, or whether the change of form was due merely to growth and absorption. In the fourth stage, the tail, appendages, and papillae disappear, and the anus is proportionately smaller than in the preceding instars.

All this time, the parasite, with its head orientated towards that of the host, lies in the haemocoele of the latter against the distended mesenteron, which, under the pressure, becomes much constricted. The *Aphidius* remains apparently healthy and retains some power of movement when irritated. It secretes silk as usual; but, immediately after the cocoon is woven, its development is in some way arrested, for the contents of the gut are never voided and metamorphosis does not take place. Death occurs only when the Cynipid larva is almost fully fed, and to the last the tissues remain fresh and undiscoloured. The *Charips*, which until then has been apneustic, makes its way out through the host's thorax, and its tracheal system becomes functional. Within the next twelve hours the hyperparasite devours the remains of the host, and prepares for metamorphosis within the cocoon inside the aphid's skin already woven by the *Aphidius*.

The full-grown larva is an apodous form with a well-developed head and thirteen body segments tapering somewhat posteriorly. The skin is soft and smooth, and the gut contents, seen through the white fat-body, give it a greenish colour. The buccal armature consists of labrum, mandibles, maxillae, and labium. The labrum is crescentic and bears eight small papillae. The mandibles are strongly chitinised, notched, and connected by powerful muscles to the endoskeleton of the head. The maxillae each bears a raised disc on which are three minute papillae, one of which terminates in a short seta. The labium is large and oval, and is furnished with two pairs of papillae. The salivary duct, which is conspicuously dilated immediately below its orifice, opens on the floor of the mouth under the U-shaped hypopharynx. There are six pairs of open spiracles in the full-grown larva, namely, between the first and second segments, and on segments 3, 4, 5, 7 and 9. Two larvae, out of the considerable number examined, had, in addition, a pair of spiracles on segment 8.

Pupation lasts from three to four weeks, and the total period of development seems to be from thirty to thirty-five days. The imago, when ready to emerge, gnaws a hole in the aphid's skin and creeps out. The adult insects feed on the honey dew of the aphides, which is either sipped from the leaves where it has fallen or else from the anus of the living insect.

The economic importance of this hyperparasite is probably not great, but to some extent it must be considered injurious, since it checks the *Aphidius* in its destruction of plant-lice. The inter-relations of aphides with their parasites and hyperparasites form a

bionomical complex of considerable intricacy. Thus, this Cynipid hyperparasite, equally with its Braconid host, is liable to destruction by the Chalcid or Proctotrypid ecto-hyperparasites; and, from observations made in the course of this work, the evidence points to the conclusion that where the incidence of Chalcid and Proctotrypid hyperparasitism is high, few of the Cynipids survive, for they are not only dependant upon the occurrence of the host, but share its vulnerability to other parasites throughout their larval life. I attribute to this the fact that from collections of parasitised aphides, made in the field, there were proportionately more Cynipid emergences in June than in July. Most of the hyperparasites obtained from later collections were external-feeding Chalcids or Proctotrypids; and the inference is that the later broods of Cynipids suffered from a second parasitisation of their host by other hyperparasites.

A method of testing Triode Vacuum Tubes. By E. V. Appleton, M.A., St John's College.

[*Read* 22 November 1920.]

The circuit to be described affords a simple and convenient method of measuring the slope of the principal voltage-current characteristic of a triode vacuum tube. It is now well known that the effectiveness of such a tube as a relay or amplifier depends primarily on the efficiency with which the magnitude of the thermionic current can be controlled by means of the grid or intermediary electrode; that is to say, we are mainly concerned in practice with the rate of variation of anode current with grid voltage, and not with the absolute magnitudes of either quantity. The characteristic mentioned is determined if sufficient correlated values of the anode current I and the grid voltage v are known. To obtain such data accurate ammeters and voltmeters are required. Moreover, even if these values for the characteristic are known, the accurate determination of dI/dv by the ordinary graphical methods takes considerable time. As the dimensions of the slope of the characteristic are those of a conductance it seems preferable to dispense with ammeters and voltmeters and determine this quantity directly in terms of a standard resistance.

A circuit designed to determine the mean value of the mutual conductance ($\Delta I/\Delta v$) over any range of grid voltage was suggested by the writer two years ago*. With this method static currents and voltages were used. In developing the circuit for use since then a great increase in both accuracy and utility has been found to accompany the use of small alternating voltages of acoustic frequency. With the sensitive alternating current detectors now available it is possible to decrease the limits of the variables involved so that the value of $\Delta I/\Delta v$ obtained by this method differs inappreciably from the value of dI/dv obtained from a graphical analysis of the complete characteristic. We shall assume that these values are identical in the following discussion.

The circuit used is shown in Fig. 1. An alternating voltage of acoustic frequency from the alternator A is applied between the grid and the filament of the tube and also across the two non-reactive resistances R and r, the former of which is variable while the latter is small and constant. The anode and grid batteries (B_1 and B_2) are suitably fixed to give the operating conditions for

* See J. A. Fleming, *The Thermionic Valve and its Development in Radio-Telegraphy and Telephony*, p. 142.

which the conductance is required. To determine the value of dI/dv the resistance R is varied until the potential difference between the points C and D is constant. The value of dI/dv is then given simply as $1/R$. An amplifying-telephone is connected across the resistance r, the constancy of the potential difference between C and D being indicated by a note of minimum intensity in the telephone.

To determine the actual voltage amplification obtainable in any particular case (e.g. in a resistance-coupled amplifier such as has been suggested for the measurement of small ionization currents) a suitable high resistance R_0 may be included in the anode circuit while the test is made. The voltage amplification then available is given by $R_0 \frac{dI}{dv}$.

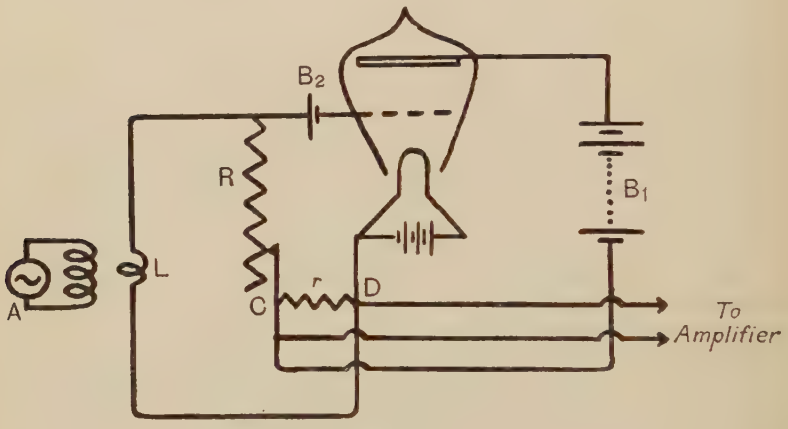


Fig. 1.

Fig. 2 shows some typical results obtained with a hard valve (Type R). For comparison the actual v-I characteristic is shown in the same figure.

Proof of Formula. Before the alternator current is switched on the potential difference between the ends of the resistance r is seen to be $I_0 \frac{Rr}{R+r}$ where I_0 is the stationary value of the thermionic anode current. If the alternating voltage applied between the grid and filament is $v_0 \sin pt$ the change in the potential difference between C and D may be regarded as the algebraic sum of two oppositely directed changes. The first is due to the direct effect of the applied alternating voltage and is equal to $\frac{rv_0 \sin pt}{r+R}$; the second may be regarded as due to the alteration of anode current from I_0

to $I_0 + \frac{dI}{dv} v_0 \sin pt$. For these two oppositely directed effects to be equal in magnitude we have

$$\frac{rv_0 \sin pt}{r+R} = \frac{Rr}{R+r}\left(I_0 + \frac{dI}{dv} v_0 \sin pt\right) - I_0 \frac{Rr}{R+r},$$

that is

$$\frac{dI}{dv} = \frac{1}{R}.$$

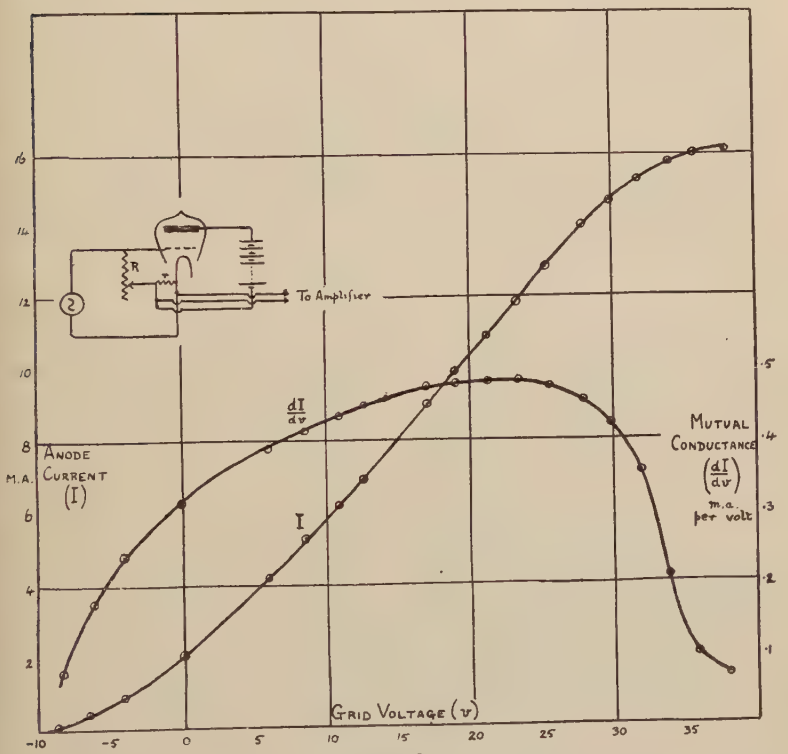


Fig. 2.

It will be seen that the method is self-compensating so far as the effect of the resistance r is concerned so long as the impedance of the coil L is small compared with R as is usually the case in practice. If, however, it occurs that the impedance of L is large compared with r the result is still a simple one, the value of dI/dv being then given by $1/(R+r)$. It will also be noticed that the result is independent of the angular frequency p of the applied alternating voltage. Thus a sinusoidal voltage is not essential and a buzzer source may be used.

The definiteness of the point of null telephone response depends on the fidelity with which the anode current changes follow the grid voltage variations. The circuit is therefore suggested as a possible method of testing whether any temporal 'lag' exists between these two quantities for any particular tube. So far as tubes of extreme exhaustion have been tested no such inertia has been found to exist. For cases of ultra-acoustic frequencies the ordinary wireless methods of testing the constancy of the potential across CD are applicable.

In addition to its use as an amplifier the vacuum tube has a distinct field of utility as a detector of high-frequency oscillations. When operation takes place with conditions represented by a curved portion of the characteristic a symmetrical grid voltage change produces a variation in the mean value of the anode current. As a first approximation in such a case one may consider that the magnitude of the anode current alteration is proportional to the value of d^2I/dv^2 and also to the square of the amplitude of the grid voltage variation.

It is possible to determine the value of d^2I/dv^2 directly. To do this the alternator of Fig. 1 is replaced by a small voltage battery (positive to grid) and tapping key, and the resistance CD by a milli-ammeter. The resistance R is adjusted so that the milli-ammeter deflection is unaltered when the key is closed. If the value of the resistance so obtained is R_1, we have approximately

$$\frac{1}{R_1} = \frac{dI}{dv} + \frac{1}{2}\frac{d^2I}{dv^2} \cdot \Delta v, \qquad \text{.....(1)}$$

where Δv is the E.M.F. of the small battery.

If the same adjustment is now made with the battery reversed (negative to grid) we have

$$\frac{1}{R_2} = \frac{dI}{dv} - \frac{1}{2}\frac{d^2I}{dv^2} \cdot \Delta v. \qquad \text{.....(2)}$$

Thus from (1) and (2) we have

$$\frac{dI}{dv} = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

and

$$\frac{d^2I}{dv^2} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \cdot \frac{1}{\Delta v}.$$

With such an arrangement the results are accurate to within 3 or 4 per cent. If extreme accuracy is required the milli-ammeter may be replaced by a resistance r (see Fig. 1) and a potentiometric scheme and sensitive galvanometer used to indicate when the voltage drop along CD is constant.

The Rotation of the Non-Spinning Gyrostat. By Sir George Greenhill and Dr G. T. Bennett.

[*Read* 22 November 1920.]

Dr G. T. Bennett has done me the honour of criticizing a statement in § 14, p. 13, of my *Report to the Aeronautical Committee on Gyroscopic Theory*, 1914 (cited as R. G. T.), and I am pleased to have this opportunity of meeting his objections.

The diagrams of Flatland on a sheet of paper are inadequate in a discussion concerning Rotations in Space; it is advisable then to have a physical representation at hand, such as that described in Fig. 3 of R. G. T., where the displacements can be visualised, without the confusion arising from thinking of the different sides of the sheet of paper of a diagram.

In this model of Fig. 3, of an Altazimuth suspension of a stalk, as an axle carrying a gyroscopic flywheel, the three angles θ, ψ, ϕ, introduced into Dynamics by Euler in 1760, and standard to this day in mathematical treatment, may receive the corresponding astronomical names; θ, measured from the downward vertical Oz, may be called the Nadir Distance; ψ will then be called the Azimuth; and Euler's third angle ϕ, the angle the wheel has turned over the axle OZ, may then be called the Hour Angle.

The point in dispute is concerning this hour angle ϕ.

On the model, of the stalk with altazimuth suspension, carrying a flywheel moveable about the stalk as a smooth axle, the hour angle ϕ will represent the extent to which the wheel has rubbed round relatively to the axle stalk. This displacement can be shown unmistakably by chalk marks, originally in coincidence on wheel and axle, and also on the frame, and in their subsequent divergence, to show the increase in ϕ, and also in ψ.

Dr G. T. Bennett will oblige greatly if he adheres to Euler's standard notation; so I will change his equations (1), (2), (3) into

$$\frac{d\phi}{dt} + \cos\theta \frac{d\psi}{dt} = 0, \quad \frac{d\phi}{dt} + \frac{d\psi}{dt} = \operatorname{vers}\theta \frac{d\psi}{dt} = \frac{d\sigma}{dt}, \quad \phi + \psi = \sigma,$$

when the flywheel is not set spinning. Here $d\sigma$ denotes the polar element of area described on the unit sphere by Z the end of the stalk; and, in an incomplete circuit, σ must be taken to represent the solid conical angle subtended by the spherical area bounded by the arc described by Z, and the two great circles proceeding from z to the ends of the arc.

When the circuit is complete, the solid angle described will be represented by the spherical area; but there is a discrepancy of 2π to be considered, according as the area encloses the nadir z or not, when ψ increases by 2π, or oscillates and returns to its original value; but we will not delay over this here.

When the circuit of the pole Z is incomplete, the geometrical interpretation of $\phi + \psi$ is not simple, as will be seen on reference to (6), p. 71, R. G. T.; and the angle does not appear in an inspection of the model. But ϕ and ψ separately are visible in the chalk-marks and their divergence; but the angles are in different planes.

It is only in the complete circuit, where the stalk OZ is brought back to its original position, that the angle can be visualised through which the flywheel has turned relative to the frame.

In the statement of R. G. T., § 14, criticized by Dr Bennett in his § 4, where in the model the stalk is hanging vertical, $\theta = 0$, and the stalk is then revolved in azimuth, the solid angle $\sigma = 0$, $\phi + \psi = 0$. But in a complete revolution of the stalk, $\psi = 2\pi$, and $\phi = -2\pi$; that is, the wheel has rubbed once round the axle inside; and ϕ does not represent the angle the wheel has turned in space; the wheel remains stationary with respect to the frame and the stalk turns round inside the wheel; but $\phi + \psi = 0$ at any intermediate stage.

Dynamics were never studied at Cambridge as an experimental science, so it is not likely that the model of R. G. T., Fig. 3, should exist there, to place on the lecture-table between the lecturer and his class, for him to show off the variation of Euler's angles, and to handle and feel an actual state of gyroscopic motion. The non-spinning gyroscope is imitated by a plummet at the end of a thread, as a spherical pendulum, when the wheel has no rotation; and by giving the wheel a spin R, the extension is made to the most general state of the spinning gyroscope.

But a penholder is always at hand, to serve as an illustration of the angular creep in ϕ. The action is not frictional as has been objected. Butter the finger to acquire the perfect smoothness of the text-book jargon, and the creeping action is not arrested.

The experiment is not inapt, as it shows the relative motion apparently reversed, where the axle is fixed in the wheel and runs in outside bearings; here the finger and thumb, perfectly buttery. So too with the other homely familiar experiments cited by Dr Bennett.

Euler's angles require to be interpreted on the altazimuth suspension of the stalk in Fig. 3. But Prandtl suppresses the complication of the hub and vertical spindle on ball-bearings, and replaces it by the simple economical arrangement of a fixed hook, to which the stalk is hooked up, with the rubbing surfaces well

greased. The motion of the stalk here is the same as with Kelvin's trunnion rings and knife-edge gimbals, or his short length of elastic wire, fixed to the stalk and the support; and it may be imitated in the swaying motion of the body seated in a chair, where one wall of the room is always faced; the conical motion does not make the body turn on the seat, not a music-stool.

It would carry us too far to discuss the modification in ϕ due to these and other modes of suspension, such as Hooke's joint and bevel-wheels.

But so long as the inertia of the stalk may be ignored, there is no modification in the θ, ψ angles of the axle of the gyroscopic wheel; but the angle ϕ will require separate consideration.

If however ϕ is suppressed by clamping the wheel to the stalk, or if the inertia of the stalk is taken into account, the motion is hyperelliptic and intractable.

The question quoted from the 1898 examination paper is a very good specimen of many such, scattered anonymously in college papers. Judging from the date, Dr Bennett ought to be able to lift for us the veil of anonymity.

A similar question by Maxwell in the Mathematical Tripos 1869, on vibration in its effect in causing a permanent deviation in a pendulum, has proved useful, nearly 50 years later, in the interpretation of compass deflection; and we have seen the deflection realised in an experiment devised by Mr C. C. Mason.

Quaternions come in useful for the geometrical interpretation of these questions on finite rotation; as for instance in the resultant rotation due to successive rotations of a spherical triangle through the exterior angles. Then there is a theorem given by Dr W. Burnside in the *Messenger of Mathematics*, XXIII., on the resultant screw displacement due to two half turns about non-intersecting axes.

The 1898 question, on the rotation that can be given to a body on a smooth axle by a conical motion, may be quoted as an answer to Aristotle's challenge—to rotate a smooth sphere—*ἥκιστα δὲ κινητικὸν ἡ σφαῖρα διὰ τὸ μηδὲν ἔχειν ὄργανον πρὸς τὴν κίνησιν.*

To the preceding remarks of Sir George Greenhill, Dr G. T. Bennett replies as follows:—

From Sir George Greenhill's comments it is happily apparent that the difference of four right angles between a result in his Report and a result in my paper is not a miscalculation of either but a discrepancy turning on a verbal ellipsis. My work has regard only to the movement of the gyroscope relatively to 'fixed space'; whereas his is limited to the movement of the gyroscope relatively to the rotating stalk of his altazimuth suspension.

In the matter of his experimental illustration with the pen-holder I was far from conjecturing that it was to be regarded as ideally smooth; and if that indeed is to be the case it must be further described as having precisely no angular velocity about its axis, if it is to imitate the gyroscope successfully.

It may be remarked that the suspension of a pendulous gyroscope by an altazimuth mounting is open to one grave objection: that the gyroscope loses one of its three degrees of freedom in its central position. And if the gyro-axis swings close past the vertical it involves in consequence excessively rapid changes in the azimuth movement of the suspension. The gunner who attempts to deal with aircraft passing overhead, using the ordinary gun-mounting, finds a corresponding difficulty with his training gear. *Διώκει παῖς ποτανὸν ὄρνιν.*

On the representation of the simple group of order 660 *as a group of linear substitutions on* 5 *symbols.* By Dr W. Burnside, Honorary Fellow of Pembroke College.

[*Read* 22 November 1920.]

Except in the cases of two and of three variables, it is only few groups of linear substitutions of finite order the forms of which have been exhibited explicitly. This, it is hoped, will justify the following calculations, which seem to offer one or two points of interest. In particular the existence of a cubic three-spread in space of four dimensions, which admits a group of 660 collineations into itself, is perhaps noteworthy. The well-known cubic three-spread of Segre, defined by

$$(x_0 + x_1 + x_2 + x_3 + x_4)^3 - x_0^3 - x_1^3 - x_2^3 - x_3^3 - x_4^3 = 0,$$

admits a group of 720 collineations. That which admits the group of 660 collineations is defined by

$$x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_0 = 0.$$

The modular group, for $p = 11$, is a simple group of order 660. Its characteristics have been calculated*, and it is known to admit two representations as an irrational irreducible group on five variables. It is proposed here to set up these two representations. In one of them the multipliers of an operation P of order 11 are $\alpha, \alpha^9, \alpha^4, \alpha^3, \alpha^5$; where α is a primitive 11th root of unity. Moreover the group contains an operation S of order 5 such that

$$SPS^{-1} = P^9.$$

Now when P has the above multipliers, there is only one representation of the cyclical group $\{S, P\}$ as a group of degree 5; and by suitably choosing the variables this can be brought to the form which is generated by

$$x_0' = \alpha x_0, \quad x_1' = \alpha^9 x_1, \quad x_2' = \alpha^4 x_2, \quad x_3' = \alpha^3 x_3, \quad x_4' = \alpha^5 x_4,$$

and $x_0' = x_1, \quad x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4, \quad x_4' = x_0.$

The variables for the required irreducible representation Γ may be therefore chosen so that Γ contains this subgroup. When the method† for finding the number of invariants of the nth degree of a group is applied to Γ it is found that the group has one cubic invariant. Now it is very easily verified that the only cubic invariant for the above metacyclical group $\{S, P\}$ is

$$x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_0,$$

and this must therefore be an invariant for Γ.

* Burnside, *Theory of Groups*, p. 502. † *loc. cit.* p. 301.

In Γ there are 5 operations A of order two, and characteristic unity, such that $ASA = S^{-1}$. Moreover Γ has no subgroup which contains an operation of order 11 and an operation of order 2. Hence P and any one of these 5 operations A will generate Γ.

If $$\xi_i = x_0 + \omega^{-i}x_1 + \omega^{-2i}x_2 + \omega^{-3i}x_3 + \omega^{-4i}x_4,$$
$$(i = 0, 1, 2, 3, 4), \quad \omega^5 = 1,$$

the canonical form of S is

$$\xi_0' = \xi_0, \ \xi_1' = \omega\xi_1, \ \xi_2' = \omega^2\xi_2, \ \xi_3' = \omega^3\xi_3, \ \xi_4' = \omega^4\xi_4,$$

and the most general operation of order two which will transform S into its inverse, while its characteristic is unity, is

$$\xi_0' = \xi_0, \ \xi_1' = a\xi_4, \ \xi_2' = b\xi_3, \ \xi_3' = b^{-1}\xi_2, \ \xi_4' = a^{-1}\xi_1.$$

Now the above invariant, when expressed in terms of the ξ's, is a numerical multiple of

$$\xi_0^3 + 2(1 + \omega + \omega^4)\xi_0\xi_1\xi_4 + 2(1 + \omega^2 + \omega^3)\xi_0\xi_2\xi_3 + (\omega + 2\omega^2)\xi_1\xi_2^2 + (\omega^4 + 2\omega^3)\xi_4\xi_3^2 + (\omega^3 + 2\omega)\xi_1^2\xi_3 + (\omega^2 + 2\omega^4)\xi_4^2\xi_2.$$

The conditions that the above substitution of order 2 shall leave this unchanged are

$$ab^2(\omega + 2\omega^2) = \omega^4 + 2\omega^3,$$
$$a^2b^{-1}(\omega^3 + 2\omega) = \omega^2 + 2\omega^4,$$

and these equations clearly have 5 solutions. Now a and b must be rational functions of α and ω, for otherwise the group could not be one of finite order. They may be expressed in that form as follows.

Put

$$\alpha + \alpha^{-1} = \lambda_0, \ \alpha^2 + \alpha^{-2} = \lambda_1, \ \alpha^4 + \alpha^{-4} = \lambda_2,$$
$$\alpha^3 + \alpha^{-3} = \lambda_3, \ \alpha^5 + \alpha^{-5} = \lambda_4,$$
$$\mu_i = \lambda_0 + \omega^i\lambda_1 + \omega^{2i}\lambda_2 + \omega^{3i}\lambda_3 + \omega^{4i}\lambda_4,$$
$$(i = 1, 2, 3, 4).$$

Direct multiplication then gives

$$\mu_1\mu_4 = \mu_2\mu_3 = (\omega + 2\omega^2)(\omega^2 + 2\omega^4)(\omega^3 + 2\omega)(\omega^4 + 2\omega^3) = 11,$$
$$\mu_1^2 = (\omega + 2\omega^2)(\omega^2 + 2\omega^4)\mu_2,$$
$$\mu_2^2 = (\omega^2 + 2\omega^4)(\omega^4 + 2\omega^3)\mu_4,$$
$$\mu_4^2 = (\omega^4 + 2\omega^3)(\omega^3 + 2\omega)\mu_3,$$
$$\mu_3^2 = (\omega^3 + 2\omega)(\omega + 2\omega^2)\mu_1.$$

From these results it follows at once that

$$a = \frac{\mu_3}{(\omega + 2\omega^2)(\omega^3 + 2\omega)}, \quad b = \frac{\mu_1}{(\omega + 2\omega^2)(\omega^2 + 2\omega^4)},$$

involving

$$a^{-1} = \frac{\mu_2}{(\omega^2 + 2\omega^4)(\omega^4 + 2\omega^3)}, \quad b^{-1} = \frac{\mu_4}{(\omega^3 + 2\omega)(\omega^4 + 2\omega^3)}$$

is a solution of the above equations, the other solutions being $a\omega'$, $b\omega'^2$, where ω' is any primitive fifth root of unity. With these values of a and b, the operation A, when expressed in terms of the original variables is

$$\begin{aligned}5x_i' &= \sum_j \omega^{ij}\xi_j' = \xi_0 + \omega^i a\xi_4 + \omega^{2i}b\xi_3 + \omega^{3i}b^{-1}\xi_2 + \omega^{4i}a^{-1}\xi_1\\ &= x_0(1 + a\omega^i + b\omega^{2i} + b^{-1}\omega^{3i} + a^{-1}\omega^{4i})\\ &\quad + x_1(1 + a\omega^{i-4} + b\omega^{2i-3} + b^{-1}\omega^{3i-2} + a^{-1}\omega^{4i-1})\\ &\quad + x_2(1 + a\omega^{i-3} + b\omega^{2i-1} + b^{-1}\omega^{3i-4} + a^{-1}\omega^{4i-2})\\ &\quad + x_3(1 + a\omega^{i-2} + b\omega^{2i-4} + b^{-1}\omega^{3i-1} + a^{-1}\omega^{4i-3})\\ &\quad + x_4(1 + a\omega^{i-1} + b\omega^{2i-2} + b^{-1}\omega^{3i-3} + a^{-1}\omega^{4i-4})\\ &\qquad (i = 0, 1, 2, 3, 4).\end{aligned}$$

On entering the values of a, b, a^{-1}, b^{-1}, it is found that there are only five distinct coefficients, viz.

$$-\tfrac{5}{11}(2\lambda_0 + 4\lambda_2 + 3\lambda_3 + 2\lambda_4),$$

and those derived from this by cyclical permutation of indices. Finally, writing

$$\beta_i = 2\lambda_i + 4\lambda_{i+2} + 3\lambda_{i+3} + 2\lambda_{i+4},$$

the substitution A is

$$\begin{aligned}-11x_0' &= \beta_0x_0 + \beta_3x_1 + \beta_1x_2 + \beta_4x_3 + \beta_2x_4,\\ -11x_1' &= \beta_3x_0 + \beta_1x_1 + \beta_4x_2 + \beta_2x_3 + \beta_0x_4,\\ -11x_2' &= \beta_1x_0 + \beta_4x_1 + \beta_2x_2 + \beta_0x_3 + \beta_3x_4,\\ -11x_3' &= \beta_4x_0 + \beta_2x_1 + \beta_0x_2 + \beta_3x_3 + \beta_1x_4,\\ -11x_4' &= \beta_2x_0 + \beta_0x_1 + \beta_3x_2 + \beta_1x_3 + \beta_4x_4.\end{aligned}$$

This substitution and P, viz.

$$x_0' = \alpha x_0,\ \ x_1' = \alpha^9x_1,\ \ x_2' = \alpha^4x_2;\ \ x_3' = \alpha^3x_3,\ \ x_4' = \alpha^5x_4,$$

generate Γ. Since the invariant has rational coefficients, if it is unchanged by a substitution T, it must necessarily be unchanged by $\bar{T}$ derived from T by changing the sign of $\sqrt{-1}$ in each of the coefficients of T. It follows that Γ and $\bar{\Gamma}$ consist of the same set of substitutions, the correspondence between the substitutions and the operations of the abstract group being distinct for the two representations.

On the representation of algebraic numbers as a sum of four squares. By L. J. MORDELL. (Communicated by Professor H. F. BAKER.)

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Professor Landau in a recent paper* entitled "Über die Zerlegung total positiver Zahlen in Quadrate" states that about twenty years ago, Professor Hilbert† gave without proof the theorem that "Every number in an algebraic field (provided that neither it nor any of its conjugate numbers are negative real quantities or if all of the conjugate fields are imaginary) can be expressed as the sum of the squares of four numbers of the field." This is an extension of the well-known theorem‡ due to Fermat and proved by Lagrange, that every positive integer§ can be expressed as the sum of the squares of four other integers. From this result it immediately follows that every positive fraction can be expressed as the sum of the squares of four other fractions, a theorem included in Hilbert's theorem, which is of course only a very special case‖ in the arithmetical theory of quadratic forms with coefficients in a given algebraic field. The development of this theory however, is a matter of great difficulty, if only from the fact that it requires a knowledge of the laws of quadratic reciprocity in the field, the investigation of which, in even the simplest general cases, requires a lengthy and detailed although very interesting discussion¶.

Professor Landau gives a simple proof for the quadratic field based upon elementary algebra. He states however that he does not know whether the theorem holds universally. This indicates that a proof of Hilbert's general theorem is not easy, a view which is confirmed by considering the general theory underlying the question. The following proof for a cubic field may therefore be of interest**.

§ (1). Let then x be the root of an irreducible cubic equation

$$x^3 - ax^2 + bx - c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

* *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, 1919, pp. 392–396.

† *Grundlagen der Geometrie*, § (38).

‡ Bachmann, *Zahlentheorie*, vol. 4, p. 151.

§ The terms integers, fractions, etc. refer to rational quantities unless otherwise stated.

‖ See also an account of some of A. Meyer's work on equations of the form $ax^2 + by^2 + cz^2 + dt^2 + eu^2 = 0$ in Bachmann, *Zahlentheorie*, vol. 4, pp. 259, 266.

¶ See for example Hilbert, *Ueber die Theorie des relativquadratischen Zahlkörpers*, *Mathematische Annalen*, vol. 51, pp. 1–127. An elementary introduction is given in Sommer "Vorlesungen über Zahlentheorie," section 5, of which there is a French translation.

** I am very greatly indebted to Professor Landau for having gone through this paper and for having suggested a number of improvements in the exposition.

where there is no loss of generality in supposing that a, b, c and all the rational numbers dealt with in this paper are integers, except when obviously otherwise. Any number f in the cubic field can be expressed in the form

$$Df = Ax^2 + Bx + C$$

where A, B, C and D are integers, of which D is positive. Hilbert's theorem then asserts that

$$f_1, f_2, f_3, f_4$$

where

$$D_1 f_1 = A_1 x^2 + B_1 x + C_1, \text{ etc.}$$

and A_1, B_1, etc. are integers, can be found so that

$$f = f_1^2 + f_2^2 + f_3^2 + f_4^2$$

provided that neither f nor any of its two conjugate numbers are negative real quantities. We shall refer shortly to this theorem by saying that H holds for f.

The case when $f = 0$ can be dismissed at once by noting that

$$0 = 0^2 + 0^2 + 0^2 + 0^2.$$

§ (2). If H holds for two numbers F_1, F_2, it also holds for their product since as is well known

$$(f_1^2 + f_2^2 + f_3^2 + f_4^2)(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = \psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2$$

where $\psi_1 = f_1\phi_1 + f_2\phi_2 + f_3\phi_3 + f_4\phi_4$ etc.;

and also for their quotient since

$$F_1/F_2 = F_1 F_2/F_2^2.$$

In particular if n is any positive rational fraction, H holds for nf if it holds for f.

Obviously H holds for the number $x^2 + q$ if q is a positive integer which can be expressed as the sum of the squares of three integers. This is always* the case except when q is of the form $4^\beta(8m + 7)$; and we shall say for shortness that any number, positive or negative, of the form $4^\beta(8m + 7)$ is of the form M. From the definition it is necessary but not sufficient that

$$M \equiv 0, 4, \text{ or } 7 \pmod 8 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

In particular the highest power of 2 dividing M must have an even exponent. Similar results hold for $x^2 + q$ when q is a fraction (positive of course)

$$\lambda/\mu = \lambda\mu/\mu^2$$

such that $\lambda\mu$ is not of the form M, and we can still say in this case that the fraction q is not of the form M. It is also clear that H holds for

$$ax^2 + bx + c = [(2ax + b)^2 + 4ac - b^2]/4a$$

* Bachmann, *Zahlentheorie*, vol. 4, p. 146.

if a is positive and the discriminant $4ac - b^2$ is neither negative nor of the form M.

It will be useful hereafter to prove now that if q is any integer, an even integer p can be found so that $q - p^2$ is not of the form M. This follows from (A), for if

$$q \equiv 1, 2 \pmod 4, \text{ take } p \equiv 0 \pmod 2,$$

if $q \equiv 3 \pmod 8$, take $p \equiv 0 \pmod 4$,

if $q \equiv 7 \pmod 8$, take $p \equiv 2 \pmod 4$.

If however $q \equiv 0 \pmod 4$

put $q = 4q_1,\ p = 2p_1$;

then since

$$q - p^2 = 4(q_1 - p_1^2)$$

the result follows by induction since if

$$q_1 \equiv 0 \pmod 4,$$

we can put $q_1 \equiv 4q_2,\ p_1 = 2p_2$, etc.

§ (3). Hilbert's problem can be simplified by writing

$$y = Ax^2 + Bx + C$$

so that x can also be expressed rationally in terms of y by aid of equation (1), while y is also the root of an irreducible cubic*. Hence the whole question can be reduced to proving H for x where

$$x^3 - ax^2 + bx - c = 0 \quad \text{.....(1)}$$

where the real roots of the cubic and hence also c are positive. The cubic in x can be written as

$$x = \frac{(a+p)x^2 + (q-b)x + c}{x^2 + px + q} \quad \text{.....(2)}$$

where p and q are entirely arbitrary. If we can find rational values for p and q so that the discriminants

$$\rho = 4q - p^2 \text{ and } \sigma = 4c(a+p) - (q-b)^2 \quad \text{.....(3)}$$

are both positive and neither of them of the form M, then H holds for both the numerator and denominator of x and hence for x.

Suppose now that a, b and c are all positive, including in particular the case when all the values of x are real. Then by considering the region common to the two parabolas

$$4\eta - \xi^2 = 0 \text{ and } 4c(a+\xi) - (\eta - b)^2 = 0 \quad \text{.....(4)}$$

it is obvious that real and hence also rational values of p and q can be found to make each of the discriminants in (3) positive. Moreover, if we can find any rational values† of p and q for which neither of the

* Except in the trivial case $A = B = 0$.

† It is by no means obvious that such values exist. For example, one of the two expressions, $k + 7$, $512 - k^2$ is always of the form M. The same applies to the two expressions pq, $512q^2 - (7q - p)^2$.

discriminants in (3) is of the form M, it is clear that the same holds when p and q are replaced by

$$p + 2^N\lambda, \quad q + 2^N\mu$$

if N is sufficiently large and λ, μ are fractions with odd denominators, and that λ, μ, N can be taken so as to bring the point

$$(p + 2^N\lambda, q + 2^N\mu)$$

within the common region of the parabolas (4), that is, ρ and σ will be both positive.

We can summarise the rest of this section § (3) by saying that we shall show that suitable values for p and q can be found when either a or b or c is odd, so that we must discuss the case when a, b and c are all even. It is then shown that suitable values can be found for p and q when the highest power of 2 dividing c is the first or second, and that the case when c is divisible by any power of 2 can be reduced to one of the preceding cases. It then follows that H is true for x in the case of a cubic with three real roots.

Suppose first then that a is odd and that 4^γ is the highest power of 4 contained in c. Put $c = 4^\gamma C$ and take

$$q - b = 2^{\gamma+1} Q, \; p = 2P$$

so that

$$\sigma/4^{\gamma+1} = C(a + 2P) - Q^2, \quad \rho/4 = q - P^2.$$

Consider first the case when C is even so that it must be divisible by 2 only and not by 4. Then because a is odd, σ is not of the form M if Q is even, whatever P may be.

Also by § (2) we can find a value for P for which ρ is not of the form M so that H holds for x.

If, however, C is odd, take P even but Q even or odd according as

$$Ca \equiv 1 \text{ or } 3 \pmod 4),$$

so that σ is not of the form M. The only restriction on P is that it should be even, and from § (2) we can find even values of P for which $\rho/4 = q - P^2$ is not of the form M when q is given. Hence H holds for x if a is odd.

Writing now $x = c/y$ so that H holds for x if H holds for y and conversely, we see from

$$y^3 - by^2 + acy - c^2 = 0$$

that if b is odd, H holds for y and hence for x. Hence we need only consider the case when both a and b are even.

Should now c be odd, take in (3) p and q both odd, then

$$\rho \equiv 3 \pmod 8), \quad \sigma \equiv 3 \pmod 8)$$

so that we need only prove H when a, b, c are all even.

If c is divisible by 2 but not by 4 put

$$c = 2C,\ a = 2A,\ b = 2B,\ p = 2P,\ q = 2Q$$

then $$\rho/4 = 2Q - P^2,\ \sigma/4 = 4C(A + P) - (Q - B)^2.$$

If now A is even, take P and $Q - B$ both odd. Then since C is odd,

$$\sigma/4 \equiv 3 \pmod 8.$$

Should, however, $\rho/4$ be of the form M, in this case

$$\rho/4 \equiv 7 \pmod 8$$

for these values of P and Q, then since the only restriction on Q is that $Q - B$ should be odd, the change of Q into $Q + 2$ changes $\rho/4$ into a number $\rho_1/4$, such that

$$\rho_1/4 \equiv 3 \pmod 8$$

while σ is changed into a number σ_1 still satisfying the congruence

$$\sigma_1/4 \equiv 3 \pmod 8.$$

If however A is odd, take P even and $Q - B$ odd. Then

$$\sigma/4 \equiv 3 \pmod 8.$$

The only restriction on P is that it should be even and from § (2) we can take it so that ρ is not of the form M. Hence H holds if c is divisible by 2 but not by 4.

H holds also if c is divisible by 4 but not by 8. For putting

$$x = c/2y$$

we find

$$y^3 - \tfrac{1}{2}by^2 + \tfrac{1}{4}acy - \tfrac{1}{8}c^2 = 0,$$

where all the coefficients are integers and $c^2/8$ is divisible by 2 but not by 4. Hence H holds for y and also for x.

Suppose now that c is divisible by 8 or say 2^n but not by 2^{n+1}. Then put again

$$x = c/2y$$

so that as before

$$y^3 - \tfrac{1}{2}by^2 + \tfrac{1}{4}acy - \tfrac{1}{8}c^2 = 0.$$

Hence if $b/2$ is odd, H holds for y and hence for x. If however $b/2$ is even, put

$$a = 2A,\ b = 4B,\ c = 8C$$

so that the equation (1) becomes

$$x^3 - 2Ax^2 + 4Bx - 8C = 0,$$

or, putting $x = 2z$

$$z^3 - Az^2 + Bz - C = 0.$$

But C is divisible now by 2^{n-3} and not by 2^{n-2}. Hence if this process be continued, we shall arrive at an equation of the form

$$x^3 - ax^2 + bx - c = 0 \quad \text{.....(1)}$$

where either a is odd or c is not divisible by 8. But H has already been proved for these cases so that we have proved H in the case when a, b and c are positive and in particular for the algebraic fields arising from a cubic with three real roots.

§ (4). If however a and b are not both positive, in which case equation (1) has imaginary roots, it is clear from equation (2) that if p and q are taken to satisfy the inequality $4q > p^2$ and t is a sufficiently large positive number, then $x+t$ can be expressed as the quotient of two positive definite quadratics in x. Hence as $x+t$ is also the root of a cubic of the type (1) where a, b and c are positive, it is clear that H holds for $x+t$ provided t is a sufficiently large positive number—integral or fractional.

We have now from equation (1)

$$(x+\xi)(x^2-(\xi+a)x+k) = (-\xi^2-a\xi+k-b)x+c+\xi k \qquad \ldots(5)$$

where k and ξ are arbitrary rational quantities. We take k so large that the discriminant $4k-(\xi+a)^2$ is positive and not of the form M (this is always possible as k need not be an integer, take it say, a fourth of an integer), and so that $-\xi^2-a\xi+k-b$ is also positive.

Suppose ξ is not negative and put

$$\xi_1 = \frac{k\xi+c}{-\xi^2-a\xi+k-b}; \qquad \ldots(6)$$

then ξ_1 is positive and also $\xi_1 > \xi$ since

$$\xi^3+a\xi^2+b\xi+c > 0$$

because the real root of the cubic (1) is positive. Also H holds for $x+\xi$ if it holds for $x+\xi_1$.

Take now $\xi = 0$, and find $\xi_2, \xi_3 \ldots$ from $\xi_1, \xi_2 \ldots$ with of course suitable values of $k_1, k_2 \ldots$, in the same way as ξ_1 was found from ξ. Then $\xi_1, \xi_2, \xi_3 \ldots$ form a monotonic increasing sequence whose limit is infinity. For if it were finite say L, then it follows immediately from equation (6) that

$$L^3+aL^2+bL+c = 0$$

which is impossible as we have assumed that the cubic (1) has no negative root.

Hence after a certain stage (H) holds for $x+\xi_\lambda$. It is obvious then from (5) that H holds for x, so that we have also proved H for the case of the field arising from a cubic with only one real root.

Note.

The general law of quadratic reciprocity in any field was given without proof by Hilbert in the *Göttinger Nachrichten* for 1898, page 380 in his paper "Ueber die Theorie der relativ-Abel'schen Zahlkörper." In his paper "Mathematische Probleme" in the *Nachrichten* for 1900, the investigation of the theory of the general quadratic form with algebraic coefficients is proposed as problem 11.

The law of quadratic reciprocity above was proved by Furtwängler in the third part of his paper "Die Reziprozitätsgesetze für Potenzreste mit Primzahlexponenten in algebraischen Zahlkörpern" in the *Mathematische Annalen*, Bd. 74, 1913.

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On a Gaussian Series of Six Elements. By L. J. Rogers. (Communicated by Prof. G. H. Hardy.)

[*Read* 26 January 1920.]

§ 1. The symbol $H(\alpha, \beta, \lambda, \mu, \gamma)$ will be used for the infinite series

$$1+\frac{\alpha\,.\,\beta\,.\,\lambda\,.\,\mu}{1\,.\,\gamma\,.\,(\kappa^2-\theta^2)}+v_2+v_3+\ldots \quad \ldots\ldots\ldots(1),$$

where $$\frac{v_{n+1}}{v_n}=\frac{(\alpha+n)(\beta+n)(\lambda+n)(\mu+n)}{(1+n)(\gamma+n)(\kappa+n-\theta)(\kappa+n+\theta)},$$

and $$2\kappa=\alpha+\beta+\lambda+\mu+1-\gamma.$$

Since $$\frac{v_{n+1}}{v_n}=1+\frac{1}{n}(\alpha+\beta+\lambda+\mu-1-\gamma-2\kappa)+O\left(\frac{1}{n^2}\right)$$

$$=1-\frac{2}{n}+O\left(\frac{1}{n^2}\right),$$

it follows that the series (1) is always convergent, provided that the elements $\alpha, \beta, \ldots$ are all finite, and $\gamma, \kappa \pm \theta$ are not negative integers.

When $\mu=\gamma$ the series reduces to $1+\frac{\alpha\,.\,\beta\,.\,\lambda}{1\,.\,(\kappa^2-\theta^2)}+\ldots,$

where $$2\kappa=\alpha+\beta+\lambda+1,$$

which series will be written

$$H(\alpha, \beta, \lambda) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

It is not necessary to introduce θ into the functional notation.

For the sake of conciseness it will be convenient to write $\alpha_n, \beta_n, \ldots \kappa_n$ for $\alpha+n, \beta+n, \ldots \kappa+n$.

§ 2. Corresponding to the well-known formula in hypergeometric series, it is easily seen that

$$H(\alpha, \beta, \lambda, \mu_1, \gamma_1)-H(\alpha, \beta, \lambda, \mu, \gamma)$$

$$=\frac{\alpha\beta\lambda(\gamma-\mu)}{\gamma(\gamma+1)(\kappa^2-\theta^2)}H(\alpha_1, \beta_1, \lambda_1, \mu_1, \gamma_2)\ldots\ldots\ldots(1).$$

Moreover

$$\{(\kappa-\mu)^2-\theta^2\}H(\alpha_1, \beta_1, \lambda_1, \mu, \gamma_1)-(\kappa^2-\theta^2)H(\alpha, \beta, \lambda, \mu, \gamma)$$

$$=\frac{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)\mu}{\gamma(\gamma+1)}H(\alpha_1, \beta_1, \lambda_1, \mu_1, \gamma_2)\ldots\ldots(2).$$

To prove this write the equation in the form

$$\{(\kappa-\mu)^2-\theta^2\}\left\{1+\frac{B_1}{\kappa_1^2-\theta^2}+\frac{B_2}{(\kappa_1^2-\theta^2)(\kappa_2^2-\theta^2)}+\cdots\right\}$$
$$-(\kappa^2-\theta^2)\left\{1+\frac{A_1}{\kappa^2-\theta^2}+\frac{A_2}{(\kappa^2-\theta^2)(\kappa_1^2-\theta^2)}+\cdots\right\}$$
$$=C_0+\frac{C_1}{\kappa_1^2-\theta^2}+\frac{C_2}{(\kappa_1^2-\theta^2)(\kappa_2^2-\theta^2)}+\ldots,$$

where the A's, B's and C's are independent of θ, it being observed that, in the first and third series in (2), κ becomes $\kappa+1$.

The left-hand side is

$$\mu^2-2\kappa\mu+B_1\left\{1+\frac{(\kappa-\mu)^2-\kappa_1^2}{\kappa_1^2-\theta^2}\right\}$$
$$+\frac{B_2}{\kappa_1^2-\theta^2}\left\{1+\frac{(\kappa-\mu)^2-\kappa_2^2}{\kappa_2^2-\theta^2}\right\}+\cdots$$
$$-A_1-\frac{A_2}{\kappa_1^2-\theta^2}-\frac{A_3}{(\kappa_1^2-\theta^2)(\kappa_2^2-\theta^2)}+\ldots.$$

Hence

$$C_0=\mu^2-2\kappa\mu+B_1-A_1$$
$$=\mu(\gamma-\alpha-\beta-\lambda-1)+\frac{(\alpha+1)(\beta+1)(\lambda+1)\mu}{\gamma+1}-\frac{\alpha\beta\lambda\mu}{\gamma}$$
$$=\frac{\mu}{\gamma(\gamma+1)}(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda),$$

while $$C_n=B_{n+1}+B_n\{(\kappa-\mu)^2-(\kappa+n)^2\}-A_{n+1}.$$

Now $$A_{n+1}=\frac{\alpha\beta\lambda\mu_n}{(n+1)\gamma}B_n, \text{ and } B_{n+1}=\frac{\alpha_{n+1}\beta_{n+1}\lambda_{n+1}\mu_n}{(n+1)\gamma_{n+1}}B_n,$$

so that
$$\frac{C_n}{B_n}=\frac{\alpha_{n+1}\beta_{n+1}\lambda_{n+1}\mu_n}{(n+1)\gamma_{n+1}}-(2\kappa-\mu+n)\mu_n-\frac{\alpha\beta\lambda\mu_n}{(n+1)\gamma}$$
$$=\frac{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}{\gamma(\gamma+n+1)}\mu_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

as may be easily verified, thereby giving the right-hand side of (2) and establishing the relation.

We may now obtain a continued fraction of the form

$$\frac{1}{1-}\frac{e_1x}{1-}\frac{e_2x}{1-}\frac{e_3x}{1-} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

for the ratio $H(\alpha, \beta, \lambda, \mu_1, \gamma_1)/H(\alpha, \beta, \lambda, \mu, \gamma)$.

For from (1)

$$1-\frac{H(\alpha, \beta, \lambda, \mu, \gamma)}{H(\alpha, \beta, \lambda, \mu_1, \gamma_1)}=1-\frac{e_1}{\kappa^2-\theta^2}\frac{H(\alpha_1, \beta_1, \lambda_1, \mu_1, \gamma_2)}{H(\alpha, \beta, \lambda, \mu_1, \gamma_1)},$$

where $$e_1 = \frac{\alpha\beta\lambda(\gamma-\mu)}{\gamma(\gamma+1)}.$$

But from (2), by changing μ, γ into μ_1, γ_1, which does not alter κ, we have

$$(\kappa^2-\theta^2)\frac{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)}{H(\alpha_1,\beta_1,\lambda_1,\mu_1,\gamma_2)} = (\kappa-\mu)^2-\theta^2-e_2\frac{H(\alpha_1,\beta_1,\lambda_1,\mu_2,\gamma_3)}{H(\alpha_1,\beta_1,\lambda_1,\mu_1,\gamma_2)},$$

where $$e_2 = \frac{(\gamma-\alpha+1)(\gamma-\beta+1)(\gamma-\lambda+1)(\mu+1)}{(\gamma+1)(\gamma+2)}.$$

These two results lead to

$$\frac{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)}{H(\alpha,\beta,\lambda,\mu,\gamma)} = \frac{1}{1-}\frac{e_1}{(\kappa-\mu-1)^2-\theta^2-}\frac{H(\alpha_1,\beta_1,\lambda_1,\mu_2,\gamma_3)}{H(\alpha_1,\beta_1,\lambda_1,\mu_1,\gamma_2)}e_2.$$

The ratio of H-functions on the right-hand side differs from that on the left-hand side only in that α, β, λ, μ are increased each by unity, and γ by 2. This does not alter $\kappa-\mu$, so that we have finally

$$\frac{H(\alpha,\beta,\lambda,\mu+1,\gamma+1)}{H(\alpha,\beta,\lambda,\mu,\gamma)} = \frac{1}{1-}\frac{e_1}{(\kappa-\mu-1)^2-\theta^2-}\frac{e_2}{1-}\frac{e_3}{(\kappa-\mu-1)^2-\theta^2-}\cdots$$

$$= \frac{1}{1-}\frac{e_1x}{1-}\frac{e_2x}{1-}\frac{e_3x}{1-}\cdots,$$

where $e_3 = \dfrac{\alpha_1\beta_1\lambda_1(\gamma_1-\mu)}{\gamma_2\gamma_3}$, $e_4 = \dfrac{(\gamma_2-\alpha)(\gamma_2-\beta)(\gamma_2-\lambda)\mu_2}{\gamma_3\gamma_4}$, etc.

and $$\frac{1}{x} = (\kappa-\mu-1)^2-\theta^2 \qquad \text{.....(5).}$$

It is remarkable that if we change μ into $\gamma-\mu$, the coefficients e all become symmetrical in α, β, λ, μ, and $\kappa-\mu-1$ becomes $\frac{1}{2}(\alpha+\beta+\lambda+\mu-1-2\gamma)$, so that

$$\frac{H(\alpha,\beta,\lambda,\gamma-\mu+1,\gamma+1)}{H(\alpha,\beta,\lambda,\gamma-\mu,\gamma)}$$

is symmetrical in α, β, λ, μ.

Hence

$$\frac{H(\alpha,\beta,\lambda,\gamma-\mu,\gamma)}{H(\alpha,\beta,\gamma-\lambda,\mu,\gamma)} = \frac{H(\alpha,\beta,\lambda,\gamma+1-\mu,\gamma+1)}{H(\alpha,\beta,\gamma+1-\lambda,\mu,\gamma+1)} \qquad \text{...(6),}$$

where the first of these fractions is the same function of γ as the second is of $\gamma+1$.

§ 3. From § 2 we have

$$\frac{(\kappa-\mu-1)^2-\theta^2}{\kappa^2-\theta^2}\,\frac{H(\alpha_1,\beta_1,\lambda_1,\mu_1,\gamma_2)}{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)} = \frac{1}{1-}\,\frac{e_2}{(\kappa-\mu-1)^2-\theta^2-1-}\,\frac{e_3}{} \ldots\ldots(1).$$

Now change $\alpha, \beta, \lambda, \mu, \gamma$ respectively into

$$\gamma-\alpha,\ \gamma-\beta,\ \gamma-\lambda,\ \gamma-\mu-1,\ \gamma-1.$$

It will then be seen that $\kappa-\mu-1$ becomes $-(\kappa-\mu-1)$, κ becomes $-(\kappa-\gamma-1)$, e_2 becomes e_1, and generally e_n becomes e_{n-1}.

We have in (1) then the same continued fraction as in § 2 (4), and hence

$$\frac{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)}{H(\alpha,\beta,\lambda,\mu,\gamma)} = \frac{(\kappa-\mu-1)^2-\theta^2}{(\kappa-\gamma-1)^2-\theta^2}\,\frac{H(\gamma_1-\alpha,\gamma_1-\beta,\gamma_1-\lambda_1,\gamma-\mu,\gamma_1)}{H(\gamma-\alpha,\gamma-\beta,\gamma-\lambda,\gamma-\mu,\gamma)},$$

or

$$\frac{H(\gamma-\alpha,\gamma-\beta,\gamma-\lambda,\gamma-\mu,\gamma)}{H(\alpha,\beta,\lambda,\mu,\gamma)} = \frac{(\kappa-\mu-1)^2-\theta^2}{(\kappa-\gamma-1)^2-\theta^2}\,\frac{H(\gamma_1-\alpha,\gamma_1-\beta,\gamma_1-\lambda,\gamma-\mu,\gamma_1)}{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)} \ldots\ldots(2).$$

Let $\lambda=0$, then

$$H(\gamma-\alpha,\gamma-\beta,\gamma-\mu) = \frac{\frac{1}{4}(\alpha+\beta-\mu-1-\gamma)^2-\theta^2}{\frac{1}{4}(\alpha+\beta+\mu-1-3\gamma)^2-\theta^2}\,H(\gamma_1-\alpha,\gamma_1-\beta,\gamma-\mu),$$

or, changing α, β, μ into $\gamma-\alpha, \gamma-\beta, \gamma-\mu$,

$$H(\alpha,\beta,\mu) = \frac{\frac{1}{4}(\alpha+\beta-\mu+1)^2-\theta^2}{\frac{1}{4}(\alpha+\beta+\mu+1)^2-\theta^2}\,H(\alpha_1,\beta_1,\mu) \ldots(4).$$

The series $H(\alpha, \beta, \mu)$ has been investigated by Saalschütz, *Zeitschrift f. Math.* XXXV., for the case when α is a negative integer, and generally by Dougall, *Proc. Edinburgh Math. Soc.* XXV., § 12 (20). For special cases of the continued fraction § 2 (5), see *Proc. London Math. Soc.* series 2, vol. 4, pp. 83, 394.

§ 4. *Heinean forms.*

The results of the foregoing sections have their counterparts, when every factor in the terms in H, are replaced by corresponding sines or hyperbolic sines.

If the fundamental series be taken as

$$1+\frac{\sinh m\alpha \sinh m\beta \sinh m\lambda \sinh m\mu}{\sinh m \sinh m\gamma \sinh m(\kappa-\theta) \sinh m(\kappa+\theta)}+v_2+\dots$$

then, if $e^{-2m}=q$, the factors in v_1 take the form $1-q^a$, etc., there being an extra factor

$$\exp m(\alpha+\beta+\lambda+\mu-1-\gamma-2\kappa), \ =e^{-2m}=q.$$

Moreover

$$\frac{v_{n+1}}{v_n}=\frac{(1-q^{\alpha+n})(1-q^{\beta+n})(1-q^{\lambda+n})(1-q^{\mu+n})}{(1-q^{n+1})(1-q^{\gamma+n})(1-q^{\kappa+n-\theta})(1-q^{\kappa+n+\theta})}q,$$

so that the series is obviously convergent if $|q|<1$, and no indices of powers of q are zero.

The formula corresponding to § 2 (1) is readily obtained, giving

$$e_1=\frac{\sinh m\alpha \sinh m\beta \sinh m\gamma \sinh(\gamma-\mu)}{\sinh m\gamma \sinh m(\gamma+1)}.$$

That corresponding to § 2 (2) is more difficult, but it depends on the fact that, by employing the somewhat intricate identity

$$\begin{aligned}&\frac{\sinh m(\alpha+n+1) \sinh m(\beta+n+1) \sinh m(\lambda+n+1)}{\sinh m(n+1) \sinh m(\gamma+n+1)}\\ &\quad-\frac{\sinh m\alpha \sinh m\beta \sinh m\lambda}{\sinh m(n+1) \sinh m\gamma}\\ &\quad-\sinh m(\alpha+\beta+\lambda-\gamma+n+1)\\ &\quad=\frac{\sinh m(\gamma-\alpha) \sinh m(\gamma-\beta) \sinh m(\gamma-\lambda)}{\sinh m\gamma \sinh m(\gamma+n+1)},\end{aligned}$$

we get, corresponding to $\frac{C_n}{B_n}$ in § 2 (3) the value

$$\frac{\sinh m(\mu+n) \sinh m(\gamma-\alpha) \sinh m(\gamma-\beta) \sinh m(\gamma-\lambda)}{\sinh m\gamma \sinh m(\gamma+n+1)}.$$

This leads to a relation corresponding to § 2 (5), where

$$\frac{1}{x}=\sinh^2 m(\kappa-\mu-1)-\sinh^2 m\theta,$$

and the e's are altered to corresponding hyperbolic functions as explained above.

§ 5. *Convergence of the continued fraction in* § 2 (5), *when* x *is negative and the* e*'s are all positive.*

A continued fraction of the type

$$\frac{1}{1+}\,\frac{e_1 t}{1+}\,\frac{e_2 t}{1+}\cdots$$

converges, if when reduced to the form

$$\frac{1}{1+}\,\frac{1}{d_1+}\,\frac{1}{d_2+}\cdots,$$

either or both the series

$$1+d_2+d_4+\ldots,\quad d_1+d_3+d_5+\ldots$$

are divergent.

That is $$1+\frac{1}{t}\left(\frac{e_1}{e_2}+\frac{e_1 e_3}{e_2 e_4}+\frac{e_1 e_3 e_5}{e_2 e_4 e_6}+\ldots\right),$$

and $$\frac{1}{t}\left(\frac{1}{e_1}+\frac{e_2}{e_1 e_3}+\frac{e_2 e_4}{e_1 e_3 e_5}+\ldots\right),$$

must be either or both divergent.

In the first the ratio of the general term to the preceding is $\frac{e_{2n-1}}{e_{2n}}$ and in the second it is $\frac{e_{2n}}{e_{2n+1}}$.

The first ratio is

$$\frac{(\alpha+n-1)(\beta+n-1)(\lambda+n-1)(\gamma-\mu+n-1)(\gamma+2n)}{(\gamma-\alpha+n)(\gamma-\beta+n)(\gamma-\lambda+n)(\mu+n)(\gamma+2n-2)}$$

$$=1-\frac{1}{n}\{1+2(\gamma+\mu+1-\alpha-\beta-\lambda)\}+O\left(\frac{1}{n^2}\right),$$

while the second is

$$\frac{(\gamma-\alpha+n)(\gamma-\beta+n)(\gamma-\lambda+n)(\gamma+2n+1)}{(\alpha+n)(\beta+n)(\lambda+n)(\gamma-\mu+n)(\gamma+2n-1)}$$

$$=1-\frac{1}{n}\{1-2(\gamma+\mu+1-\alpha-\beta-\lambda)\}+O\left(\frac{1}{n^2}\right).$$

It is evident that one series converges and the other diverges and therefore the continued fraction converges.

In the Heinean case these ratios are approximately

$$q^{(\alpha+\beta+\lambda-\gamma-\mu-1)},\quad q^{(\gamma+\mu+1-\alpha-\lambda-\mu)},$$

one or other of which must be greater than 1. Hence one series must diverge and the continued fraction converges.

Note on Ramanujan's trigonometrical function $c_q(n)$, and certain series of arithmetical functions. By Prof. G. H. HARDY.

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1. Ramanujan's memoir 'On certain trigonometrical sums and their applications in the theory of numbers'* is devoted to the study of the function

$$c_q(n) = \sum_p e^{-2np\pi i/q} = \sum_p \cos\frac{2np\pi}{q} \quad \text{.....(1·1),}$$

where p runs through the numbers less than and prime to q. Ramanujan proves that

$$c_q(n) = \Sigma\, \delta\mu\left(\frac{q}{\delta}\right) \quad \text{.....(1·2),}$$

where $\mu(n)$ has its usual meaning† and δ is a common divisor of q and n‡. He then proceeds to express a number of the most important arithmetical functions of n as series of the form

$$\sum_{q=1}^{\infty} a_q c_q(n),$$

where a_q is independent of n. From among the many interesting results which he obtains I may quote the following:—

$$\sigma_s(n) = \frac{n^s}{\zeta(s+1)} \sum \frac{c_q(n)}{q^{s+1}} \quad \text{.....(1·3),}$$

and in particular

$$\sigma(n) = \frac{\pi^2 n}{6} \sum \frac{c_q(n)}{q^2} \quad \text{.....(1·31),}$$

$$0 = \sum \frac{c_q(n)}{q} \quad \text{.....(1·32);}$$

$$d(n) = -\sum \frac{\log q}{q} c_q(n);$$

$$\phi(n) = \frac{6n}{\pi^2} \sum \frac{\mu(q)\, c_q(n)}{\phi_2(q)} \quad \text{.....(1·4);}$$

$$r(n) = \pi \sum \frac{(-1)^{q-1} c_{2q-1}(n)}{2q-1} \quad \text{.....(1·5).}$$

* *Trans. Camb. Phil. Soc.*, vol. 22, 1918, pp. 259–276.

† $\mu(m) = 0$ unless m is a product of ρ different primes, when $\mu(m) = (-1)^\rho$.

‡ The formula (1·2) seems to have been first stated explicitly by Ramanujan (*l.c.* p. 260). It is given for $n = 1$ by Landau (*Handbuch*, 1909, p. 572) and by Jensen ('Et nyt Udtryk for den talteoretiske Funktion $\Sigma\mu(n) = M(m)$,' *Saertryk af Beretning om den 3 Skandinaviske Matematiker-Kongres*, Kristiania, 1915). The deduction of the general formula from that given by Landau is trivial; but Ramanujan makes so many beautiful applications of the sums that they may well be associated with his name.

Here $d(n)$ is the number of divisors of n, $\sigma(n)$ their sum, and $\sigma_s(n)$ the sum of their s-th powers; $\phi(n)$ is the number of numbers less than and prime to n;

$$\phi_2(n) = n^2\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p_1^2}\right)\ldots.$$

when $n = p^a p_1^{a_1}\ldots$; and $r(n)$ is the number of representations of n as the sum of two squares.

These series have a peculiar interest because they show explicitly the source of the irregularities in the behaviour of their sums. Thus, for example, the formula (1·31) may be written in the form

$$\sigma(n) = \frac{\pi^2 n}{6}\left\{1+\frac{(-1)^n}{2^2}+\frac{2\cos\frac{2}{3}n\pi}{3^2}+\frac{2\cos\frac{1}{2}n\pi}{4^2}\right.$$
$$\left.+\frac{2(\cos\frac{2}{5}n\pi+\cos\frac{4}{5}n\pi)}{5^2}+\frac{2\cos\frac{1}{3}n\pi}{6^2}+\ldots\right\};$$

and we see at once that the most important term in $\sigma(n)$ is $\frac{1}{6}\pi^2 n$, and that irregular variations about this average value are produced by a series of harmonic oscillations of decreasing amplitude.

2. Ramanujan's proofs of his principal formulae are very interesting and ingenious, but are not, I think, the simplest or the most natural. In this note I prove a number of them by a different method. This method occurred to Mr Littlewood and me in the course of our researches connected with Waring's and Goldbach's problems, and in which Ramanujan's sums play an important part. I also include a few new results which are suggested naturally by our analysis.

The multiplicative property of $c_q(n)$.

3. The first step is to prove that

$$c_{qq'}(n) = c_q(n)\,c_{q'}(n) \quad \ldots\ldots\ldots\ldots\ldots\ldots(3\cdot1),$$

whenever q and q' are prime to one another. For this we observe that

$$c_q(n)\,c_{q'}(n) = \sum_p e^{-2np\pi i/q}\sum_{p'} e^{-2np'\pi i/q'} = \sum_{p,p'} e^{-2nP\pi i/qq'},$$

where

$$P = pq' + p'q.$$

But it is plain that every value of P is prime to both q and q', and that no two values of P are congruent to modulus qq'. Also the total number of values of P is $\phi(q)\,\phi(q') = \phi(qq')$. Hence P runs through a system of values congruent, to modulus qq', to the $\phi(qq')$ numbers less than and prime to qq'. This plainly establishes the truth of (3·1).

The value of $c_q(n)$.

4. I show next that the value of $c_q(n)$ is given by (1·2). Let us write

$$C_q(n) = \Sigma\, \delta\mu\left(\frac{q}{\delta}\right) \ldots\ldots\ldots\ldots\ldots\ldots (4{\cdot}1).$$

If q and q' are prime to one another, we have

$$C_q(n)\, C_{q'}(n) = \sum_{\delta,\,\delta'} \delta\delta' \mu\left(\frac{q}{\delta}\right) \mu\left(\frac{q'}{\delta'}\right),$$

where δ is a common divisor of q and n and δ' one of q' and n. Clearly $\delta\delta'$ runs through the common divisors of qq' and n, each once only. Also q/δ and q'/δ' are prime to one another. Hence

$$C_q(n)\, C_{q'}(n) = \sum_D D\mu\left(\frac{qq'}{D}\right),$$

where $D = \delta\delta'$ runs through all common divisors of qq' and n; and so

$$C_q(n)\, C_{q'}(n) = C_{qq'}(n) \ldots\ldots\ldots\ldots\ldots\ldots (4{\cdot}2).$$

Since $c_q(n)$ and $C_q(n)$ each possess the multiplicative property, we need only verify their identity when q is a power of a prime, say ϖ^k. Now

$$C_{\varpi^k}(n) = \sum_p e^{-2np\pi i/\varpi^k}$$

where p runs through the $\varpi^{k-1}(\varpi - 1)$ numbers less than and prime to ϖ^k. These may be expressed in the form

$$p = \varpi^{k-1} z + p',$$

where $z = 0, 1, \ldots, \varpi - 1$ and p' runs through the numbers less than and prime to ϖ^{k-1}. Hence

$$c_{\varpi^k}(n) = \sum_z e^{-2nz\pi i/\varpi} \sum_{p'} e^{-2np'\pi i/\varpi^k};$$

and the sum with respect to z is zero unless $\varpi \mid n$*. Thus

$$c_{\varpi^k}(n) = 0 \qquad (k > 1,\ \varpi \nmid n)$$

and

$$c_{\varpi^k}(\varpi\nu) = \varpi \sum_{p'} e^{-2\nu p'\pi i/\varpi^{k-1}} = \varpi c_{\varpi^{k-1}}(\nu).$$

Now plainly

$$c_\varpi(n) = -1 \quad (\varpi \nmid n), \qquad c_\varpi(n) = \varpi - 1 \quad (\varpi \mid n) \ldots (4{\cdot}3).$$

Hence

$$c_{\varpi^2}(n) = 0 \quad (\varpi \nmid n), \qquad c_{\varpi^2}(n) = -\varpi \quad (\varpi \mid n,\ \varpi^2 \nmid n),$$
$$c_{\varpi^2}(n) = \varpi(\varpi - 1) \quad (\varpi^2 \mid n);$$

and generally

$$c_{\varpi^k}(n) = 0 \quad (\varpi^{k-1} \nmid n), \qquad c_{\varpi^k}(n) = -\varpi^{k-1} \quad (\varpi^{k-1} \mid n,\ \varpi^k \nmid n),$$
$$c_{\varpi^k}(n) = \varpi^{k-1}(\varpi - 1) \quad (\varpi^k \mid n) \ldots\ldots\ldots (4{\cdot}4).$$

* Following Landau, I write $\varpi \mid n$ for 'ϖ is a divisor of n' and $\varpi \nmid n$ for 'ϖ is not a divisor of n.'

It may be immediately verified that these are also the values of $C_{\varpi^k}(n)$. Hence $c_q(n) = C_q(n)$ when q is a power of a prime, and therefore generally.

Summation of the series (1·3).

5. It is plain that $|c_q(n)|$ does not exceed the sum of the divisors of n. It therefore follows from (3·1) that, if $s > 1$,

$$\Sigma \frac{c_q(n)}{q^s} = \prod_{\varpi} \chi_{\varpi},$$

where

$$\chi_{\varpi} = 1 + \frac{c_{\varpi}(n)}{\varpi^s} + \frac{c_{\varpi^2}(n)}{\varpi^{2s}} + \dots.$$

If $\varpi \nmid n$, $\qquad \chi_{\varpi} = 1 - \varpi^{-s}$.

If ϖ^a is the highest power of ϖ which divides n,

$$\chi_{\varpi} = 1 + \frac{\varpi - 1}{\varpi^s} + \frac{\varpi(\varpi - 1)}{\varpi^{2s}} + \dots + \frac{\varpi^{a-1}(\varpi - 1)}{\varpi^{as}} - \frac{\varpi^a}{\varpi^{(a+1)s}}$$

$$= 1 - \varpi^{a-(a+1)s} + \varpi^{-s}(\varpi - 1)\frac{1 - \varpi^{a-as}}{1 - \varpi^{1-s}}$$

$$= (1 - \varpi^{-s})\frac{1 - \varpi^{(a+1)(1-s)}}{1 - \varpi^{1-s}}.$$

Hence

$$\Sigma \frac{c_q(n)}{q^s} = \prod_{\varpi}(1 - \varpi^{-s}) \prod_{\varpi \mid n} \frac{1 - \varpi^{(a+1)(1-s)}}{1 - \varpi^{1-s}}$$

$$= \frac{\sigma_{1-s}(n)}{\zeta(s)} = \frac{n^{-(s-1)}\sigma_{s-1}(n)}{\zeta(s)}.$$

This is Ramanujan's formula, with s in the place of $s - 1$.

The formula (1·32) lies, as Ramanujan points out, somewhat deeper, since $s = 1$ does not lie inside the region of absolute convergence. I have nothing to add to Ramanujan's remarks concerning this limiting case*.

Summation of the series (1·4).

6. Let us write, with Ramanujan,

$$\phi_s(n) = n^s(1 - p^{-s})(1 - p_1^{-s})\dots,$$

where p, p_1, ... are the different prime factors of n, so that $\phi_1(n) = \phi(n)$. Then, if $s > 1$, we have

$$\Sigma \frac{\mu(q)\, c_q(n)}{\phi_s(q)} = \prod \chi_{\varpi},$$

where

$$\chi_{\varpi} = 1 + \frac{\mu(\varpi)\, c_{\varpi}(n)}{\phi_s(\varpi)};$$

* *l.c.*, p. 265.

so that

$$\chi_\varpi = 1 + \frac{1}{\varpi^s - 1} \quad (\varpi \nmid n), \qquad \chi_\varpi = 1 - \frac{\varpi - 1}{\varpi^s - 1} \quad (\varpi \mid n).$$

It follows that

$$\begin{aligned}\Sigma \frac{\mu(q)\, c_q(n)}{\phi_s(q)} &= \prod_{\varpi \nmid n} \left(\frac{\varpi^s}{\varpi^s - 1}\right) \prod_{\varpi \mid n} \left(\frac{\varpi^s - \varpi}{\varpi^s - 1}\right) \\ &= \prod \frac{1}{1 - \varpi^{-s}} \prod_{\varpi \mid n} \{1 - \varpi^{-(s-1)}\} \\ &= \zeta(s)\, n^{-(s-1)}\, \phi_{s-1}(n).\end{aligned}$$

Thus

$$\phi_{s-1}(n) = \frac{n^{s-1}}{\zeta(s)} \Sigma \frac{\mu(q)\, c_q(n)}{\phi_s(q)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6{\cdot}1).$$

This formula is equivalent to Ramanujan's formula (9·6), and reduces to (1·4) when $s = 2$.

The case $s = 1$.

7. The case in which $s = 1$, which is not discussed by Ramanujan, is of particular interest.

We observe first that

$$n^{-(s-1)}\, \phi_{s-1}(n)\, \zeta(s) = \{1 - p^{-(s-1)}\}\, \{1 - p_1^{-(s-1)}\} \ldots \zeta(s)$$

tends to zero, when $s \to 1$, unless n is a prime p or a power of a prime p, in which case its limit is $\log p$. Thus

$$\lim_{s \to 1} n^{-(s-1)}\, \phi_{s-1}(n)\, \zeta(s) = \Lambda(n),$$

$\Lambda(n)$ having its usual significance in the theory of primes*. If then we suppose that the series (6·1) is convergent for $s = 1$, and that its sum is continuous, we obtain

$$\Lambda(n) = \Sigma \frac{\mu(q)\, c_q(n)}{\phi(q)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}1).$$

It is not difficult to prove that this formula is incorrect.

In order to prove this, and to obtain the correct formula, I consider the function

$$f(s) = \Sigma \frac{\mu(q)\, c_q(n)}{q^{s-1}\, \phi(q)},$$

supposing first that σ, the real part of s, is greater than unity. It is plain that

$$f(s) = \prod_{\varpi} \theta_\varpi,$$

where

$$\theta_\varpi = 1 - \frac{c_\varpi(n)}{\varpi^{s-1}(\varpi - 1)},$$

so that

$$\theta_\varpi = 1 + \frac{1}{\varpi^{s-1}(\varpi - 1)} \quad (\varpi \nmid n), \qquad \theta_\varpi = 1 - \frac{1}{\varpi^{s-1}} \quad (\varpi \mid n).$$

* $\Lambda(n) = \log \varpi$ if $n = \varpi^m$, and $\Lambda(n) = 0$ otherwise.

It follows that

$$f(s) = \prod_{\varpi+n}\left\{1+\frac{1}{\varpi^{s-1}(\varpi-1)}\right\}\prod_{\varpi|n}\left(1-\frac{1}{\varpi^{s-1}}\right)$$
$$= \prod\left\{1+\frac{1}{\varpi^{s-1}(\varpi-1)}\right\}\prod_{\varpi|n}\left\{\frac{(\varpi-1)(\varpi^{s-1}-1)}{\varpi^s-\varpi^{s-1}+1}\right\}$$
$$= g(s)\,h(s),$$

say. Now

$$\frac{g(s)}{\zeta(s)} = \prod\left[\left\{1+\frac{1}{\varpi^{s-1}(\varpi-1)}\right\}\left(1-\frac{1}{\varpi^s}\right)\right]$$
$$= \prod\{1+O(\varpi^{-\sigma-1})\};$$

and this product is uniformly convergent throughout any half plane $\sigma \geqslant \delta > 0$, so that $g(s)/\zeta(s)$ is regular for $\sigma > 0$. Thus $g(s)$ is regular for $\sigma > 0$, except for a simple pole at $s = 1$, with residue 1; and its properties for $\sigma > 0$ are substantially the same as those of $\zeta(s)$.

If n is a prime p, or a power of a prime p,

$$h(s) = \frac{(p-1)(p^{s-1}-1)}{p^s-p^{s-1}+1}$$

has a simple zero for $s = 1$, and

$$h'(1) = \frac{p-1}{p}\log p.$$

Thus in this case $f(s)$ is regular for $s = 1$, and

$$f(1) = \frac{p-1}{p}\log p.$$

In all other cases (except when $n = 1$), $h(s)$ has a zero of at least the second order, and

$$f(1) = 0.$$

We have now only to apply to $f(s)$ the arguments which, when applied to the function $1/\zeta(s)$, lead to the proof of the well known theorem expressed by the equation

$$\Sigma\frac{\mu(n)}{n} = 0^* \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7{\cdot}2),$$

in order to conclude that the series for $f(s)$ is convergent for $s = 1$, and that its value is $f(1)$. That is to say

$$\Sigma\frac{\mu(q)\,c_q(n)}{\phi(q)} = \frac{p-1}{p}\log p$$

* See for example Landau, *Handbuch*, pp. 593 *et seq.*; Hardy and Littlewood, 'New proofs of the prime-number theorem and similar theorems,' *Quarterly Journal*, vol. 46, 1915, pp. 215–219.

or zero, according as n is or is not of the form p^k; or

$$\Sigma \frac{\mu(q)\, c_q(n)}{\phi(q)} = \frac{\phi(n)\, \Lambda(n)}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}3),$$

which contradicts (7·1).

It should be noticed that, if we are merely concerned to prove that (7·1) is false, all the difficult part of the preceding argument may be omitted. For, if (7·1) were true, $f(s)$ would certainly tend to the limit $\Lambda(n)$ when $s \to 1$, and the mere knowledge of the value of $f(1)$ is enough to show that this is not the case.

The simplest examples of the formulae (6·1) and (7·3) are obtained by taking $n = 2$, when

$$c_q(n) = \mu(q) + 2\mu(\tfrac{1}{2}q)\,*$$

is equal to $\mu(q)$ if q is odd and to $-\mu(q)$ if q is oddly even. We then obtain

$$\Sigma \frac{(-1)^{q-1}\,|\mu(q)|}{\phi_s(q)} = (1-2^{1-s})\,\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - \ldots,$$

which tends to the limit $\log 2$ when $s \to 1$; but

$$\Sigma \frac{(-1)^{q-1}\,|\mu(q)|}{\phi(q)} = \tfrac{1}{2}\log 2.$$

The last result may be written in the form

$$1 - \frac{1}{\phi(2)} + \frac{1}{\phi(3)} + \frac{1}{\phi(5)} - \frac{1}{\phi(6)} + \frac{1}{\phi(7)} - \frac{1}{\phi(10)} + \ldots = \tfrac{1}{2}\log 2.$$

The series $\Sigma \dfrac{c_q(n)}{\phi(q)}$.

8. Another very interesting series, the consideration of which is suggested naturally by the work which precedes, is the series

$$\Sigma \frac{\mu(q)}{\phi(q)} = 1 - \frac{1}{\phi(2)} - \frac{1}{\phi(3)} - \frac{1}{\phi(5)} + \frac{1}{\phi(6)} - \ldots.$$

I shall prove that this series is convergent, and has the sum 0: more generally,

$$\Sigma \frac{c_q(n)}{\phi(q)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8{\cdot}1).$$

Let $$f(s) = \Sigma \frac{c_q(n)}{q^{s-1}\,\phi(q)};$$

and let us write $s - 1 = z$. Then

$$f(s) = \Pi\, \chi_\varpi,$$

where $$\chi_\varpi = 1 + \frac{c_\varpi(n)}{\varpi^z\,\phi(\varpi)} + \frac{c_{\varpi^2}(n)}{\varpi^{2z}\,\phi(\varpi^2)} + \ldots.$$

* We agree to regard $\mu(x)$, when x is not an integer, as zero.

If $\varpi + n$ $$\chi_\varpi = 1 - \frac{\varpi^{-z}}{\varpi - 1};$$

while if ϖ^a is the highest power of ϖ which divides n

$$\chi_\varpi = 1 + \varpi^{-z} + \varpi^{-2z} + \ldots + \varpi^{-az} - \frac{\varpi^{-(a+1)z}}{\varpi - 1}.$$

Thus

$$f(s) = \prod_{\varpi+n}\left(1 - \frac{\varpi^{-z}}{\varpi - 1}\right)\prod_{\varpi|n}\left(1 + \varpi^{-z} + \ldots + \varpi^{-az} - \frac{\varpi^{-(a+1)z}}{\varpi - 1}\right)$$
$$= g(s)\, h(s),$$

say. It is plain that $h(s)$ is a finite Dirichlet's series in s. On the other hand, if

$$\zeta_n(s) = \prod_{\varpi+n}\left(\frac{1}{1-\varpi^{-s}}\right) = \zeta(s)\prod_{\varpi|n}(1-\varpi^{-s}),$$

$$g(s)\,\zeta_n(s) = \prod_{\varpi+n}\left\{\left(1 - \frac{\varpi^{-(s-1)}}{\varpi - 1}\right)\Big/(1-\varpi^{-s})\right\}$$
$$= \prod_{\varpi+n}\{1 + O(\varpi^{-\sigma-1}) + O(\varpi^{-2\sigma})\}$$

is regular and bounded in any half-plane $\sigma \geqslant \frac{1}{2} + \delta > \frac{1}{2}$. Thus $g(s)$ behaves, in any such half-plane, substantially like the reciprocal of $\zeta_n(s)$, or of $\zeta(s)$; and so therefore does $f(s)$. And so we can prove (8·1) by substantially the same argument as that which proves (7·2).

For $n = 1$ we obtain

$$\Sigma \frac{\mu(q)}{\phi(q)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8\cdot2).$$

It should be observed that, in this series, the terms cancel in pairs, since

$$\frac{\mu(2q)}{\phi(2q)} = -\frac{\mu(q)}{\phi(q)}$$

if q is odd. Naturally this does not in itself establish the truth of (8·2).

If $n = 2$ we find that $c_q(n)$ is equal to $\mu(q)$ if q is odd, to $-\mu(q)$ if q is an odd multiple of 2, to $2\mu(\frac{1}{2}q)$ if q is an odd multiple of 4, and to zero if $q \equiv 0 \pmod 8$. Thus we obtain

$$1 + \frac{1}{\phi(2)} - \frac{1}{\phi(3)} - \frac{2}{\phi(4)} - \frac{1}{\phi(5)} - \frac{1}{\phi(6)} - \frac{1}{\phi(7)} - \frac{1}{\phi(10)}$$
$$- \frac{1}{\phi(11)} + \frac{2}{\phi(12)} - \ldots = 0.$$

9. The following results may be proved in substantially the same way:—

$$\Sigma \frac{\mu(q) \log q}{\phi(q)} = 0,$$

$$\Sigma \frac{\mu(q) (\log q)^2}{\phi(q)} = 4A \log 2;$$

where
$$A = \Pi \left\{1 - \frac{1}{(\varpi - 1)^2}\right\},$$

ϖ running through all odd primes*;

$$\sum_{1,3,5,\ldots} \frac{\mu(q)}{\phi(q)} = 0, \qquad \sum_{1,3,5,\ldots} \frac{\mu(q) \log q}{\phi(q)} = -2A;$$

$$\Sigma \frac{(-1)^{\nu(q)}}{q} = 0,$$

where $\nu(q)$ is the number of different prime factors of q;

$$\Sigma \frac{(-1)^{\nu(q)} \log q}{q} = 0;$$

$$\sum_{1,3,5,\ldots} \frac{(-1)^{\nu(q)}}{q} = 0, \qquad \sum_{1,3,5,\ldots} \frac{(-1)^{\nu(q)} \log q}{q} = -2A;$$

$$\Sigma \frac{2^{\nu(n)} \mu(n)}{n} = \Sigma \frac{2^{\nu(n)} \mu(n) \log n}{n} = \Sigma \frac{2^{\nu(n)} \mu(n) (\log n)^2}{n} = 0;$$

$$\Sigma \frac{2^{\nu(n)} \mu(n) (\log n)^3}{n} = -24A \log 2.$$

* $A = \cdot 66016\ldots$ is the constant which appears to play a fundamental part in 'Goldbach's Problem'. See two notes, by Shah and Wilson and by Hardy and Littlewood, in vol. XIX (1919) of the *Proceedings of the Cambridge Philosophical Society* (pp. 238–244 and 245–254).

On the distribution of primes. By H. CRAMÉR, Stockholm.

(Communicated by Prof. G. H. HARDY.)

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Throughout the whole of this paper, I shall assume the truth of the Riemann hypothesis concerning the roots of the Zeta function, viz. $\zeta(\sigma+it)\neq 0$ *for* $\sigma>\frac{1}{2}$.

This being so, it is known that

$$\pi(x)=Li(x)+O(x^{\frac{1}{2}}\log x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$p_n\sim n\log n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where $\pi(x)$ denotes the number of primes less than or equal to x, and p_n denotes the nth prime. The last relation is independent of the Riemann hypothesis. But very little is known as to the behaviour of the difference

$$\Delta_n=p_{n+1}-p_n$$

between two successive primes, for large values of n. It follows from the "Prime Number Theorem" (1) or (2) that

$$\frac{\Delta_1+\Delta_2+\ldots\Delta_n}{n}\sim\log n\sim\log p_n$$

and from (1) that

$$\Delta_n=O(p_n^{\frac{1}{2}}\log^2 p_n).$$

I have recently shown* that the last relation may be replaced by

$$\Delta_n=O(p_n^{\frac{1}{2}}\log p_n).$$

So far as I know, this is all that is actually known about Δ_n. It is very probable that $\Delta_n=2$ for an infinity of values of n; but this has not yet been proved, and it has not even been proved that $\Delta_n<\frac{1}{2}\log p_n$ or $\Delta_n>2\log p_n$ for an infinity of values of n. The object of the present paper is to prove the following theorem, which gives an upper limit for the frequency of certain large values of Δ_n.

Theorem. *Let* $h(x)$ *be the number of primes* $p_n\leq x$ *satisfying the inequality*

$$p_{n+1}-p_n>p_n^k,$$

where $0<k\leq\frac{1}{2}$. *Then*

$$h(x)=O(x^{1-\frac{1}{2}k+\epsilon})$$

for every positive ϵ.

* "Some theorems concerning prime numbers," *Arkiv för Matematik, Astronomi och Fysik*, Band 15 (1920), No. 5, pp. 1—32.

It is interesting to remark that we may obtain by a very trivial argument (*viz.* that the sum of all the $h(x)\,\Delta_n$'s which are greater than p_n^k must be less than $x + x^k$) the evaluation

$$h(x) = O(x^{1-k}),$$

but it seems impossible to improve this even by direct deduction from the Prime Number Theorem.

The proof of the theorem given here depends on the theory of the function $\Sigma e^{\rho z}$, which I have studied in some recent papers*. We denote here and in the sequel by $\rho = \frac{1}{2} + i\gamma$ an arbitrary zero of the function $\zeta(s)$, situated in the *upper half* of the plane of the complex variable $s = \sigma + it$.

In order to prove the theorem, we shall require a set of lemmas. It seems convenient to remark that all the sets of points we shall have to deal with in the proof consist of a finite number of finite intervals (and perhaps a finite number of isolated points). Hence their measure may be taken to be the measure in the elementary sense.

Lemma 1. *If*

$$\phi(v, t) = \sum_{\gamma \leqq v} e^{\gamma i t},$$

we have

$$\int_{x-2}^{x+2} |\phi(v, t)|\, dt = O\left(v^{\frac{1}{2}} (\log v)^{\frac{3}{2}}\right)$$

uniformly for $x > 2$.

Proof. We have

$$|\phi(v, t)|^2 = \sum_{\gamma \leqq v} \sum_{\gamma' \leqq v} e^{(\gamma - \gamma') it},$$

and thus

$$\int_{x-2}^{x+2} |\phi(v, t)|^2 dt = \sum_{\gamma \leqq v} \sum_{\gamma' \leqq v} \int_{x-2}^{x+2} e^{(\gamma - \gamma') it}\, dt$$

$$< \sum_{\gamma \leqq v} \sum_{\gamma' \leqq v} \operatorname{Min}\left(\frac{2}{|\gamma - \gamma'|}, 4\right).$$

The number of numbers γ' in the interval $(\gamma + \nu, \gamma + \nu + 1)$ is $O(\log(\gamma + \nu))$†. It follows that our sum is

$$O\left(\sum_{\gamma \leqq v} \log \gamma\right) + O\left[\sum_{\gamma \leqq v-1} \left(\frac{\log(\gamma+1)}{1} + \frac{\log(\gamma+2)}{2} + \cdots + \frac{\log(\gamma + [v-\gamma])}{[v-\gamma]}\right)\right]$$

$$= O(v (\log v)^3),$$

* *l.c.* (footnote [1]); *Comptes rendus*, t. 168, p. 539; and *Mathematische Zeitschrift*, Band 4, pp. 104—

† Landau, *Handbuch*, p. 337.

uniformly in x. Hence the truth of the lemma follows by the use of Schwarz's inequality.

Lemma 2. *Let us put*

$$y = \frac{1}{t}\, e^{-\alpha t}$$

and consider the interval $x - 2 \leqq t \leqq x + 2$, *where* $x > 2$ *and* $\frac{1}{2} \leqq \alpha < 1$. *Then the set* S_x *of points* t *belonging to this interval such that*

$$\left| \sum_{\gamma} \cos \gamma t\, e^{-\gamma y} \right| \geqq \tfrac{1}{8} e^{\frac{1}{2}t}$$

is of measure M_x *such that*

$$M_x = O\left(x^2 e^{-\frac{1-\alpha}{2}x}\right).$$

Proof. We have

$$\left| \sum_{\gamma} \cos \gamma t\, e^{-\gamma y} \right| \leqq \left| \sum_{\gamma} \left(e^{i\gamma t} \,.\, e^{-\gamma y} \right) \right| = \left| y \int_0^{\infty} \phi(v, t)\, e^{-vy}\, dv \right|$$

$$\leqq y \int_0^{\infty} |\, \phi(v, t)\, |\, e^{-vy}\, dv.$$

It follows by Lemma 1 that, if we denote by y_1 and y_2 the values of y at the points $x - 2$ and $x + 2$ respectively,

$$\int_{x-2}^{x+2} \left| \sum_{\gamma} \cos \gamma t\, e^{-\gamma y} \right| dt = O\left(\int_0^{\infty} y_1 v^{\frac{1}{2}} (\log v)^{\frac{3}{2}} e^{-vy_2}\, dv \right)$$

$$= O\left(y_1 y_2^{-\frac{3}{2}} \left(\log \frac{1}{y_2} \right)^{\frac{3}{2}} \right)$$

$$= O\left(x^2 e^{\frac{1}{2}\alpha x}\right).$$

Thus the measure of the set of points t belonging to the interval of integration, such that

$$\left| \sum_{\gamma} \cos \gamma t\, e^{-\gamma y} \right| \geqq \tfrac{1}{8} e^{\frac{1}{2}t},$$

must be of the form

$$M_x = O\left(x^2 e^{\frac{1}{2}\alpha x} \,.\, e^{-\frac{1}{2}x}\right)$$

$$= O\left(x^2 e^{-\frac{1-\alpha}{2}x}\right).$$

Lemma 3. *In the set* $\bar{S}_x$, *complementary to the set* S_x *of Lemma 2, we have*

$$\sum_{1}^{\infty} \frac{\eta(n)}{n\left((t - \log n)^2 + y^2\right)} = \frac{\pi + \theta}{ty},$$

where $|\, \theta \,| < 1$ *for all sufficiently large values of* x. *Here* $\eta(n)$ *denotes the arithmetical function defined for integral values of* $n \geqq 1$ *by*

$$\eta(n) = \frac{1}{m} \quad (n = p^m,\ p \text{ prime}),$$

$$\eta(n) = 0 \quad (\text{otherwise}).$$

Proof. In a recently published paper*, I have proved the formula

$$y \sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2+y^2)} = \frac{\pi}{t} - 2\pi \Re \sum_\gamma \frac{e^{(\rho-1)(t+iy)}}{t+iy} + o\left(\frac{1}{t}\right),$$

where y may denote any positive decreasing function of t which tends to zero as t tends to infinity. On the Riemann hypothesis, and assuming y to be the function of Lemma 2, this formula reduces to

$$\begin{aligned} & y \sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2+y^2)} \\ &= \frac{\pi}{t} - \frac{2\pi}{t} e^{-\frac{1}{2}t}\left(1+O\left(\frac{y}{t}\right)\right) \sum_\gamma \cos\left(\gamma t - \tfrac{1}{2}y - \arg(t+iy)\right) e^{-\gamma y} \\ &= \frac{\pi}{t} - \frac{2\pi}{t} e^{-\frac{1}{2}t}\left(1+O\left(\frac{y}{t}\right)\right)\left(\sum_\gamma \cos\gamma t\, e^{-\gamma y} + O\left(y \sum_\gamma e^{-\gamma y}\right)\right) + o\left(\frac{1}{t}\right). \end{aligned}$$

But we have

$$\sum_\gamma e^{-\gamma y} = O\left(\sum_1^\infty \log n\, e^{-ny}\right) = O\left(\frac{1}{y}\log\frac{1}{y}\right).$$

Hence

$$y \sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2+y^2)} = \frac{\pi}{t}\left(1 - 2e^{-\frac{1}{2}t} \sum_\gamma \cos\gamma t\, e^{-\gamma y}\right) + o\left(\frac{1}{t}\right).$$

By the definition of the set S_x, we have

$$\left| 2e^{-\frac{1}{2}t} \sum_\gamma \cos\gamma t e^{-\gamma y} \right| < \tfrac{1}{4}$$

for all values of t belonging to the complementary set $\bar{S}_x$. Since $\bar{S}_x$ is contained in the interval $(x-2,\ x+2)$, we conclude that

$$\sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2+y^2)} = \frac{\pi+\theta}{ty},$$

where $|\theta| < 1$ for all sufficiently large values of x.

Lemma 4. *We denote by $f(v)$ the function*

$$f(v) = \sum_{n \leqq v} \eta(n)$$

introduced by Riemann, and we consider the interval $(x-2,\ x+2)$ of the two preceding lemmas. The set of points t belonging to this interval such that

$$\sum_{t < \log n \leqq t+y} \eta(n) = f(e^{t+y}) - f(e^t) \geqq \frac{9y}{t} e^t$$

is, for all sufficiently large values of x, a subset of the set S_x of Lemma 2, so that its measure is of the form

$$O\left(x^2 e^{-\frac{1-\alpha}{2}x}\right).$$

* *l.c.*, footnote *, p. 272.

Proof. To prove this lemma, it is only necessary to show that

$$\sum_{t<\log n\leqq t+y} \eta(n) < \frac{9y}{t}\, e^t$$

if t belongs to the complementary set $\bar{S}_x$. This follows immediately from Lemma 3, for we have, since $\eta(n) \geqq 0$,

$$\sum_{t<\log n\leqq t+y} \frac{\eta(n)}{n\left((t-\log n)^2+y^2\right)} < \frac{\pi+1}{ty}$$

if x is sufficiently great and t belongs to $\bar{S}_x$. Hence

$$\sum_{t<\log n\leqq t+y} \eta(n) < \frac{\pi+1}{ty}\,.\,2y^2.\,e^{t+y} < \frac{9y}{t}\, e^t$$

for all sufficiently large values of x.

Lemma 5. *Let S be a set of points of measure M, situated in a finite interval ab. Then it is possible to divide ab into sub-intervals of length δ (the two extreme intervals being possibly less than δ) in such a way that not more than M/δ of the points of division belong to S.*

Proof. Consider any such division of ab, and denote by $\alpha\beta$ an arbitrary sub-interval of length δ. Let x be a point in $\alpha\beta$, and denote by $\phi(x)$ the number of points of S which are "congruent" to x according to the adopted division of ab. Then it is clear that

$$\int_\alpha^\beta \phi(x)\, dx = M.$$

Thus there must be at least one point x_o in $\alpha\beta$, such that $\phi(x_o) \leqq M/\delta$. Starting the division of ab from this point, we see that the conditions stated in the lemma are fulfilled.

Lemma 6. *Let us denote bg σ_x the set of points t, belonging to the interval $(x-1,\ x+1)$, such that*

$$\pi\left(e^{t+cy}\right) - \pi\left(e^{t-cy}\right) \leqq \frac{y}{t}\, e^t,$$

where c is a positive constant. Here $\pi(v)$ denotes as usual the number of primes less than or equal to v. Then it is possible to give such a value to c that the measure μ_x of σ_x satisfies the relation

$$\mu_x = O\left(x^4 e^{-\frac{1-\alpha}{2}x}\right).$$

Proof. We shall prove this lemma by first excluding from the interval $(x-2,\ x+2)$ a certain set, the measure of which satisfies

the relation just stated for μ_x, and then proving that we may choose c so that

$$\pi(e^{t+cy}) - \pi(e^{t-cy}) > \frac{y}{t} e^t \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in the part of the remaining set which belongs to $(x-1, x+1)$.

By Lemma 5, we are able to divide $(x-2, x+2)$ into subintervals of the length

$$y_2 = \frac{1}{x+2} e^{-\alpha(x+2)}$$

in such a way that not more than

$$\frac{M_x}{y_2}$$

of the points of division belong to the set S_x of Lemma 2.

Supposing this to have been done, we exclude from the interval $(x-2, x+2)$ the set Σ_x defined in the following way: we take first the whole set S_x and then, denoting by t_0 any of the just mentioned points of division belonging to S_x, the interval $(t_0 - 2x^2y_2, t_0 + 2x^2y_2)$. The measure of Σ_x is thus less than

$$M_x + 2x^2y_2 \cdot \frac{M_x}{y_2} = O\left(x^4 e^{-\frac{1-\alpha}{2}x}\right).$$

In order to prove the lemma, we now have to show that c may be so chosen that (3) is valid for any t in $(x-1, x+1)$ not belonging to Σ_x. It is to be noticed that the definition of Σ_x in no way involves c.

Since t does not belong to Σ_x, it does not belong to S_x. Thus we have by Lemma 3

$$\sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2 + y^2)} > \frac{\pi - 1}{ty} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

for all sufficiently large values of x. We put

$$\sum_1^\infty = \sum_{\log n \leqq t-1} + \sum_{t-1 < \log n \leqq t-x^2y_2} + \sum_{t-x^2y_2 < \log n \leqq t-cy_2} + \sum_{t-cy_2 < \log n \leqq t+cy_2}$$

$$+ \sum_{t+cy_2 < \log n \leqq t+x^2y_2} + \sum_{t+x^2y_2 < \log n \leqq t+1} + \sum_{t+1 < \log n}$$

$$= A_1 + A_2 + \ldots A_7 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Then we have

$$A_1 < \sum_{\log n \leqq t-1} \frac{\eta(n)}{n} < \sum_{n \leqq e^t} \frac{1}{n} = o\left(\frac{1}{ty}\right) \ldots\ldots\ldots\ldots (6),$$

and

$$A_7 = \sum_{\log n > t+1} \frac{\eta(n)}{n((t-\log n)^2+y^2)} < \sum_{\log n > t+1} \frac{\eta(n)}{n(\log n - t)^2}$$

$$= \sum_{\log n > t+1} \frac{\eta(n)}{n}\left(\frac{1}{(\log n)^2+t^2} + \frac{2t\log n}{((\log n)^2+t^2)(\log n - t)^2}\right)$$

$$< \sum_{\log n > t+1} \frac{\eta(n)}{n}\left(\frac{1}{(\log n)^2} + \frac{2t}{\log n}\right) = o\left(\frac{1}{ty}\right) \quad \ldots\ldots\ldots\ldots(7),$$

since the series $\Sigma \frac{\eta(n)}{n\log n}$ is convergent.

In order to obtain similar evaluations for A_2 and A_3, we consider the division of $(x-2,\ x+2)$ into sub-intervals of the length y_2 which we have used for the definition of Σ_x. Since t does not belong to Σ_x, it follows from this definition that none of the points of division situated in the interval $(t-x^2y_2,\ t+x^2y_2)$ belong to the set S_x. Hence, denoting by t_0 any such point of division, and by y_0 the corresponding value of y, we obtain, by Lemma 4,

$$\sum_{t_0 < \log n \leqq t_0+y_2} \eta(n) \leqq \sum_{t_0 < \log n \leqq t_0+y_0} \eta(n) < \frac{9y_0}{t_0} e^{t_0}.$$

Thus, if we consider first A_3, and group together the terms belonging to the same sub-interval (considered as interval of variation of $\log n$), we obtain

$$A_3 < K\frac{y}{t} e^t . e^{-t}\left(\frac{1}{c^2y^2} + \frac{1}{(c+1)^2y^2} + \frac{1}{(c+2)^2y^2} + \ldots\right) < \frac{2K}{cty} \quad \ldots(8),$$

supposing $c>1$ and denoting by K a constant independent of c and t.

Grouping together the terms of A_2 in a similar way, we have

$$A_2 < e^{-(t-1)} \sum_{t-1 < \log n \leqq t-x^2y_2} \frac{1}{(t-\log n)^2}$$

$$= O\left[e^{-t} . ye^t\left(\frac{1}{x^4y^2} + \frac{1}{(x^2+1)^2y^2} + \frac{1}{(x^2+2)^2y^2} + \ldots\right)\right]$$

$$= O\left(\frac{1}{x^2y}\right) = o\left(\frac{1}{ty}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

since the number of terms in each group is of the form $O(ye^t)$.

Of the remaining terms, A_6 may obviously be treated in the same way as A_2, and A_5 in the same way as A_3. Thus we see, from (4)—(9), that it is possible to determine two absolute constants c_0 and x_0 such that

$$A_4 = \sum_{t-cy_2 < \log n \leqq t+cy_2} \frac{\eta(n)}{n((t-\log n)^2+y^2)} > \frac{\pi-2}{ty}$$

for all values of t in $(x-1, x+1)$ not belonging to Σ_x, so long as $c > c_0$ and $x > x_0$. Hence, *a fortiori*,

$$\sum_{t-cy<\log n\leqq t+cy} \frac{\eta(n)}{n((t-\log n)^2+y^2)} > \frac{\pi-2}{ty}$$

and

$$\frac{e^{-(t-cy)}}{y^2} \sum_{t-cy<\log n\leqq t+cy} \eta(n) > \frac{\pi-2}{ty},$$

$$\sum_{t-cy<\log n\leqq t+cy} \eta(n) > (\pi-2)\frac{y}{t}e^{t-cy} \quad \ldots\ldots\ldots(10).$$

But we have

$$\sum_{t-cy<\log n\leqq t+cy} \eta(n) = \pi(e^{t+cy}) - \pi(e^{t-cy})$$

$$+ \sum_{\nu=2}^{\frac{t+cy}{\log 2}} \frac{1}{\nu}\left[\pi\left(e^{\frac{1}{\nu}(t+cy)}\right) - \pi\left(e^{\frac{1}{\nu}(t-cy)}\right)\right],$$

and

$$\sum_{\nu=2}^{\frac{t+cy}{\log 2}} \frac{1}{\nu}\left[\pi\left(e^{\frac{1}{\nu}(t+cy)}\right) - \pi\left(e^{\frac{1}{\nu}(t-cy)}\right)\right]$$

$$= O\left[\sum_{2}^{t} \frac{1}{\nu}\left(e^{\frac{1}{\nu}(t+cy)} - e^{\frac{1}{\nu}(t-cy)} + 1\right)\right] = O\left[\sum_{2}^{t} \frac{1}{\nu}\left(\frac{y}{\nu}e^{\frac{1}{2}t} + 1\right)\right]$$

$$= O(\log t) = o\left(\frac{y}{t}e^{t}\right).$$

Hence, if c is fixed and greater than c_0, we obtain from (10),

$$\pi(e^{t+cy}) - \pi(e^{t-cy}) > \frac{y}{t}e^{t},$$

for all sufficiently large values of x, and for all values of t in $(x-1, x+1)$ not belonging to Σ_x.

Proof of the theorem.

Consider the set of points e^t, where t passes through all the points t of the set σ_x of Lemma 6. This set belongs to the interval (e^{x-1}, e^{x+1}), and its measure is less than $e^{x+1}\mu_x$. Hence if, in Lemma 6, we write x in the place of e^x, t in the place of e^t and k in the place of $1-\alpha$, this lemma will take the following form:

Let us denote by σ_x' the set of points t, belonging to the interval $\left(\frac{x}{e}, ex\right)$, such that

$$\pi\left(t+\frac{c}{\log t}t^k\right) - \pi\left(t-\frac{c}{\log t}t^k\right) \leqq \frac{t^k}{\log^2 t},$$

where c is a positive constant and $0 < k \leqq \frac{1}{2}$. Then it is possible to give such a value to c that the measure μ_x' of σ_x' satisfies the relation

$$\mu_x' = O(x^{1-\frac{1}{2}k} \log^4 x).$$

Suppose now that c has been properly fixed. Then it is clear that, if x is sufficiently large, and if p_n denotes a prime belonging to the interval $(\frac{1}{2}x, x)$ and satisfying the inequality

$$p_{n+1} - p_n > p_n^k,$$

then the interval (p_n, p_{n+1}) will contribute more than $\frac{1}{2}p_n^k$, and *a fortiori* more than $\frac{1}{2}(\frac{1}{2}x)^k$, to σ_x'. Hence we obtain

$$\begin{aligned} \tfrac{1}{2}(\tfrac{1}{2}x)^k (h(x) - h(\tfrac{1}{2}x)) &= O(x^{1-\frac{1}{2}k+\epsilon}), \\ h(x) - h(\tfrac{1}{2}x) &= O(x^{1-\frac{3}{2}k+\epsilon}). \end{aligned}$$

If we replace in the last relation x first by $\frac{1}{2}x$, then by $\frac{1}{4}x$, and so on, and add together all the relations obtained in this way, we get

$$h(x) = O(x^{1-\frac{3}{2}k+\epsilon}).$$

Hence our theorem is proved.

CAMBRIDGE, 20 *July*, 1920.

Note on the parity of the number which enumerates the partitions of a number. By Major P. A. MacMahon.

[*Read* 25 October 1920.]

In a letter received by me from the late S. Ramanujan about a year ago he stated that it was his intention to calculate the number of the partitions of 1000 by the direct approximate formula which had been successfully used by him and G. H. Hardy in the calculation of the case $n = 200$. He enquired, at the same time, if I knew of any simple way of ascertaining whether the sought number is even or uneven as this information would be of importance to him. This note has arisen in consequence of this enquiry.

I shew, in particular, that in the case of the partitions of 1000 the parity can be found, in a few minutes, from certain congruence relations.

It is easy to derive, from the theory of the self-conjugate partitions of n, that

$$\frac{1}{\prod_1^\infty{}^m (1-q^m)} \equiv \prod_1^\infty{}^m (1+q^{2m-1}) \bmod 2.$$

Thence

$$\frac{1}{\prod_1^\infty{}^m (1-q^m)} \equiv \frac{\prod_1^\infty{}^m \{(1+q^{4m-1})(1+q^{4m-3})(1-q^{4m})\}}{\prod_1^\infty{}^m (1-q^{4m})} \bmod 2.$$

The numerator of the fraction on the right is one of Jacobi's elliptic products which has the series expression

$$1 + q + q^3 + q^6 + q^{10} + \ldots + q^{\frac{1}{2}s(+1)} + \ldots .$$

Wherefore if p_n denote the number of partitions of n

$$\Sigma p_n q^n \equiv (1 + q + q^3 + q^6 + q^{10} + \ldots)\, \Sigma p_n q^{4n} \bmod 2.$$

Put $p_n \equiv a_n \bmod 2$.

Comparison of the coefficients of like powers of q yields the four relations

$$\begin{aligned}
a_{4n} \equiv\ & a_n + a_{n-7} + a_{n-9} + a_{n-30} + a_{n-34} + a_{n-69} + a_{n-75} + a_{n-124} \\
& + a_{n-132} + a_{n-195} + a_{n-205} + a_{n-282} + a_{n-294} + \ldots , \\
a_{4n+1} \equiv\ & a_n + a_{n-5} + a_{n-11} + a_{n-26} + a_{n-38} + a_{n-63} + a_{n-81} + a_{n-116} \\
& + a_{n-140} + a_{n-185} + a_{n-215} + a_{n-270} + a_{n-306} + \ldots , \\
a_{4n+2} \equiv\ & a_{n-1} + a_{n-2} + a_{n-16} + a_{n-19} + a_{n-47} + a_{n-52} + a_{n-94} + a_{n-101} \\
& + a_{n-157} + a_{n-166} + a_{n-236} + a_{n-247} + a_{n-331} + \ldots , \\
a_{4n+3} \equiv\ & a_n + a_{n-3} + a_{n-13} + a_{n-22} + a_{n-42} + a_{n-57} + a_{n-87} + a_{n-108} \\
& + a_{n-148} + a_{n-175} + a_{n-225} + a_{n-258} + a_{n-318} + \ldots .
\end{aligned}$$

These relations may be written

$$\left.\begin{aligned} a_{4n} &= \sum_0^\infty{}^s\, a_{n-(8s^2\pm s)} \\ a_{4n+1} &= \sum_0^\infty{}^s\, a_{n-(8s^2\pm 3s)} \\ a_{4n+2} &\equiv \sum_0^\infty{}^s\, a_{n-1-(8s^2\pm 7s)} \\ a_{4n+3} &\equiv \sum_0^\infty{}^s\, a_{n-(8s^2\pm 5s)} \end{aligned}\right\}.$$

In fact the four relations are connected with the four elliptic products

$$\prod_1^\infty{}^m (1-q^{16m-9})(1-q^{16m-7})(1-q^{16m}),$$

$$\prod_1^\infty{}^m (1-q^{16m-11})(1-q^{16m-5})(1-q^{16m}),$$

$$q\prod_1^\infty{}^m (1-q^{16m-15})(1-q^{16m-1})(1-q^{16m}),$$

$$\prod_1^\infty{}^m (1-q^{16m-13})(1-q^{16m-3})(1-q^{16m}).$$

The formulae which have been obtained soon involve high numbers and are therefore suitable for the calculation of parity of p_n when m is large.

As an example of the use of the formulae I append the calculation of a_{1000}, the parity of p_{1000}, making use of the enumeration of the partitions of n, as far as $n = 200$, calculated by me in connection with the valuable paper by G. H. Hardy and S. Ramanujan*.

We use the first of the four relations but we first of all require the parities of p_{250}, p_{243}, p_{241}, p_{220}, p_{216}.

From the third relation

$$\begin{aligned} a_{250} &\equiv a_{61} + a_{60} + a_{46} + a_{43} + a_{15} + a_{10} \\ &\equiv 1+1+0+1+0+0 \equiv 1. \end{aligned}$$

From the fourth relation

$$\begin{aligned} a_{243} &\equiv a_{60} + a_{57} + a_{47} + a_{38} + a_{18} + a_3 \\ &\equiv 1+0+0+1+1+1 \\ &\equiv 0. \end{aligned}$$

From the second relation

$$\begin{aligned} a_{241} &\equiv a_{60} + a_{55} + a_{49} + a_{34} + a_{22} \\ &\equiv 1+0+1+0+0 \\ &\equiv 0. \end{aligned}$$

* *Proc. Lond. Math. Soc.* Vol. xv *et seq.*

From the first relation

$$\begin{aligned} a_{220} &\equiv a_{55} + a_{48} + a_{46} + a_{25} + a_{21} \\ &\equiv 0 + 1 + 0 + 0 + 0 \\ &\equiv 1. \\ a_{216} &\equiv a_{54} + a_{47} + a_{45} + a_{24} + a_{20} \\ &\equiv 1 + 0 + 0 + 1 + 1 \\ &\equiv 1. \end{aligned}$$

Thence

$$\begin{aligned} a_{1600} &\equiv a_{250} + a_{243} + a_{241} + a_{220} + a_{216} + a_{181} + a_{175} + a_{126} + a_{118} + a_{55} + a_{45} \\ &\equiv 1 + 0 + 0 + 1 + 1 + 1 + 0 + 0 + 1 + 0 + 0 \\ &\equiv 1, \end{aligned}$$

establishing, in about five minutes work, that p_{1000} is an uneven number.

PROCEEDINGS
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Cambridge Philosophical Society.

Note on constant volume explosion experiments. By S. Lees, M.A., St John's College.

[*Read* 24 January 1921.]

§ 1. Much work has been done in recent years in determining the values of the volumetric heats of gases at high temperatures. In the main, the experiments have been made either at constant pressure with the gas heated externally, or at constant volume with the heating produced by explosion in a closed vessel. A comprehensive collection of available data on this subject, with estimates of the probable degrees of accuracy involved, has been given by Mr D. R. Pye of Trinity College*.

Such differences as occur in the results from the two methods seem to be in one direction, the internal energy for a gas obtained by constant pressure methods being slightly less than the value obtained by constant volume methods. It seemed possible to the author that the variations of temperature experienced at any instant in different parts of an explosion vessel, might account for some part of the difference in the values of the internal energy obtained by the two methods.

In this connection, the following extract from the first *British Association Report on Gaseous Explosions* (1908) may be quoted:

> If the volumetric heat of the gas were constant, the equalisation of these temperature differences by convection and conduction, could it take place without loss of heat, would cause no change of pressure. The volumetric heat is, however, not constant, but may quite possibly be 50 per cent. greater in the hottest than in the coldest part of the mass. The attainment of thermal equilibrium must, in fact, cause a change of pressure....The amount of the change might be the subject of rough calculation, taking an assumed distribution of temperature and assuming values for the volumetric heat. Such a calculation in the present state of knowledge would only be of value as showing the possible order of magnitude of the quantity sought, and the assumptions made could therefore be of a character to make the calculation fairly simple. More accurate knowledge both of temperature distribution and of thermal capacity will enable greater accuracy to be attained in the estimation of this correction, which will be of such a kind that a method of successive approximation can be pursued, the revised values of thermal capacity resulting from its application being applied to a more accurate calculation of the correction, if necessary.

* See *Automobile Engineer*, Feb., 1920.

It may be stated at the outset that the corrections on this account to be applied to the values of total energy obtained from constant volume experiments appear to be of very small amounts, and to be well within the limits of experimental error. In spite of this, the method of computation given below may be of interest to workers on this subject.

§ 2. In discussing the state of affairs after a constant volume explosion, some simplifying assumptions will be made. The constituent gases of the exploded mixture are each assumed to follow the ideal gas law

$$pV = RT, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where p is pressure, V is volume, T is absolute temperature, and R is the same constant for a gramme-molecule of each of the constituent gases. It is further assumed that these constituent gases are intimately mixed, so that there is no variation in the chemical composition of the gas mixture at different points of the interior of the vessel. Finally it is assumed that the volumetric heat of each constituent follows a linear law of increase with temperature. Thus a given mixture will also follow a linear law of increase of volumetric heat.

§ 3. The calculations involved are made more simple if a mass M of a constituent gas be replaced by $N = M/m$, where m is its molecular weight. The mass M is thus replaced by its equivalent in gramme-molecules. In the case of any portion of a chemically homogeneous gas mixture, the weights of the constituent gases may be replaced by the number of gramme-molecules in each case, and we may speak of the number of gramme-molecules of the mixture without any difficulty.

To take into account the variation of temperature at any instant of the mixture at different parts of the interior of the explosion vessel, we may divide up the contents of the vessel as follows:

N_1 gramme-molecules of mixture at absolute temperature T_1,
N_2 ,, ,, ,, ,, ,, T_2,
N_3 ,, ,, ,, ,, ,, T_3,
etc. etc.

If N be the total number of gramme-molecules of the mixture in a vessel of volume V, the partial volumes V_1, V_2, etc. of the constituents are given by

$$pV_1 = RN_1T_1, \quad pV_2 = RN_2T_2, \quad pV_3 = RN_3T_3, \text{ etc.},$$

where p is the uniform pressure in the vessel. Thus

$$pV = p(V_1 + V_2 + V_3 + \text{etc.}) = R(N_1T_1 + N_2T_2 + N_3T_3 + \text{etc.})$$
$$= R\Sigma(N_1T_1). \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

In most experiments of this kind, the pressure p is used as a means of measuring the mean temperature T_m. For a chemically homogeneous mixture, this mean temperature will obviously be defined as

$$T_m = \frac{\Sigma(N_1 T_1)}{\Sigma N_1} = \frac{\Sigma(N_1 T_1)}{N}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where N is the total number of gramme-molecules in the vessel. Hence from equations (2)

$$pV = RNT_m, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

which since V, R, N are constants, indicates how p may be used to measure T_m.

§ 4. The total internal energy E of a gramme-molecule of the gaseous mixture at a uniform temperature T (absolute), assuming linear increase of volumetric heat with temperature, is given by

$$E = A_0 T + \tfrac{1}{2} B T^2,$$

where A_0 and B are constants. If as usual we measure E from a standard temperature T_0, the internal energy E must be written

$$E = A_0 (T - T_0) + \tfrac{1}{2} B (T^2 - T_0^2) = C + A_0 T + \tfrac{1}{2} B T^2, \quad \ldots(5)$$

where $$C = -(A_0 T_0 + \tfrac{1}{2} B T_0^2).$$

On dividing this value of E by $(T - T_0)$ we get the mean specific heat (at constant volume) per gramme-molecule, i.e. the mean volumetric heat, between T and T_0, in the form

$$A_0 + \tfrac{1}{2} B (T + T_0) = A + \tfrac{1}{2} B T, \quad \ldots\ldots\ldots\ldots\ldots(6)$$

where $$A = A_0 + \tfrac{1}{2} B T_0.$$

The true volumetric heat at temperature T is given by dE/dT and is therefore equal to

$$A_0 + BT. \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

§ 5. Reckoned from T_0, the internal energy of the mixture defined in § 3 will be

$$C\Sigma N_1 + A_0 \Sigma N_1 T_1 + \tfrac{1}{2} B \Sigma N_1 T_1^2 = CN + A_0 N T_m + \tfrac{1}{2} B \Sigma N_1 T_1^2. \quad \ldots\ldots(8)$$

In explosion experiments, this is usually assumed to be equal to the internal energy of the whole mass taken at the mean temperature T_m, i.e. is assumed to be equal to

$$CN + A_0 N T_m + \tfrac{1}{2} B N T_m^2. \quad \ldots\ldots\ldots\ldots\ldots(9)$$

The first two terms of expression (8) are the same as the first two terms of expression (9). We have, therefore, to compare $\Sigma N_1 T_1^2$ with $N T_m^2$. To do this, put

$$T_1 = T_m + t_1, \quad T_2 = T_m + t_2, \quad T_3 = T_m + t_3, \text{ etc.} \quad \ldots(10)$$

so that $NT_m = \Sigma N_1 T_1 = \Sigma N_1 (T_m + t_1) = NT_m + \Sigma N_1 t_1$, from (3),

i.e. $$\Sigma N_1 t_1 = 0. \quad \text{.....(11)}$$

We then have

$$\Sigma N_1 T_1^2 = \Sigma N_1 (T_m + t_1)^2 = T_m^2 \Sigma N_1 + 2T_m \Sigma N_1 t_1 + \Sigma N_1 t_1^2, \quad (12)$$

so that from (11)

$$\Sigma N_1 T_1^2 = NT_m^2 + \Sigma N_1 t_1^2. \quad \text{.....(13)}$$

Hence expression (8) is greater than expression (9) by the essentially positive quantity

$$\tfrac{1}{2} B \Sigma N_1 t_1^2. \quad \text{.....(14)}$$

To get the true value of the internal energy corresponding to uniform temperature T_m, we must therefore diminish the actually

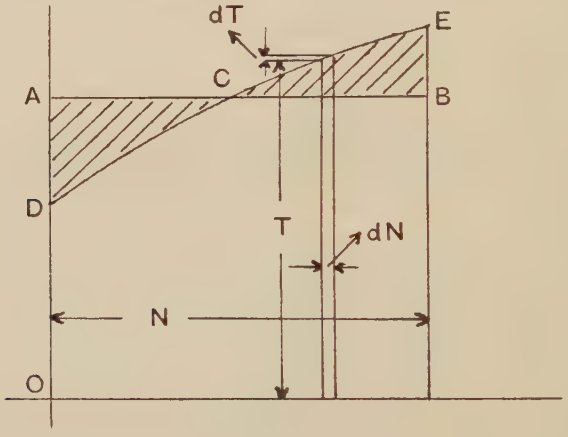

Fig. 1.

measured value (8) by expression (14). This means that the explosion experiments will give higher values, in general, for the internal energy, than constant pressure experiments, ideal accuracy in other respects being postulated for the determinations.

§ 6. There remains the problem of estimating the order of this difference of internal energy values, i.e. the order of magnitude of expression (14).

The problem of temperature distribution at any instant inside an explosion vessel requires the determination of the number dN of gramme-molecules of mixture* whose temperature lies between T and $T + dT$. If this be known, the temperature distribution can be represented by a graph like fig. 1. The graph is such that the abscissa of any point on the curve DCE represents the total number of gramme-molecules whose absolute temperature is not greater

* To emphasise the simple algebraic character of the argument, the calculus notation has been avoided in §§ 3 to 5.

than T. From its nature, therefore, the curve must always slope upwards from D to E. Referring to this diagram, it is easily seen from the definition (3) of T_m, that the shaded areas ACD and CEB, on opposite sides of the line AB, must be equal.

In the absence of more accurate knowledge as to the shape of the curve DCE, we may take it as approximately a straight line. This will, at any rate, give some kind of approximation to the truth. On this assumption, we shall apply our expressions to some results obtained by the late Prof. B. Hopkinson. In one of his explosion vessel experiments, Hopkinson* found for the instant of

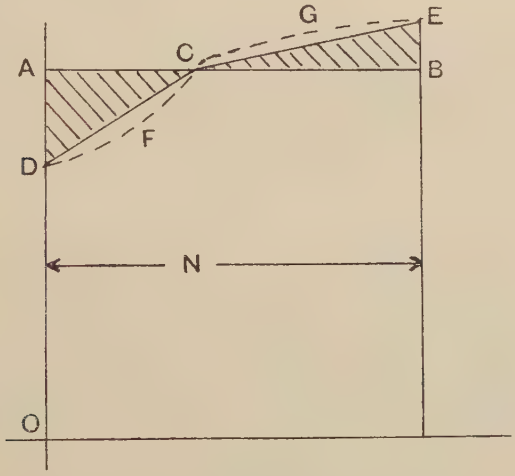

Fig. 2.

maximum pressure (corresponding in this case to a mean temperature of 1600° C.) a maximum temperature at the centre of the vessel of about 1900° C., whilst the temperature was probably as low as 1100° C. in the immediate neighbourhood of the walls. The difference between the maximum temperature and the mean being less than the difference between the minimum temperature and the mean, the curve of temperature distribution has been arbitrarily assumed as DCE in fig. 2. In this diagram, DC and CB are taken as straight lines, whilst the point C on AB has been so chosen that the area ADC equals the area CEB. It will readily be verified for the temperatures obtained by Hopkinson that $AC = 3N/8$, whilst $CE = 5N/8$.

For continuously varying temperatures throughout the gas, expression (14) must be replaced by

$$\tfrac{1}{2}B \int t^2 dN, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

* *Proc. Roy. Soc.*, 1906.

the integral being taken over the shaded areas previously referred to. It is useful to note that the integral in (15) for each piece of the shaded area is exactly twice the first moment* of the area of the piece about AB. The expression (15) thus becomes in our case numerically equal to

$$B\left\{\frac{(500)^2 \times 3N}{6 \times 8} + \frac{(300)^2 \times 5N}{6 \times 8}\right\} = 25000BN. \ldots\ldots(16)$$

The mean volumetric heat per gramme-molecule for air, from 0° C. to 1600° C., is probably about 5·55 gramme-calories. Thus for air at 1600° C., the internal energy reckoned from 0° C. is about $8880N$. The increase in value of the volumetric heat with temperature does not appear to follow exactly the linear law, but in the neighbourhood of 1600° C. we may take the increase as being roughly 0·5 gramme-calories per 1000° C. Thus from expression (6) we see that B can be taken as 1×10^{-3}. Thus expression (16) becomes $25N$, which expressed in terms of $8880N$ is 0·283 %. With the assumed conditions, this is the amount by which the observed internal energy at 1600° C. should be diminished to give the corrected value corresponding to uniform temperature throughout.

It is conceivable that the true shape of the temperature distribution curve may be something like the dotted curve $DFCGE$ of fig. 2, in which case the correction would be materially increased. It is also possible that the temperature differences in the gas may diminish during the early stages of cooling at a much slower rate than the mean temperature. On the whole, however, it would seem probable that the correction ought not to exceed 1 % of the value of the internal energy, and might very well be much less.

§ 7. *Conclusion.* An effort has been made to investigate the effect of temperature variations in an explosion vessel, on the values of the total internal energy measured. Instead of approaching the problem from the point of view indicated in the *B.A. Report*†, by modifying the temperature for a given internal energy determination, the author has modified the internal energy for a given mean temperature, and has given reasons which seem to indicate that the correction is probably not more than 1 % at a temperature of 1600° C. This order of correction is within the limits of probable error of experiments at the present time‡.

* For $\int t^2 dN = 2\int \frac{t}{2} . dS$, where $dS = tdN$.

† *Loc. cit.*

‡ See Pye, *loc. cit.*

On the Latent Heats of Vaporisation. By Eric Keightley Rideal, M.A., Trinity Hall.

[*Read* 28 February 1921.]

A number of attempts have been made to associate the latent heats of evaporation (L) with the natural periods of atomic vibration (v) as calculated by the methods of Lindemann*, Einstein, Bernouilli and Nernst or determined experimentally by Rubens. There is, however, no relationship between the infra red vibration frequencies of the elements and their latent heats of evaporation. On the assumption, however, that intermolecular, chemical and physical actions take place with energy transfer in quanta and not continuously, it should be possible to derive the latent heat of evaporation of an element from some natural vibration frequency or spectral line. Although this spectral line need not necessarily be in the infra red portion of the spectrum. It is especially desirable to test this hypothesis in the case of the latent heats of evaporation, since this is a typical physical process and a conformity to theory would confirm the supposed identity of physical and chemical forces and at the same time from the thermodynamic equations connecting the latent heats of evaporation and the vapour pressure together with the molecular kinetic effusion equations of Herz†, Marcelin and Langmuir it would be possible to calculate the indetermined integration constant of the vapour pressure formula and thence the so-called Nernst chemical constants.

According to the quantum theory applied to the energy change involved in evaporation considered as a chemical process the latent heat of evaporation per gm. mol. should be given by the expression

$$L = Nh\,(v_{\text{products}} - v_{\text{reactants}}),$$

where h is Planck's constant, $v_{\text{p.}}$ the vibration frequency of the products the metal vapour and $v_{\text{r.}}$ of the reactants, the solid or liquid evaporating. The evaporation of a metal may be imagined to take place by two different processes. We may remove a complete atom from the metal to the vapour state, then the activating frequency of the metal atom in the vapour state will correspond to some very small energy transfer, since experiment has indicated that practically all vapour atoms coming in contact with the solid metal stick, *i.e.* as far as reactivity with the solid metal is concerned practically all the atoms are active, thus v products will correspond to a radiation far in the infra red and is probably not far removed

* *Physical Chemistry*, McLewis, vol. III.
† *Ann. Physik*, 17, 177 (1882).

from the natural infra red vibration frequency itself. The frequency of the radiation necessary to activate an atom in the solid will correspond to some line in the spectral series, each representing a different degree of activation of the element, which should be capable of experimental determination in the spectrum. In the attached tables are given a few of the lines in the spectrum of the various elements as observed by Rowland, Kayser and others (c/o Landolt-Bornstein Tabellen) and the latent heats calculated therefrom (L_1). It will be noted that the line corresponding to the activation frequency of the atom in the solid is generally fairly widely separated from the other lines and is thus at or near the head of a series. The choice of the particular fundamental line, however, is somewhat arbitrary and it is consequently necessary to find some alternative method for calculation of the latent heats.

If we consider that a metal is built up of space lattices of alternating valence electrons and positive nuclei, the activation of a metal atom can be assumed to take place in two distinct steps, the activation of the positive nucleus in the metal and the activation of the valence electron attached to it. The activation frequencies of the positive nucleus and the valence electron respectively are given by the $\nu_{\text{infra red}}$ and $\nu_{\text{ultra violet}}$ (or the photo electric frequency) frequencies whilst the activation frequency of the atom as a nucleus-electron complex corresponds to the so-called "radiation potential" of the element. It has already been indicated by Haber* that the transfer of a quantum of energy in a solid results in the activation of two atoms, being virtually a cleavage of a diatomic molecule linked by an electron into two active atoms. The latent heat of evaporation will therefore be given by the expression:

$$L = \tfrac{1}{2}NL\{\nu_{\text{radiation}} - \{\nu_{\text{infra red}} + \nu_{\text{ultra violet}}\}\}.$$

In the following tables are given the latent heats of evaporation calculated on this basis (L_2), the infra red radiation frequency being calculated from Lindemann's melting point formula, the ultra violet frequency from Haber's relationship

$$M\nu^2_{\text{red}} = m\nu^2_{\text{ultra violet}},$$

where M and m are the atomic and electronic masses respectively. Where experimental values for $\nu_{\text{radiation}}$ (or the radiation potential) are not available there have been calculated from the approximate relationship $\nu_{\text{radiation}} = \frac{\nu_{\text{ultra violet}}}{2.3}$. In the last column are given the values of (L_3) calculated from vapour pressure data collected by Johnston (*Jour. Ind. and Eng. Chem.* 9, 876, 1917), Gebhardt (*Ann.* 40, 438, 1913) and Langmuir (*Phys. Rev.* 2, 329, 1913).

* *Ber. Deutsch. phys. ges.* **13**, 1117 (1911).

Metal	Line of atomic activation with those next to it λ in $\mu\mu$.	L_1	Infra red $Nh\nu$ in calories	Ultra violet $Nh\nu$ in calories	Resonance $Nh\nu$ in calories observed	L_2	L_3 observed latent heat
K	1248, 1252, *1516·6*, 2706	18,575	225	64,100	27,480	18,420	17,900
Na	819, 1138, *1267*, 1845	23,718	382	88,640	48,090	20,466	20,500
Li	670, 8, *812*, 1868	34,264	936	104,600	—	30,000	29,000
Rb	1366, 1475, *1529*, 3581	17,644	136	53,700	—	15,240	17,780
Cs	1012, 1354, *1470*, 2931	18,511	889	57,300	—	16,180	15,900
Cd	643·8, *1039·6*, 1398	27,219	281	126,350	87,930	24,350	25,000
Zn	636·4, *1105·5*, 1305·5	25,590	408	140,600	90,000	25,500	28,800
Ag	518·4, *880·7*, 1182·9	32,000	664	141,500	—	40,300	33,700
Au	338, *405·5*, 563·3	69,800	408	187,500	—	51,700	65,500
Cu	327·4, *402·3*, 578·2	70,373	627	214,500	—	61,100	75,400
Fe	346, 357, *372*, 386	76,000	777	250,650	—	71,160	77,800
Cr	360·5, 364, 367·6, 390·3	72,230	777	240,400	—	68,370	68,200
Pb	*391·6*, 404·9, 412·9 560, *665*	42,820	180	112,600	—	32,000	45,400
Sn	*374·6*, *452*, 556	75,800	210	98,260	—	27,900	69,000
		63,100					
Al	*572*, 1125	49,300	702	157,000	—	44,700	52,600
Tl	377, *535*, 1151	55,130	172	105,500	—	30,000	40,500
Hg	615, 1014, *1367·4*	20,700	202	124,900	114,210	5,450	13,500
Pt	—	—	408	244,900	—	69,400	128,000
Mo	—	—	590	246,000	—	70,000	177,800
W	—	—	468	273,500	—	77,450	218,500
Ni	367, *372*, 377·5	76,140	768	250,800	—	71,230	76,500
Mn	478, *482·3*, 601·3	58,600	702	224,270	—	63,700	56,300

It will be noted that for the metals in which the natural constants have actually been determined there is a very fair agreement in the three values of (L). Marked divergencies are, however, to be noted in the case of the polyvalent elements Hg, Sn and Tl, and in those metals of high boiling points. In this latter case the fact that thermionic emission is strongly marked is an indication that the Haber relationship does not give the correct periodicity of the binding electron, but one of the electrons liberated by thermal agitation. Closer agreement is scarcely to be expected since, in many cases, the whole basis of calculation is the Lindemann melting point and the Haber vibrational relationship, both only approximations.

There is, however, a sufficiently good agreement to afford evidence for the validity of the quantum relationship in this physical molecular energy interchange. It would be of interest to determine whether the extension of this hypothesis, involving new assumptions, were also valid, *i.e.* whether the vapour pressure of a metal, *e.g.* potassium, could be raised by illumination with either monochromatic radiation of $\lambda = 1516{\cdot}6\ \mu\mu$ or with a dichromatic

illumination of frequencies $2{\cdot}3\ 10^{12}$ and $6{\cdot}85\ 10^{14}$, the respective infra red and ultra violet activating frequencies.

The application of the hypothesis to the calculation of the latent heat of evaporation of non-conductors is more difficult, owing to our lack of knowledge of the intermolecular forces. The latent heats are, however, all relatively small and the infra red absorption bands generally well marked, thus CCl_4 possesses a marked absorption band at $4{\cdot}5\ \mu$ equivalent to the value $L = 6400$ cal. per gm. mol. which agrees with the observed value of

$$7100 - 700 = 6400 \text{ cal.}$$

In the case of naphthalene an absorption band would be anticipated at $3{\cdot}36\ \mu$, whilst actual observation indicates a strong absorption at $3{\cdot}25\ \mu$. In the case of water the complex infra red spectrum leaves little doubt as to the existence of polymers in solution. That the formation of di- and tri-hydrol takes place through the oxygen atom is indicated by the equivalence of the latent heat of evaporation to the interatomic energy provided by a vibration frequency of $3{\cdot}0\ \mu$ (9,540 cal. per gm. mol.) an absorption band which is always noted in compounds containing the hydroxyl group.

The so-called chemical constants of Nernst necessary for the evaluation of the equilibrium constants and reaction affinities of reactions occurring in non-condensed systems are related to the undetermined constant of the vapour pressure formula by the relationship

$$C_0 = \frac{i + \log_e R}{2{\cdot}3023},$$

where i is the integration constant of the Clapeyron-Clausius equation

$$\frac{L_m}{RT^2} = \frac{\partial \log P}{\partial T},$$

or

$$\frac{\partial \log C}{\partial T} = \frac{L_m - RT}{RT^2};$$

putting

$$L_m - RT = L_0 + \alpha T + \beta T^2 + \ldots,$$

$$\frac{\partial \log_e C}{\partial T} = \frac{L_0}{RT^2} + \frac{\alpha}{RT} + \ldots,$$

$$\log_e C = \frac{-L_0}{RT} + \frac{\alpha}{R} \log T + i,$$

or

$$\log_e P = \frac{-L_0}{RT} + \left\{\frac{\alpha_0 + R}{R}\right\} \log_e T + \frac{\beta_0 T}{R} + \ldots i + \log_e R.$$

When equilibrium is established between a liquid or solid and

its vapour the number of molecules striking the surface in unit time is equal to the number leaving.

According to the effusion equation developed by Herz, Knudsen and Langmuir* the number of molecules striking unit area per second in a gas at pressure p, temperature T and of molecular weight M is given by the equation

$$\mu = \frac{P}{\sqrt{2\pi RMT}},$$

where μ is the number of gm. mols. striking a square cm. per second. In a recent communication† it was shown that the rate of a unimolecular reaction

$$\frac{\partial n}{\partial t} = v \,.\, e^{-\frac{Nhv}{RT}}$$

had a physical significance in that v was the time of molecular relaxation. This relationship shown to be experimentally true by Dushman and Langmuir‡ assumes merely that intermolecular energy changes involving chemical forces obey the quantum relationship. The above data indicate that the latent energy of evaporation is the work done in overcoming forces of a nature similar if not identical with the usual chemical manifestations. We may therefore regard the vaporisation of a substance from a liquid or solid surface as a monomolecular chemical reaction.

The number of gm. molecules per square cm. of surface is

$$\frac{1}{N\pi d^2},$$

where N is the number of molecules per gm. mol. and d the molecular diameter.

The rate of evaporation in gm. mols. per square cm. per sec. is accordingly given by the expression

$$\frac{v}{N\pi d^2} e^{-\frac{Nhv}{RT}}.$$

This can be equated to the number of gm. mols. striking a square cm. per second, or

$$\frac{P}{\sqrt{2\pi RMT}} = \frac{v}{N\pi d^2} e^{-\frac{Nhv}{RT}}.$$

On taking logs of each side we obtain

$$\log_e P = -\frac{Nhv}{RT} + \tfrac{1}{2}\log T + \log_e \frac{v\sqrt{2\pi MR}}{Nd^2\pi}.$$

* *Phys. Rev.* 2, 331 (1913).
† Rideal, *Phil. Mag.* xi, 461 (1920).
‡ *Jour. Amer. Chem. Soc.* 42, 2190 (1920).

We have also seen that the expression Nhv can be put equal to the latent heat of evaporation (L).

If p be measured in atmospheres (10^6 bars.) and L in calories the expression becomes

$$\log_e P = \frac{-L}{RT} + \tfrac{1}{2}\log T + \log_e \frac{v\sqrt{M}}{N \cdot \pi d^2 \cdot 43{\cdot}75}.$$

The value of the chemical constant C_0 is thus equal to

$$\frac{\log_e R + \log_e \dfrac{v\sqrt{M}}{N \cdot \pi d^2 \cdot 43{\cdot}75}}{2{\cdot}3023},$$

or taking the generally accepted value of $N = 6{\cdot}062\ 10^{23}$ we obtain

$$C_0 = 0{\cdot}2980 + \log_{10} \frac{v\sqrt{M}}{d^2} \cdot 1{\cdot}200\ 10^{-26}.$$

This relationship may be tested by calculation of the chemical constant in those cases where both L and d are known, and comparing these values with those determined in the usual way from the vapour pressure curve.

In the case of CO_2

$M = 44$, $L_{25^\circ\,C.} = 3100$*, $\therefore$ $L = 2600$, $v = 2{\cdot}61 \cdot 10^{13}$,

and $$d = 4{\cdot}56\ 10^{-8}\ \text{cm.}\dagger$$

$$\therefore\ \log_{10} \frac{v\sqrt{M}}{d^2} \cdot 1{\cdot}200\ 10^{-26} = \log_{10} 9{\cdot}972\ 10^2,$$

or $$C_0 = 3{\cdot}29.$$

The actual value of C_0 from vapour pressure data is

$$C_0 = 3{\cdot}20.$$

For hydrogen Eucken finds $L = 229$ cal. per gm. mol.

$$d = 2{\cdot}68\ 10^{-8}\ \text{cm.}$$

Hence $$v = 2{\cdot}446\ 10^{12},$$

and $$C_0 = 2{\cdot}10 \quad \text{(observed value 2·20).}$$

For water, inserting the values

$$d = 4{\cdot}54\ 10^{-8}\ \text{cm.},\quad L = 10{,}200\ \text{cal.},$$

we find $$v = 1{\cdot}09\ 10^{14},$$

we obtain $$C_0 = 3{\cdot}73 \quad \text{(observed value 3·70).}$$

The influence of molecular size is well exhibited in the case of benzene. Surface tension data indicate that the molecular diameter

* Landolt-Bornstein Tabellen.

† Jeans, *The dynamical theory of gases.*

across the benzene ring is $10{\cdot}6\ 10^{-8}$ cm. (the distance between two carbon atoms being $6{\cdot}2\ 10^{-8}$ cm., hence the diameter is

$$\frac{\sqrt{3}}{2} 2\ .\ 6\ .\ 2\ 10^{-8})$$

and that the benzene molecule lies flat upon the surface of a liquid.

Inserting the values

$$L_0 = 8000 \text{ cal.}, \quad M = 78,$$

$$C_0 = 3{\cdot}187 \quad \text{(observed value } 3{\cdot}20).$$

A high value for the chemical constant is found in the case of iodine, where $C = 4{\cdot}0$. The molecular weight is

$$M = 254, \quad d = 3{\cdot}96\ 10^{-8} \text{ cm.}*$$

L per atom 2146 cal. or per mol. 4292 cal., hence

$$v = 4{\cdot}58\ 10^{13}, \quad \text{and} \quad C_0 = 4{\cdot}09.$$

Similar close agreements are to be found in the other cases where the values of L and of d are known.

The above derivation for the chemical constant leads to the dimensional expression $i = Ml^{-1}t^{-2}$, where M, l and t are the respective dimensions of mass, length and time.

If we insert temperature into the dimensional expression we obtain $i = Ml^{-1}t^{-2}\theta^{-1}$.

This can be compared with the expression derived by Nernst†,

$$i = \log \frac{(2\pi m)^{\frac{3}{2}} v^3}{k^{\frac{1}{2}}} = Ml^{-1}t^{-2}\theta,$$

and with that obtained by Sackur‡ and by Tetrode§,

$$i = \log \frac{(2\pi m)^{\frac{3}{2}} k^{\frac{5}{2}}}{h^3} = Ml^{-1}t^{-2}\theta^{-\frac{5}{2}}.$$

It will be noted that there is a lack of agreement in these expressions as far as the temperature is concerned. Lindemann has developed from dimensional considerations the expression

$$i = \log \frac{m^{\frac{3}{2}} k^{\frac{5}{2}}}{h^3} \cdot \theta^{\frac{5}{2} - (b' - a')/R},$$

where $b' = \frac{5}{2}R$ approximately (the atomic heat of the gas at 0° K.) and a' is the atomic heat of the solid at 0° K., which is either zero or a very small quantity. With elevation of the temperature the

* Rankine, *Proc. Roy. Soc.* A, 83, 516 (1910); *Phil. Mag.* 29, 552 (1915).
† *Grundlagen des neuen Warmesatzes*, Halle, 1918, p. 138.
‡ *Ann. der Physik*, 40, 67 (1918).
§ *Ibid.*, 434, 39, 255 (1912).

value of $b' - a'$ gradually decreases and the power of θ increases from $-\frac{5}{2}$ at 0° K. to 0 at the critical point where gas and solid become identical.

It will be noted that the above expression derived from the latent heat equation and the effusion formula gives us a general method of calculation of the thermodynamic potential differences or the difference in the fugacities of a substance distributed in two phases and is not necessarily confined to the particular case where one phase consists of the material in the zero thermodynamic environment of a perfect gas.

On the function $[x]$. By Viggo Brun (Dröbak, Norway). (Communicated by Prof. G. H. Hardy.)

[*Read* 24 January 1921.]

§ 1. In the *Proceedings of the Cambridge Philosophical Society* (Vol. xix, Part 5, 1919) Mr Shah and Mr Wilson have discussed some formulae proposed for calculating the number of Goldbachian decompositions of an even number. They have also mentioned a formula of mine, saying: "The formula to which Brun's argument leads is ...(11)" (page 243). In reality I have not enunciated this formula, as Mr Hardy and Mr Littlewood justly remark in their "Note on Messrs Shah and Wilson's paper."

But it may appear at first sight as if my method should naturally lead to the formula (11), and I should like to explain why one will not find it so on examining the matter further.

The formula in question is deduced from the sieve method of Eratosthenes, employed twice. We will here simplify the question, examining only the common sieve of Eratosthenes. Let us determine the number of primes under 14. We write the 14 numbers

$$1 \quad \underline{2} \quad \underline{3} \quad \underline{4} \quad 5 \quad \underline{\underline{6}} \quad 7 \quad \underline{8} \quad \underline{9} \quad \underline{10} \quad 11 \quad \underline{\underline{12}} \quad 13 \quad \underline{14}$$

effacing first the numbers 2, 4, 6, 8, 10, 12, 14 and then the numbers 3, 6, 9, 12. The uneffaced numbers are the number 1 and the primes between $\sqrt{14}$ and 14. We find that

$$\pi(14) - \pi(\sqrt{14}) + 1 = 14 - \left[\frac{14}{2}\right] - \left[\frac{14}{3}\right] + \left[\frac{14}{2\cdot 3}\right]$$
$$= 14 - 7 - 4 + 2 = 5,$$

where $\pi(x)$ denotes the number of primes not exceeding x, and $[x]$ denotes the number of integers not exceeding x. This formula can easily be generalised; it is not difficult to see how*. We could now say that it was "natural" to put $[x] = x$, and we should then get the approximate formula

$$\pi(14) - \pi(\sqrt{14}) + 1 = 14 - \frac{14}{2} - \frac{14}{3} + \frac{14}{2\cdot 3} = (1 - \tfrac{1}{2})(1 - \tfrac{1}{3})\,14;$$

and generalising, we should get

$$\pi(x) - \pi(\sqrt{x}) + 1 = (1 - \tfrac{1}{2})(1 - \tfrac{1}{3}) \ldots \left(1 - \frac{1}{p_r}\right) x,$$

when p_r denotes the greatest prime under $\sqrt{x}$. But this formula

* See Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, I. p. 67.

is erroneous, the correct asymptotic formula being according to Mertens

$$\pi(x) - \pi(\sqrt{x}) + 1 = k \cdot (1 - \tfrac{1}{2})(1 - \tfrac{1}{3}) \ldots \left(1 - \frac{1}{p_r}\right) x,$$

where $$k = \tfrac{1}{2}e^C = 0 \cdot 89\ldots,$$

C being the Eulerian constant. For this, and also for other reasons, I have multiplied the formula in question by an undetermined constant. I also attempted to approximate to this constant empirically; but the value 1·598 thus obtained was seriously in error. The right value 1·320 has been determined independently by Stäckel (*Sitz. der Heidelberger Akad.*, Abth. A, 1916) and by Hardy and Littlewood (*l.c. supra*).

§ 2. We have seen an example of the error of putting $[x] = x$. Let us study the function $[x]$ more closely. If we draw the curve $y = [x]$, we see that it forms a discontinuous line like the steps of a staircase. The same can be said of the curve $y = \pi(x)$, but here the steps are not regular.

Let us try to express our discontinuous function by another discontinuous, but simpler, function. We write

$$\Phi(x) = \begin{cases} 1 & (x \geqq 1), \\ 0 & (0 < x < 1). \end{cases}$$

We see that

$$[x] = \Phi(x) + \Phi\left(\frac{x}{2}\right) + \Phi\left(\frac{x}{3}\right) + \Phi\left(\frac{x}{4}\right) + \ldots;$$

e.g. $$[3 \cdot 5] = \Phi(3 \cdot 5) + \Phi\left(\frac{3 \cdot 5}{2}\right) + \Phi\left(\frac{3 \cdot 5}{3}\right) = 1 + 1 + 1 = 3.$$

This formula has been employed by Lipschitz in the *Comptes Rendus* of 1879. But our function will also express other discontinuous functions, such as $\pi(x)$: thus

$$\pi(x) = \Phi\left(\frac{x}{2}\right) + \Phi\left(\frac{x}{3}\right) + \Phi\left(\frac{x}{5}\right) + \Phi\left(\frac{x}{7}\right) + \Phi\left(\frac{x}{11}\right) + \ldots:$$

e.g. $$\pi(6) = \Phi\left(\frac{6}{2}\right) + \Phi\left(\frac{6}{3}\right) + \Phi\left(\frac{6}{5}\right) = 1 + 1 + 1 = 3.$$

§ 3. Let us try to approximate to the function Φ by a continuous function. We will choose the function

$$1 - e^{-x^s} = \frac{x^s}{1} - \frac{x^{2s}}{2!} + \frac{x^{3s}}{3!} - \cdots,$$

where s is a positive integer not less than 2. It is not difficult to see that $1 - e^{-x^s}$ is very near 0 when $x < 1$ and very near 1 when

$x > 1$. At the point of discontinuity $x = 1$, $1 - e^{-x^s} = 1 - e^{-1}$. We are now able to approximate to the function $[x]$: thus

$$[x] = \frac{x^s}{1}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots\right) - \frac{x^{2s}}{2!}\left(1 + \frac{1}{2^{2s}} + \frac{1}{3^{2s}} + \frac{1}{4^{2s}} + \ldots\right) + \ldots + R,$$

or
$$[x] = \frac{\zeta(s)\, x^s}{1} - \frac{\zeta(2s)\, x^{2s}}{2!} + \frac{\zeta(3s)\, x^{3s}}{3!} - \ldots + R.$$

It is not difficult to find an expression for R, valid for all values of x which are not nearer to the integers (the points of discontinuity of $[x]$) than ϵ, where $\epsilon < \frac{1}{2}$. We can give R the following form

$$R = 8\theta 2^{-s\epsilon/x+1},$$

where $-1 < \theta < 1$ and where $s > x$, $s > 2$.

We can now give our formula the following two forms

$$[x] = \frac{\zeta(s)\, x^s}{1} - \frac{\zeta(2s)\, x^{2s}}{2!} + \frac{\zeta(3s)\, x^{3s}}{3!} - \ldots + 8\theta 2^{-s\epsilon/x+1} \quad \ldots(1),$$

valid for all positive values of x not in the intervals $(1-\epsilon, 1+\epsilon)$, $(2-\epsilon, 2+\epsilon)$, $(3-\epsilon, 3+\epsilon)$, etc.; or

$$[x] = \lim_{s\to\infty}\left(\frac{\zeta(s)\, x^s}{1} - \frac{\zeta(2s)\, x^{2s}}{2!} + \frac{\zeta(3s)\, x^{3s}}{3!} - \ldots\right) = \lim_{s\to\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(ns)\, x^{ns}}{n!} \quad \ldots(2),$$

valid for all non-integral x.

If we wish to deduce results from the method of Eratosthenes, it is advantageous to make use of these formulae. Later on I shall perhaps show some of these applications.

§ 4. We are also able to find an approximate formula for $\pi(x)$, viz.

$$\pi(x) = \frac{\eta(s)\, x^s}{1} - \frac{\eta(2s)\, x^{2s}}{2!} + \frac{\eta(3s)\, x^{3s}}{3!} - \ldots + R',$$

where
$$\eta(s) = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \ldots.$$

This sum can be expressed in terms of $\zeta(s)$ by employing the identity of Euler,

$$\frac{1}{\zeta(s)} = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\ldots,$$

from which results

$$\log \zeta(s) = \eta(s) + \tfrac{1}{2}\eta(2s) + \tfrac{1}{3}\eta(3s) + \tfrac{1}{4}\eta(4s) + \ldots.$$

We can express $\eta(s)$ in terms of $\log \zeta(s)$ by using the factors $\mu(n)$ of Möbius, but we soon discover that it is more practical to study the function

$$f(x) = \pi(x) + \tfrac{1}{2}\pi(\sqrt{x}) + \tfrac{1}{3}\pi(\sqrt[3]{x}) + \tfrac{1}{4}\pi(\sqrt[4]{x}) + \dots$$

than the function $\pi(x)$ itself; and it is not difficult to deduce the formula

$$\pi(x) + \tfrac{1}{2}\pi(\sqrt{x}) + \dots$$
$$= \frac{\log \zeta(s)\, x^s}{1} - \frac{\log \zeta(2s)\, x^{2s}}{2!} + \frac{\log \zeta(3s)\, x^{3s}}{3!} - \dots + R'' \dots (3).$$

The function $f(x)$ has points of discontinuity only for integral x, and R'' can be given the same form as R, viz.

$$R'' = 8\theta 2^{-s\epsilon/x+1}.$$

It follows that

$$\pi(x) + \tfrac{1}{2}\pi(\sqrt{x}) + \dots = \lim_{s\to\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\log \zeta(ns)\, x^{ns}}{n!} \dots (4)$$

for all non-integral x.

This formula has been deduced by Helge von Koch, in Vol. XXIV of the *Acta Mathematica*, but is not mentioned in the *Handbuch* of Landau nor in the *Encyclopédie des sciences mathématiques* (Pt I, Vol. III, Fasc. 4).

It is very interesting to compare the formulae (1) and (3). The only difference between them is that (1) has $\zeta(s)$ where (3) has $\log \zeta(s)$, and in spite of this the formula (1) gives an approximation for the regular function $[x]$ while the formula (3) gives an approximation for the very irregular function $f(x)$ of Riemann.

§ 5. If we use for our function $\Phi(x)$ the formula of Kronecker (see *Encycl. des sciences math.*, Pt I, Vol. III, Fasc. 3, p. 256)

$$\Phi(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^s}{s}\, ds,$$

we obtain the famous formula of Riemann

$$f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^s \log \zeta(s)}{s}\, ds,$$

which is equivalent to that of Helge von Koch (4). We equally obtain the following formula, equivalent to (2),

$$[x] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^s\, \zeta(s)}{s}\, ds \quad \dots\dots\dots\dots\dots (5),$$

valid for all positive non-integral x.

§ 6. It is interesting to compare these two methods. It follows on comparing the two expressions employed for $\Phi(x)$ that

$$\lim_{s\to\infty}(1-e^{-x^s}) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s}\,ds \qquad (x>0,\ x\neq 1),$$

which gives a curious analogy between an integral with complex limits and a real function. The problem arises whether there is any more striking analogy between this sort of integrals and real functions, *e.g.* real integrals. And it is in reality not difficult to find a formula analogous to the well-known formula

$$\frac{1}{2\pi i}\int_C \frac{dz}{z-a} = 1 \text{ (or } 0),$$

according as a is or is not contained in C.

We find that

$$\lim_{\epsilon\to 0}\frac{\epsilon}{2}\int_A^B \frac{dx}{|x-a|^{1-\epsilon}} = 1,$$

when a is contained in the interval $A-B$, while the limit is 0 if a is not contained in the interval. Here $|x-a|$ denotes the numerical value of $x-a$, and ϵ is supposed positive.

We are naturally also able to write down a formula analogous to the famous formula of Cauchy

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a},$$

valid if a is contained in C. The analogue is

$$f(a) = \lim_{\epsilon\to 0}\frac{\epsilon}{2}\int_A^B \frac{f(x)\,dx}{|x-a|^{1-\epsilon}},$$

which is valid if a is contained in the interval $A-B$.

It is in the nature of the matter that the analogy can only be very limited.

A theorem concerning summable series. By Prof. G. H. HARDY.

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1. It is well known that *if the series* Σa_n *is summable* $(C, 1)$, that is to say if

$$s_n' \sim An \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $s_n = a_0 + a_1 + \ldots + a_n$, $s_n' = s_0 + s_1 + \ldots + s_n$,

then

$$\Sigma \frac{a_n}{n+1} = \frac{a_0}{1} + \frac{a_1}{2} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

*is convergent**. The converse is not true, as may be seen at once from trivial instances to the contrary. It is therefore interesting to frame a theorem of this kind which embodies a *necessary and sufficient* condition for summability. Such a theorem is the following.

Theorem. *The necessary and sufficient condition that* Σa_n *should be summable* $(C, 1)$ *to sum* A *is that*

$$s_n + (n+1)\, b_{n+1} \to A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where

$$b_n = \frac{a_n}{n+1} + \frac{a_{n+1}}{n+2} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

2. It is plain that we may (replacing a_0 by $a_0 - A$) suppose without loss of generality that $A = 0$. If

$$s'_n = o(n), \quad s_n = s'_n - s'_{n-1} = o(n).$$

Again, (3) and (4) involve the convergence of (2), *i.e.* involve $t_n \to B$, where $a_n = (n+1)\, c_n$ and $t_n = c_0 + c_1 + \ldots + c_n$. And

$$s_n = \sum_0^n (\nu+1)\, c_\nu = (n+1)\, t_n - \sum_0^{n-1} t_\nu = o(n).$$

Hence we may suppose $s_n = o(n)$ in proving either part of the theorem.

This being so, we have

$$\begin{aligned} b_{n+1} &= \sum_{n+1}^{\infty} \frac{s_\nu - s_{\nu-1}}{\nu+1} = \lim_{m\to\infty} \sum_{n+1}^{m} \frac{s_\nu - s_{\nu-1}}{\nu+1} \\ &= \lim_{m\to\infty} \left(\frac{s_m}{m+1} - \frac{s_n}{n+2} + \sum_{n+1}^{m-1} \frac{s_\nu}{(\nu+1)(\nu+2)} \right) \\ &= -\frac{s_n}{n+2} + \sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)}, \end{aligned}$$

$$s_n + (n+1)\, b_{n+1} = \frac{s_n}{n+2} + (n+1) \sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)}.$$

* H. Bohr, 'Bidrag til de Dirichlet'ske Rackkers Theori,' Inaugural Dissertation (Copenhagen, 1910), p. 100. The theorem follows at once from (1) and the identity

$$\sum_0^n \frac{a_\nu}{\nu+1} = 2 \sum_0^{n-2} \frac{s'_\nu}{(\nu+1)(\nu+2)(\nu+3)} + \frac{s'_{n-1}}{n(n+1)} + \frac{s'_n - s'_{n-1}}{n+1}.$$

The condition (3) is therefore equivalent to

$$\sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)} = o\left(\frac{1}{n}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

3. Suppose first that Σa_n is summable, to sum zero, *i.e.* that $s_n' = o(n)$. Then

$$\begin{aligned}
\sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)} &= \lim_{m\to\infty} \sum_{n+1}^{m} \frac{s'_\nu - s'_{\nu-1}}{(\nu+1)(\nu+2)} \\
&= \lim_{m\to\infty} \left(\frac{s'_m}{(m+1)(m+2)} - \frac{s'_n}{(n+2)(n+3)} + 2\sum_{n+1}^{m-1} \frac{s'_\nu}{(\nu+1)(\nu+2)(\nu+3)} \right) \\
&= 2\sum_{n+1}^{\infty} \frac{s'_\nu}{(\nu+1)(\nu+2)(\nu+3)} - \frac{s'_n}{(n+2)(n+3)} \\
&= 2\sum_{n+1}^{\infty} o\left(\frac{1}{\nu^2}\right) - o\left(\frac{1}{\nu}\right) = o\left(\frac{1}{\nu}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).
\end{aligned}$$

Hence (5), and therefore (3), is a *necessary* condition for summability.

4. Suppose now that (5) is satisfied. Then, by (6),

$$2\sum_{n+1}^{\infty} \frac{s_\nu'}{(\nu+1)(\nu+2)(\nu+3)} - \frac{s_n'}{(n+2)(n+3)} = o\left(\frac{1}{n}\right) \quad \ldots (7).$$

Writing

$$\phi_n = (n+2)(n+3)\sum_{n+1}^{\infty} \frac{s_\nu'}{(\nu+1)(\nu+2)(\nu+3)},$$

we obtain

$$2\phi_n - s_n' = o(n) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

But

$$\begin{aligned}
\phi_n - \phi_{n-1} &= 2(n+2)\phi_n \sum_{n+1}^{\infty} \frac{s_\nu'}{(\nu+1)(\nu+2)(\nu+3)} - \frac{s_n'}{n+3} \\
&= \frac{2\phi_n - s_n'}{n+3} = o(1),
\end{aligned}$$

by (8); and therefore $\phi_n = o(n)$ and $s_n' = o(n)$, so that the series is summable to sum 0. Thus the theorem is proved.

5. In order to show that the theorem is not without application, I apply it to the deduction of two known convergence criteria*.

* See (for A) L. Fejér, 'La convergence sur son cercle de convergence d'une série de puissances effectuant une représentation conforme du cerc le sur le plan simple,' *Comptes Rendus*, 6 Jan. 1913, and 'Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf die schlichte Ebene,' *H. A. Schwarz Festschrift*, 1914, pp. 42–53; G. H. Hardy and J. E. Littlewood, 'Some theorems concerning Dirichlet's series,' *Messenger of Mathematics*, vol. 43, 1914, pp. 134–147: and (for B) G. H. Hardy 'Theorems relating to the summability and convergence of slowly oscillating series,' *Proc. London Math. Soc.*, ser. 2, vol. 8, 1910, pp. 301–320; E. Landau 'Über die Bedeutung einiger neuerer Grenzwertsätze von Herrn Hardy und Axer,' *Prace Matematyczno-fizyczne*, vol. 21, 1910, pp. 97–177; M. Cipolla, 'Sul criterio di convergenza di Hardy,' *Rend. dell' Acc. di Napoli*, ser. 3, vol. 26, 1920, pp. 96–107, 151–160.

(A) *If* Σa_n *is summable* $(C, 1)$, *and* $\Sigma n^p |a_n|^{p+1}$ *is convergent for some positive* p, *then* Σa_n *is convergent.*

(B) *If* Σa_n *is summable* $(C, 1)$, *and either* (α) a_n *is real and*

$$a_n > -\frac{K}{n},$$

or (β)

$$a_n = O\left(\frac{1}{n}\right),$$

then Σa_n *is convergent.*

6. To prove (A) we observe that

$$b_{n+1} = \sum_{n+1}^{\infty} \frac{a_\nu}{\nu+1} = \sum_{n+1}^{\infty} \left((\nu+1)^{-\frac{2p+1}{p+1}} . (\nu+1)^{\frac{p}{p+1}} a_\nu\right),$$

and so*

$$|b_n| \leqq \left(\sum_{n+1}^{\infty} (\nu+1)^{-\frac{2p+1}{p}}\right)^{\frac{p}{p+1}} \left(\sum_{n+1}^{\infty} (\nu+1)^p |a_n|^{p+1}\right)^{\frac{1}{p+1}}$$

$$= O\left(\left(n^{-\frac{p+1}{p}}\right)^{\frac{p}{p+1}}\right) o(1) = o\left(\frac{1}{n}\right).$$

Hence $(n+1)\, b_{n+1} \to 0$, and so, by (3), $s_n \to A$.

7. To prove (B) we observe first that, if it is condition (β) that is given, we may suppose without loss of generality that a_n is *real*; for we may treat the real and the imaginary parts of the series separately. But then (β) becomes a special case of (α). It is therefore only necessary to consider condition (α). Further, we may plainly suppose that $A = 0$.

Suppose that

$$\overline{\lim}\, s_n = \lambda > 0,$$

and choose a sequence of values of n for which

$$s_n > \tfrac{1}{2}\lambda.$$

Let us denote a value of n, belonging to this sequence, by m; and choose H so that $0 < KH < \frac{1}{4}\lambda$. Then

$$s_\nu = s_m + a_{m+1} + a_{m+2} + \ldots + a_\nu$$

$$> \tfrac{1}{2}\lambda - K\left(\frac{1}{m+1} + \ldots + \frac{1}{\nu}\right) > \tfrac{1}{2}\lambda - K\frac{\nu-m}{m+1}$$

$$> \tfrac{1}{2}\lambda - KH > \tfrac{1}{4}\lambda,$$

* By the well known inequality

$$\Sigma ab \leqq (\Sigma a^k)^{\frac{1}{k}} (\Sigma b^l)^{\frac{1}{l}} \quad \left(a \geqq 0,\ b \geqq 0,\ k > 1,\ l > 1,\ \frac{1}{k} + \frac{1}{l} = 1\right).$$

if $m < \nu \leqq m + Hm$. Hence

$$\sum_{m+1}^{m+Hm} \frac{s_\nu}{(\nu+1)(\nu+2)} > \tfrac{1}{4}\lambda \sum_{m+1}^{m+Hm} \frac{1}{(\nu+1)(\nu+2)} \sim \frac{\lambda H}{4m}$$

when $m \to \infty$. But this plainly contradicts (5). Hence

$$\overline{\lim}\, s_n \leqq 0.$$

It may be shown in just the same way* that

$$\underline{\lim}\, s_n \geqq 0.$$

Hence $s_n \to 0$, and the theorem is proved.

* Using a sum $\sum_{m-Hm}^{m} \frac{s_\nu}{(\nu+1)(\nu+2)}$.

Standing Waves parallel to a Plane Beach. By H. C. Pocklington, M.A., St John's College.

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1. The object of this paper is to investigate the standing waves parallel to the shore line in the case of an infinite fluid bounded below by a plane sloping at an angle α to the horizontal. We restrict ourselves to the case where α is a sub-multiple of a right angle and use the method of images. The conditions to be satisfied are stated in section 2 and the images formed by the boundaries (beach and free surface) are found in section 3. In section 4 we write down the velocity potential and show that it satisfies all the conditions that it should. In section 5 we find that the amplitude of oscillation at the shore is increased (in consequence of the compression of the waves as they get into shallow water) in the ratio $\sqrt{(\pi/2\alpha)}:1$.

2. Let the liquid be bounded below by a plane beach sloping at an angle $\alpha = \pi/2n$. We take the shore line as axis of z and use cylindrical coordinates, so that $\theta = 0$ is the equation of the free surface of the liquid, and $\theta = \alpha$ is that of the beach. Let the period of the standing waves be $2\pi/p$. Then the velocity potential is of the form $\phi = A\Phi \cos(pt + \epsilon)$. The conditions to be satisfied by Φ are (i) that $d^2\Phi/dr^2 + d\Phi/rdr + d^2\Phi/r^2d\theta^2 = 0$, (ii) that $d\Phi/d\theta = 0$ when $\theta = \alpha$, (iii) that $d\Phi/rd\theta = -p^2\Phi/g$ when $\theta = 0$, (iv) that at infinity Φ must have the correct form for standing waves in deep water. Let $p^2/g = \lambda$.

3. If we have a value of Φ, say Φ_1, that satisfies (i) but not (ii) we can find Φ_2 so that Φ_2 satisfies (i) and $\Phi_1 + \Phi_2$ satisfies (ii) (it will of course satisfy (i)), and may call Φ_2 the image of Φ_1 with respect to lower boundary. Similarly, if Φ_1 satisfies (i) but not (iii) and we find Φ_2 to satisfy (i) and such that $\Phi_1 + \Phi_2$ satisfies (iii) (it will of course satisfy (i)) then Φ_2 may be called the image of Φ_1 with respect to the upper boundary.

If

$$\Phi_1 = \exp\{-\lambda r \sin(\theta - \beta)\} \cos\{\eta + \lambda r \cos(\theta - \beta)\} = f(-\beta, \eta, \theta)$$

say, it is clear that its image with respect to the lower boundary may be taken to be

$$\Phi_2 = \exp\{-\lambda r \sin(2\alpha - \beta - \theta)\} \cos\{\eta + \lambda r \cos(2\alpha - \beta - \theta)\}$$
$$= f(2\alpha - \beta, \eta, -\theta) = f(\beta', \eta', -\theta) \quad \text{say.}$$

The image of Φ_2 with respect to the upper boundary may be taken to be

$$\Phi_3 = \cot \beta'/2 \,.\, \exp \{-\lambda r \sin (\beta' + \theta)\} \cos \{\eta' - \pi/2 + \lambda r \sin (\beta' + \theta)\}$$
$$= \cot \beta'/2 \,.\, f(\beta', \eta' - \pi/2, \theta)$$

(and it will be found on trial that this is the only value of Φ_3 that has the same general form as that given and satisfies the conditions stated above).

4. Let

$$\Phi_1 = \exp \{-\lambda r \sin \theta\} \cos \{(n-1)\pi/4 + \lambda r \cos \theta\}$$
$$= f\{0, (n-1)\pi/4, \theta\}$$

and add to it the first $n-1$ images taken alternately with respect to the lower and upper boundaries, so that

$$\Phi = f\{0, (n-1)\pi/4, \theta\} + f\{2\alpha, (n-1)\pi/4, -\theta\}$$
$$+ \cot \alpha \,.\, f\{2\alpha, (n-3)\pi/4, \theta\} + \cot \alpha \,.\, f\{4\alpha, (n-3)\pi/4, -\theta\}$$
$$+ \cot \alpha \cot 2\alpha \,.\, f\{4\alpha, (n-5)\pi/4, \theta\} + \text{etc.},$$

the last term being

$$\cot \alpha \cot 2\alpha \ldots \cot (n-1)\alpha/2 \,.\, f\{(n-1)\alpha, 0, \theta\}$$

if n is odd and

$$\cot \alpha \cot 2\alpha \ldots \cot (n-2)\alpha/2 \,.\, f\{n\alpha, \pi/4, -\theta\}$$

if n is even.

Each term satisfies (i). Pairing the terms starting from the beginning we see that each pair satisfies (ii) and that if n is odd the odd term at the end also does so. Pairing the second term with the third, the fourth with the fifth and so on we see that each pair satisfies (iii) and that the odd term at the beginning does so, as does the odd term at the end when n is even. Also if $0 \gtrless \theta \gtrless \alpha$ the sine under the exponential sign is positive for each term except the first so that these terms vanish exponentially at the infinite part of the fluid. The first term has the correct form for standing waves. Hence Φ satisfies all the conditions and $A\Phi \cos(pt + \epsilon)$ is the velocity potential required.

5. The amplitudes at various points of the upper boundary are proportional to the values of Φ there taken positively. The value of Φ at $\theta = 0$, $r = \infty$ varies from $+1$ to -1. Hence ρ the amplitude at the origin divided by the maximum amplitude at infinity is equal to the value of Φ at the origin.

Firstly, let n be odd. Pairing the terms starting from the beginning we have

$$\rho = 2 \cos (n-1)\pi/4 + 2 \cos (n-3)\pi/4 \,.\, \cot \alpha$$
$$+ 2 \cos (n-5)\pi/4 \,.\, \cot \alpha \cot 2\alpha + \text{etc.}$$

to $(n + 1)/2$ terms, the coefficient of the last term being 1 instead of 2.

That is,

$$\rho = \exp(n-1)\,\pi i/4 + \exp(n-3)\,\pi i/4 \,.\, \cot\alpha + \text{etc.}$$

to n terms, the first term of the previous series being the sum of the first and last of this, and so on, for

$$\cot(n-r)\,\alpha = \tan r\alpha.$$

Secondly, let n be even. Pairing the terms

$$\rho = 2\cos(n-1)\,\pi/4 + 2\cos(n-3)\,\pi/4 \,.\, \cot\alpha + \text{etc. to } n/2 \text{ terms}$$
$$= \exp(n-1)\,\pi i/4 + \exp(n-3)\,\pi i/4 \,.\, \cot\alpha + \text{etc. to } n \text{ terms},$$

which is of the same form as in the case of n odd.

If $$\omega = \cos\pi/n + i\sin\pi/n,$$

we have $$\cot r\alpha = -i\omega^r(\omega^{n-r} - 1)/(\omega^r - 1),$$

so that

$$\rho = \exp(n-1)\,\pi i/4 - \exp(n-3)\,\pi i/4 \,.\, i\omega(\omega^{n-1} - 1)/(\omega - 1)$$
$$+ \exp(n-5)\,\pi i/4 \,.\, i^2\omega^3(\omega^{n-1} - 1)(\omega^{n-2} - 1)/(\omega - 1)(\omega^2 - 1)$$
$$- \text{etc. to } n \text{ terms},$$

the indices of ω in $i^q\omega^s$ being the triangular numbers in order. Also

$$i = \exp\pi i/2.$$

Hence*

$$\rho = (1-\omega)(1-\omega^2)(1-\omega^3)\ldots(1-\omega^{n-1})\exp(n-1)\,\pi i/4.$$

Being real this is equal to its conjugate

$$(1+\omega^{n-1})(1+\omega^{n-2})\ldots(1+\omega)\exp(1-n)\,\pi i/4.$$

Multiplying, we find

$$\rho^2 = (1-\omega^2)(1-\omega^4)\ldots(1-\omega^{2n-2}).$$

But ω^2, ω^4, etc. are the roots of $(x^n - 1)/(x - 1) = 0$, whence $\rho^2 = n$.

Hence the amplitude at the origin is $\sqrt{n}$ times the maximum amplitude at infinity (but this has been proved only for the case of n integral).

* *Vide* Todhunter, *Theory of Equations* (1885), p. 217 (ch. xxiv, § 292).

The Origin of the Disturbances in the Initial Motion of a Shell.
By R. H. Fowler and C. N. H. Lock.

[*Read* 28 February 1921.]

(1) The following paper* is an attempt to throw some light on one of the most obscure outstanding problems in gunnery, namely, the precise cause of the initial angular oscillations of the axis of a (stable) spinning shell. The factors that produce the initial oscillations have hitherto only been guessed at, and design, which aims at reducing these disturbances to a minimum, has been guided solely by empirical results. It is therefore a matter of some importance to analyse carefully the experimental evidence which has recently been acquired, for the purpose of discriminating between possible causes, and suggesting the proper lines of future research. On the scientific rather than the technical side, we point out the desirability for a solution, if possible, of the elastic vibrations of the gun under its firing stresses.

In experiments carried out by ourselves and others in January, 1919†, we succeeded in recording with reasonable accuracy the initial oscillations of the axis of a series of shells, of four different types, fired at a series of muzzle velocities from a pair of 3-inch guns of two different twists of rifling. Specimens of these observations will be found in the paper quoted. By extrapolation of the observed curves backwards to the neighbourhood of the muzzle of the gun, it is possible to deduce with some confidence rough values for the magnitude and direction of the initial angular velocity of the axis of the shell. The extrapolation is not a serious one, for the law of motion of the axis of the shell is well understood. By analysis of these values and their variation with the varying circumstances of projection, it is possible to throw light on the origin of the disturbances themselves.

The most obvious origin for these disturbances would appear to lie in random gas pressure variations during the last part of the travel of the shell down the rifled bore, and in random asymmetry of the blast which flows past the shell for the first few feet of its motion outside the barrel. The only reasonable alternative suggestion is that the initial angular velocity is primarily due to some form of barrel vibration. It is important to discriminate between

* This paper is published by permission of the Ordnance Committee, for whom the experimental work was carried out. The authors also thank the Admiralty Director of Scientific Research and Experiment, who recently propounded to one of them a technical problem on initial motions. This problem suggested the possible importance of these results.

† "The Aerodynamics of a Spinning Shell," *Trans. Roy. Soc.* A, vol. ccxxi, 1920, p. 295.

these causes, and it is the object of this paper to show by analysis of the available experimental data that it is barrel vibrations that are the dominant cause, at least in these particular experiments. Whether the barrel vibrations are the cause in general it is for future experiments to decide.

(2) *Consequences of the random pressure variation theory.* If random pressure variations and random blast disturbances are the dominant cause of the initial angular velocity of the axis of the shell, certain well-marked characteristics of the initial motion of the axis can be deduced. For shortness it is convenient to define the position of the axis OA relative to the direction of motion of the centre of gravity OP by the angles δ, ϕ of Fig. 1.

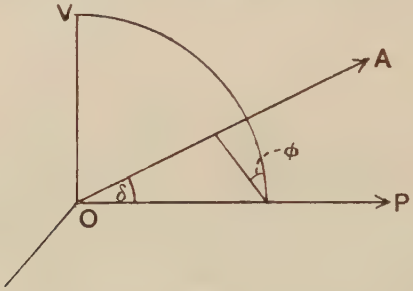


Fig. 1.

The plane POV is vertical, and the angle δ is usually called the yaw. Then the initial circumstances of the axis of the shell are defined with sufficient precision by the initial values of ϕ and $d\delta/dt$ or δ', that is to say, by ϕ_0 and δ_0'. The initial value of δ itself, obtained by extrapolation, is in practice very little different from zero, as is to be expected, and may be ignored in what follows.

Now random variations of gas pressure *across* the base of the shell and random asymmetry of the blast will result in a disturbing couple (practically impulsive) acting on the shell, whose plane of action must be expected to vary from round to round in an entirely arbitrary manner. That is to say, this theory demands that all values of ϕ_0 should be equally probable. Again, the impulsive disturbing couple which acts on the shell can hardly depend in any way on the axial spin of the shell or the twist of the rifling. The mean value of the disturbing impulsive couple, and therefore of δ_0', should thus be the same for similar shells fired at similar velocities from similar guns with different twists of rifling. These are two deductions from the theory which can be tested by the experimental results. We may say at once that neither deduction is sufficiently fulfilled. We may be certain that, though no doubt such random pressure effects occur and are appreciable, the main cause of the disturbance lies elsewhere.

(3) *Consequences of the barrel vibration theory.* On the other hand, if the main cause of the disturbances lies in the vibration of the barrel, which constrains the base of the shell to follow its movements so long as the driving band is engaged in the rifling, we can draw at once very different deductions. For a gun is not a figure of revolution, and its vibrations may well be expected to take place in or nearly in the same plane from round to round, and to be in or nearly in the same phase each time when the driving band disengages*. This effect might be expected to be all the more marked and give rise to larger disturbances when the gun has a considerable curvature (technically *droop*). It appears in such cases that the gun temporarily straightens more or less while the shell is travelling down the bore.

It follows then as a consequence of this theory that ϕ_0 should be roughly constant from round to round fired under similar conditions, or at least that the values of ϕ_0 should be highly correlated. Again, in this case the twist of the rifling may play a fundamental part in the phenomenon. For there will at least be stresses in the barrel proportional to the twist of the rifling, and we cannot say that the theory demands a value of δ_0' independent of the twist of the rifling. The further discussion of this case is unfortunately somewhat speculative at present and requires a little mathematical analysis. It is postponed until we have described the experimental evidence.

(4) *The experimental evidence.* The evidence available is deduced almost entirely from the experiments previously referred to. The following table gives the mean values of δ_0' and δ_0'/Ω† which can be deduced from that experiment, for groups fired under similar conditions from guns of two different riflings. The other entries are the reference numbers of the observations‡, Ω itself, the muzzle velocity and the twist of the rifling. The values of δ_0' for the various rounds in any one group are reasonably consistent.

It can be seen at once, by inspection of the columns for δ_0' and δ_0'/Ω, that δ_0' is distinctly greater for the gun rifled 1 turn in 30 diameters—that is, with the sharper twist of rifling—than for the gun rifled 1 in 40. On the other hand δ_0'/Ω appears to be practically the same (to the accuracy of the experiment) for both twists of rifling in each group. *The disturbing impulse, other things being*

* The case of a rifle is well-known, in which the direction of departure of the bullet is very largely affected by barrel vibrations, which are themselves affected by the presence or absence of the bayonet.

† The notation is that of our previous paper *loc. cit.* $\Omega = AN/B$, where A and B are respectively the axial and transverse moments of inertia of the shell and N is the axial spin in radians per sec. The twist of the rifling is specified as 1 complete turn in n diameters of the bore. Ω and N are inversely proportional to n.

‡ *Loc. cit.*

equal, is roughly proportional to the spin of the shell. We can even go somewhat further on these figures, for we notice that the values of δ_0'/Ω show a well-marked constancy through the whole table.

Table I.

Muzzle velocity, f.s.	Nos. of rounds	Ω radians per sec.	Mean value of δ_0' radians per sec.	Mean value of δ_0'/Ω	Twist of rifling
2130	IV 1, 2, 3, 5	150	1·7	0·014	1/40
2120	IV 24–26	200	3·0	0·015	1/30
1553	IV 7–9	110	1·5	0·014	1/40
1547	IV 16–18	146	2·1	0·015	1/30
1565	I 5–7	125	2·6	0·021	1/40
1563	I 27, 28	167	3·1	0·019	1/30
1585	II 5–7	124	1·0	0·008	1/40
1589	II 22, 23	165	2·3	0·014	1/30
1583	III 5–7	145	2·0	0·014	1/40
1567	III 22, 23	192	3·2	0·017	1/30
1084	I 9, 10	87	1·1	0·013	1/40
1119	I 22–24	120	1·9	0·016	1/30

The next table, Table II, also contains values of δ_0'/Ω for each group for which they have been deduced from the observations. The mean values of δ_0'/Ω for all groups for each gun are the same, confirming the more precise comparison of Table I.

The main purpose of Table II is to exhibit the correlation (if it exists) between the values of ϕ_0 for the various rounds of a group. It contains the mean value of ϕ_0 for each group, and the "spread" of the values of ϕ_0 in the group, the spread being the smallest angle which will include all members of the group.

The entries of the table show unmistakable regularities. In the first place the spread is in general far too small for one to suppose that the observed values of ϕ_0 represent a random distribution over the whole 360°. The groups of two rounds each provide only slight evidence on this point. But even here there is some evidence of correlation, for it is obviously equally probable that the spread of a pair will or will not exceed 90° for random distributions of ϕ_0. Of the seven such groups, the spread of one only (III 22, 23) exceeds 90° and the average spread of the seven is only 50°.

There are twenty-one groups in the table containing three or more rounds. In a group of three rounds distributed at random,

the probability of a spread of less than θ radians can easily be shown to be

$$\frac{3\theta^2}{4\pi^2}, \quad (\theta \leqslant \pi),$$

$$\frac{3\theta - 3\pi}{\pi} + \frac{3\theta\,(4\pi - 3\theta)}{4\pi^2}, \quad (\pi \leqslant \theta \leqslant \tfrac{4}{3}\pi).$$

Table II.

Mean muzzle velocity, f.s.	Nos. of rounds	Mean value of ϕ_0, degrees	Spread degrees	Mean of δ_0'/Ω
	Gun rifled 1 in 40			
2346	I 20, 21	179	20	0·022
2167	I 1–4	99	100	0·020
1565	I 5–7	30	61	0·021
1312	I 17, 18	− 55	51	—
1072	I 8–10	−105	104	—
922	I 11–14	− 83	30	0·011
2024	II 1–4	83	51	0·011
1584	II 5–7	− 3	100	0·008
1334	II 14–16	− 24	202	—
934	II 8–10	−128	46	0·014
2025	III 1–4	152	109	0·008
1583	III 5–7	157	147	0·014
1312	III 14–16	−107	40	—
1077	III 11–13	−135	53	—
931	III 8–10	−128	5	0·015
884	IV 10–12	−137	98	—
	Gun rifled 1 in 30			
1563	I 27, 28	20	19	0·019
1326	I 25, 26	45	29	0·015
1119	I 22–24	29	51	0·015
1589	II 22, 23	36	72	0·014
1119	II 17–19	6	49	0·018
1567	III 22, 23	55	110	0·017
1292	III 20, 21	37	49	0·019
1119	III 17–19	− 5	10	0·009
2121	IV 24–26	113	172	0·016
1547	IV 16–18	17	49	0·015
1078	IV 13–15	7	12	0·008
900	IV 21–23	81	72	0·010

The greatest possible value of the spread is of course $\frac{4}{3}\pi$ which occurs when the three values of ϕ_0 are of the form

$$\alpha,\ \alpha + \tfrac{2}{3}\pi,\ \alpha + \tfrac{4}{3}\pi.$$

It is thus equally probable that the spread of a group of three will or will not exceed 147° on a basis of random distribution. In the twenty-one groups the spread only twice exceeds and once equals 147°, and the average spread is only 70°. The probability with a random distribution that any *individual* spread should be less than or equal to 70° is only 0·113. The probability that the average of twenty-one spreads shall be as small as 70° will be excessively small.

We may therefore be fairly confident on the evidence of the spread that the main part of the angular velocity of the axis of the shell is due to some cause which operates in or nearly in the same plane for all rounds fired under the same conditions. This conclusion is supported by the results of two five-round groups of similar shells fired from another type of gun*. In each case the spread of the five values of ϕ_0 was less than 30°.

We are, however, justified in concluding that there are other regularities in Table II. It is arranged in order of muzzle velocities for the shells of each type, fired from the gun of each twist of rifling. Thus the table consists of eight sets of such arrangements separated by spaces. In each such set which is sufficiently complete there is a well-marked progressive change noticeable in the mean value of ϕ_0 as the velocity changes through the set.

Finally, compare together the mean values of ϕ_0 for all groups at the same muzzle velocity from the same gun, irrespective of the type of shell. We obtain the following sets of numbers. For the gun rifled 1 in 40,

(2000–2170)	99, 83, 152
(1560)	30, − 3, 157
(1320)	− 55, − 24, −107
(1070)	−105, −135
(920)	− 83, −128, −128, −137.

For the gun rifled 1 in 30,

(1570)	20, 36, 55, 17
(1320)	45, 37
(1110)	29, 6, − 5, 7.

The numbers in brackets on the left specify the muzzle velocity to which the group belongs. These mean values of ϕ_0 are obviously, with one exception, heavily correlated. We may say in fact that the main part of the initial disturbance consists of an angular velocity of the axis of the shell located in a plane which varies little from round to round or even with changes in the centre of gravity and moments of inertia of the shell, but which changes

* Gun: 16-pdr, 9-cwt.

progressively with the muzzle velocity and also with the twist of rifling. To this we may add on the evidence of Table I that the size of the disturbance appears to be roughly proportional to Ω.

On referring to the characteristics of the disturbances due to the two possible causes described in sections 2 and 3, we see that the evidence is fairly conclusively against the theory of random gas pressure variations which demands random ϕ_0's and δ_0' independent of n. On the other hand there is nothing to prevent its fitting in with the theory of barrel vibrations. We must examine this theory now in greater detail, though, as we have already said, all we can say is speculative and intended only to suggest lines of further work on the problem.

(5) *The motion of a shell with the base constrained.* Let us consider the circumstances under which a shell emerges from the muzzle of a gun. It seems probable that even for fairly small values of the clearance* the shell will cease to touch the bore, except at the driving band, from the moment when its shoulder reaches the lip of the muzzle. This is a direct consequence of the actual numerical quantities concerned—it is unnecessary to give numerical details here. Thus from the moment at which the shoulder emerges the shell may be regarded as practically free to swing about the driving band†.

Let us suppose in the first instance that owing to the firing stresses the muzzle end of the gun vibrates, so that the shell is constrained to take up the sideways velocity of the end of the gun just before it leaves the muzzle. We will assume first that the axis of the bore is unaltered in direction by this vibration. To work out the motion of the shell under such conditions we refer it to fixed rectangular axes Ox, Oy, Oz, such that Ox is parallel to the axis of the bore, Oy vertical, and Oz horizontal and to the right as viewed from behind the gun. That point in the shell whose motion is constrained (the base point) may be taken to be the centre of a section through the middle of the driving band. Let $\boldsymbol{\Lambda}$ be a unit vector representing the direction of the axis of the shell (components l, m, n) and $\mathbf{V}$ a vector representing the total constrained velocity of the base point (components u, v, w). Then at the moment when the driving band disengages, u is the muzzle velocity and v, w are the components of the barrel vibration.

The resultant action on the shell may be taken to be (1) a force $\mathbf{F}$ acting through the base point, whose position is defined by the vector $-d\boldsymbol{\Lambda}$ relative to the centre of gravity of the shell. The axial

* The excess of the minimum internal diameter of the bore over the maximum external diameter of the shell (driving band ignored).

† If the size of the initial disturbance is found to vary decidedly with the clearance, this paragraph will need modification. But the succeeding arguments hold unaltered provided less time is taken to refer to the moment at which the shell is first free to swing about the band.

distance of the centre of gravity from the base point is taken to be d; (2) a constraining couple acting about Ox.

The velocity of the centre of gravity is therefore $\mathbf{V} + d\mathbf{\Lambda}'$, and

$$\mathbf{F} = m^* \frac{d}{dt} \{\mathbf{V} + d\mathbf{\Lambda}'\}^*.$$

The moment of this force about the centre of gravity is $d\,[\mathbf{F} \,.\, \mathbf{\Lambda}]$. The moment of momentum about the centre of gravity is† $AN\mathbf{\Lambda} + B\,[\mathbf{\Lambda} \,.\, \mathbf{\Lambda}']$. The equation of angular motion is therefore (*omitting the constraining couple* which we assume to act about the axis of the bore, Ox)

$$\frac{d}{dt} \{AN\mathbf{\Lambda} + B\,[\mathbf{\Lambda} \,.\, \mathbf{\Lambda}']\} = m^* d \left[\frac{d}{dt} \{\mathbf{V} + d\mathbf{\Lambda}'\} \,.\, \mathbf{\Lambda}\right]$$

$$= m^* d \frac{d}{dt} [\{\mathbf{V} + d\mathbf{\Lambda}'\} \,.\, \mathbf{\Lambda}] - m^* d\,[\mathbf{V} \,.\, \mathbf{\Lambda}'],$$

or

$$\frac{d}{dt} \{AN\mathbf{\Lambda} + B_1\,[\mathbf{\Lambda} \,.\, \mathbf{\Lambda}']\} = m^* d\,[\mathbf{V}' \,.\, \mathbf{\Lambda}], \qquad \text{.....(1)}$$

where $B_1 = B + m^* d^2$. The y- and z-components of equation (1) are unaffected by the ignored constraining couple. When written out at length they are

$$\frac{d}{dt} \{ANm + B_1 (nl' - ln')\} = m^* d\,(w'l - u'n), \quad \text{.....(2)}$$

$$\frac{d}{dt} \{ANn + B_1 (lm' - ml')\} = m^* d\,(u'm - v'l). \quad \text{.....(3)}$$

Equations (2) and (3) are exact. If, now, we recall that the axis of the shell is only slightly inclined to the axis Ox, we may approximate by assuming that m and n are small, $l = 1$, and $l' = 0$. Writing $\zeta = m + in$ we can combine these equations into the single one

$$\frac{d}{dt} \{AN\zeta + iB_1\zeta'\} = -im^* d\,(v' + iw') - im^* d u' \zeta. \quad \text{...(4)}$$

The approximations made in obtaining (4) are certainly legitimate. To interpret (4) we must approximate further in a more speculative manner.

If barrel vibrations are of primary importance, a rough calculation, for the 3-inch shells used in these experiments, shows that $|\, v' + iw' \,|$ must not be less than about $\frac{1}{3\,0}u'$; u' is the ordinary linear acceleration of the shell at the muzzle. Also $|\, \zeta \,|$ must be of the order of $|\, \zeta' \,| \tau$, where τ is the total time of emergence of the shell from the barrel (about 0·0005 secs.) so that $|\, \zeta \,| < 0{\cdot}0015$. It follows that the last term in equation 4 is in absolute value at

* The mass of the shell is m^*. Vector and scalar products are denoted by [.] and (.) respectively. The notation is that of our previous paper, pp. 326, 327.

† *Loc. cit.* p. 327.

most $\frac{1}{20}$ of the last but one, and may be omitted*; the equation is then integrable as it stands. If we take as zero time the moment at which the shell begins to be able to rotate freely about the driving band (the moment of emergence of the shoulder), we may take $t = 0$ for the lower limit of integration and (practically) $\dot{\zeta}_0 = 0$, $\zeta_0' = 0$. Then we find

$$iB_1\zeta' + AN\zeta = -im^*d\,\{\delta\,(v + iw)\}, \qquad \text{.....(5)}$$

where $\delta\,(v + iw)$ denotes the change in $v + iw$ in the interval $(0, t)$. As the interval in which the shell disengages is small, the action is practically impulsive, so that the term in ζ' in equation 5 may be neglected to the present approximation. We thus obtain as a first rough value for ζ', at the moment τ when the band clears the muzzle,

$$\zeta'\,(\tau) = -\frac{m^*d}{B_1}\,\{\delta\,(v + iw)\}. \qquad \text{.....(6)}$$

This can of course be obtained more directly if we are content to ignore the nature of the approximations made at each stage. Since τ is of the order of 0·0005 sec. the value of $\zeta\,(\tau)$ is negligible by comparison. There are of course random gas pressure and blast effects to be superposed on the disturbance resulting from (6), but we are not concerned with these here. The alteration in $\zeta'\,(\tau)$ produced by including the term in ζ can easily be calculated and is found to be of the order of 5 %.

Thus we see that on this theory the main part of the disturbance will be the acquisition of an angular velocity of the axis $\zeta'\,(\tau)$ given by (6) whose nature depends essentially on

$$(v + iw)_\tau - (v + iw)_0.$$

This change of velocity of the free end of the muzzle in the interval $(0, \tau)$ may *a priori* be expected to be fairly constant in direction from round to round, thus satisfying the main requirement of the observations. Whether it could be really proportional to the twist of rifling it is more difficult to say; owing to the torsional strains, such proportionality is not *a priori* impossible. We would content ourselves with pointing out here that

$$(v + iw)_\tau - (v + iw)_0$$

is directly observable; the principal object of this discussion is to urge the importance of its proper determination. The foregoing discussion also suggests problems in the theory of elasticity. It would be of great interest if any sort of approximation to the elastic vibrations of a gun under firing stresses could be obtained theoretically.

* This omitted term represents the disturbing effect of the gas pressure acting through the centre of the base. It is only effective *after* a disturbance has already been set up.

Tides in the Bristol Channel. By G. I. Taylor, F.R.S.

[*Read* 24 January 1921.]

It is well known that tidal waves coming from an open ocean increase in amplitude as they reach the shallower water which usually surrounds the land. This increase is specially marked when the tidal wave enters a contracting channel. The cause of the increase is well understood, and in certain simple cases its amount has been discussed mathematically.

Unfortunately it is very seldom that the nature of the bottom or of the coast line of a channel permits us to apply these results with any hope of getting even an approximate representation of the actual state of affairs in any existing tidal basin. At any rate I do not know of any case in which it has been attempted.

One of the most striking examples of the effect of a contracting channel in increasing the height of the tidal wave is that of the Bristol Channel where there is one of the largest tides in the world. On looking at a chart of that region, I was struck by the way in which both the depth and the breadth of the channel at low water appear to contract almost uniformly from the entrance to the head. Under these circumstances it seemed worth while to work out theoretically the increase in tide which might be expected in a channel whose breadth and mean depth both decrease uniformly from the open end to the head, and both vanish there. The results might then be comparable with the observed tides in the Bristol Channel.

Theoretical Calculation.

The differential equation which represents the variation in amplitude of the tides in a channel as the mean depth and breadth vary, is given in Lamb's *Hydrodynamics**. It is

$$\frac{g}{b}\frac{d}{dx}\left(hb\frac{d\eta}{dx}\right)+\sigma^2\eta=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where η represents the rise and fall of tide, b the breadth, and h the mean depth of a section of the channel taken at a distance x from the head. $2\pi/\sigma$ is the period of the tidal oscillation and g is the acceleration due to gravity.

Two cases have been solved by Lamb, namely (1) h constant, b proportional to x, and (2) h proportional to x, b constant. For the Bristol Channel neither of these is suitable, we require both b and h to be proportional to x.

* P. 267, 4th edition.

Let h_0 be the mean depth and b_0 the breadth of a section at distance x_0 from the head of the channel. Then

$$h = h_0 x/x_0, \quad b = b_0 x/x_0.$$

Inserting these values in (1) the equation becomes

$$\frac{d}{dx}\left(x^2 \frac{d\eta}{dx}\right) + k\eta x = 0 \quad \text{.....(2),}$$

where

$$k = \sigma^2 x_0/h_0 g \quad \text{.....(3).}$$

Putting $z^2 = kx$, $\xi = \eta z$, this equation becomes

$$\frac{d^2\xi}{dz^2} + \tfrac{1}{2}\frac{d\xi}{dz} + \left(4 - \frac{1}{z^2}\right)\xi = 0.$$

The solution of this is $\xi = AJ_1(2z)$, where J_1 represents a Bessel's Function of the first order. Replacing the original variables the solution becomes

$$\eta = KJ_1\{2\sqrt{(kx)}\}/\sqrt{(kx)} \quad \text{.....(4),}$$

where K is a constant.

Comparison with tidal observations in the Bristol Channel.

For this purpose a chart of the Bristol Channel was taken and a curved line was drawn down the middle of the channel (see fig. 1).

Sections A, B, C, D, E, F, G were taken at convenient points and roughly at right angles to the centre line. These sections are shown on the sketch chart (fig. 1). The breadth b, and mean depth h, of the channel at low water at each of these sections was found. The distance x of its mid-point from the head of the channel down the curved line was also measured. The figures so obtained are given in Table I. The head of the channel was taken as being at Portishead, near Bristol.

TABLE I. *Dimensions of Bristol Channel at various sections.*

Section	Distance from Portishead, miles	Mean depth, fathoms	Breadth, miles
A	61·7	20	22
B	49·7	15	20
C	42	11	14
D	25	8	12
E	15·5	4½	7
F	8	3	4
G	0	3	1

Two diagrams were then drawn showing the relationship between b and x, and between h and x. These are shown in figs. 2

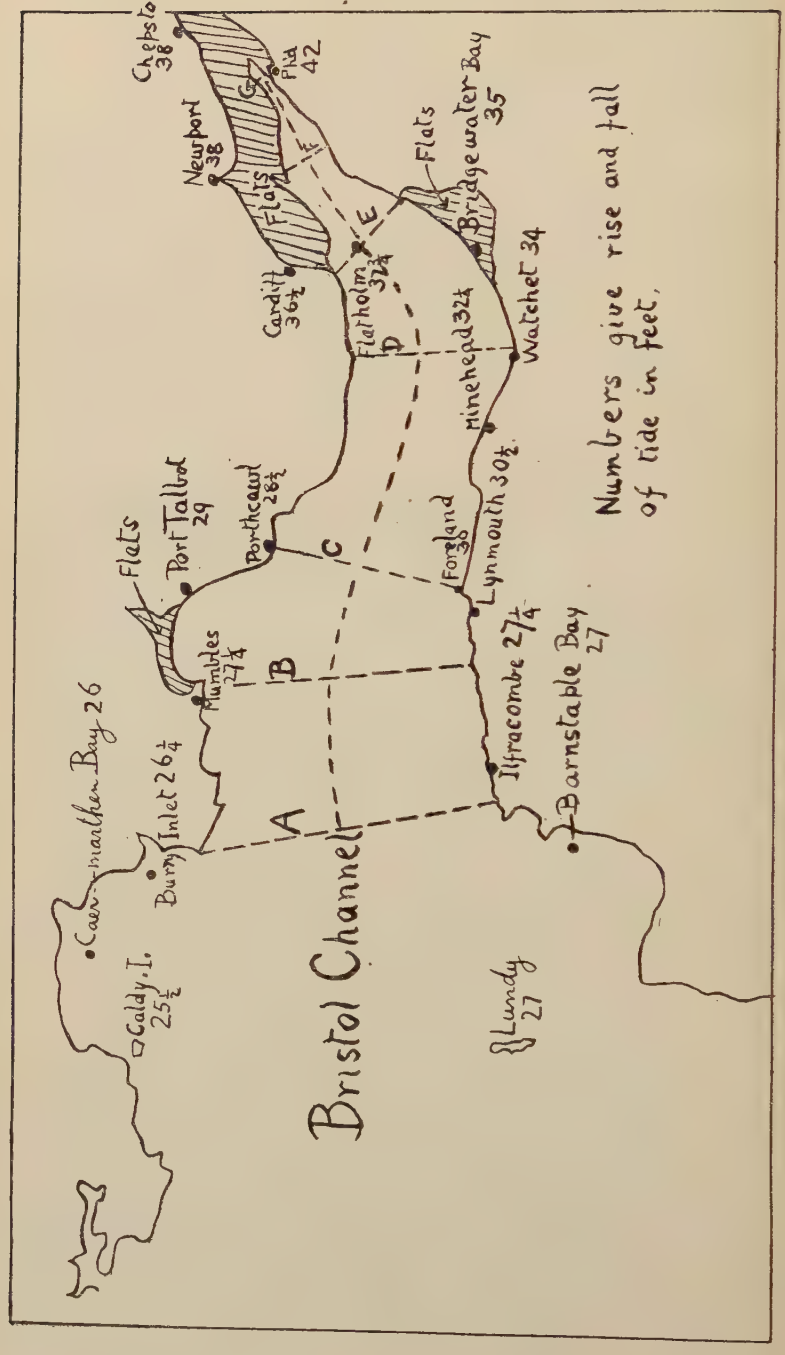
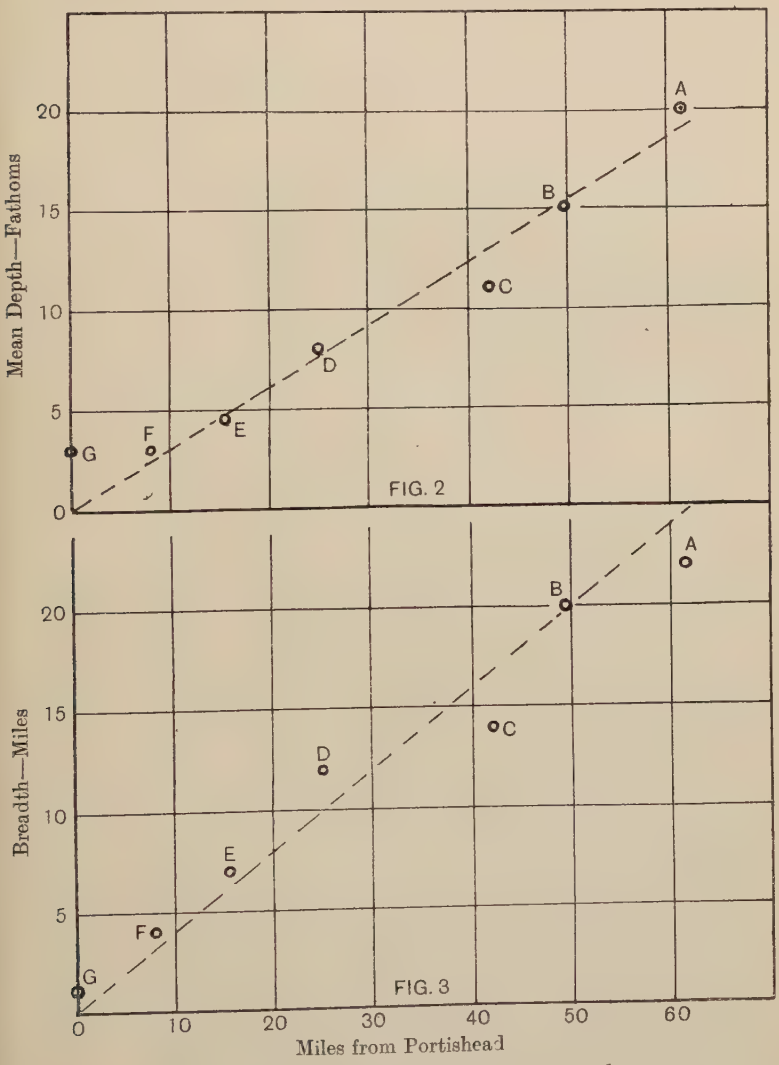

Chepstow 38
Newport 38
Flats
Penarth 42
Cardiff 30½
Flatholm 32½
Flats
Bridgewater Bay 35
Minehead 32½
Watchet 34
Porthcawl 28½
Port Talbot 29
Flats
Mumbles 27½
Foreland 30
Lynmouth 30½
Ilfracombe 27½
Barnstable Bay 27
Lundy 27
Caldy I. 25½
Carmarthen Bay 26
Burry Inlet 26½
Bristol Channel
A
B
C
D
E
F
Numbers give rise and fall of tide in feet.

Fig. 1.

and 3. It will be seen that the assumption that the breadth and the mean depth of the channel increase uniformly from the head towards the mouth is a fairly good approximation to the truth.



Figs. 2 and 3. Dimensions of Bristol Channel.

On looking at the sketch chart (fig. 1) it will be seen that the breadth increases westward of section A, but that owing to two deep bays, Barnstaple Bay and Caermarthen Bay, it ceases to

increase even approximately uniformly. For this reason the mouth of the channel has been taken at section A.

Straight lines were next drawn in figs. 2 and 3 to represent the best values to take for the uniform rates of increase in breadth and depth of the channel. These are shown as dotted lines there. The values of h_0 and b_0 in equation 3 were taken in this way as $x_0 = 80$ nautical miles, $h_0 = 25$ fathoms. Since the period of the semi-diurnal tide is 12·4 hours, $\sigma = 2\pi/12{\cdot}4$. From these data k is found to be ·0118 (miles)$^{-1}$.

If the rise and fall of tide is known at one point of the channel, equation 4 enables us to find it at all other points. In order to compare the theoretical and observed increase in tide due to the contracting walls of the channel, the simplest method appears to be to use the observed rise and fall of tide at the mouth of the Bristol Channel to determine the constant K, in equation 4, and then to apply equation 4 to calculate theoretically the tides at places up the channel at which tidal measurements have been made.

The rise and fall of tide at the time of spring tides has been measured at a number of places on the shores of the Bristol Channel. These are shown on the sketch chart (fig. 1) by means of numbers placed against the names of the places in question*.

The rise and fall of tide at section A has been taken as 27 feet, a figure which appears to agree with the measured tides in the neighbourhood. Using this figure the theoretical rise and fall of tide has been calculated for all values of x, and a curve has been drawn (see fig. 4) to show the relationship between them. On looking at this curve it will be seen that in the range with which we are concerned the curve is very nearly a straight line.

The observed tides, η, and distances, x, for the places shown on the sketch chart (fig. 1) are given in columns 3 and 2 of Table II. They are represented by means of the dots in fig. 4. It will be seen that the agreement between the theoretical tides and the observed tides is very close, much closer indeed than one might have expected when it is remembered that the rise and fall of tide in the upper part of the channel is as great, or even greater than the depth at low water. The calculated tides at the places mentioned in Table II are taken from the curve (fig. 4), and are given in column 4.

In order to compare the predicted with the observed increase in tide due to the contracting channel, 27 feet has been subtracted from the figures in each of columns 3 and 4 of Table II. The results are given in columns 5 and 6.

It appears from these results that the usual hydrodynamical theory of tides accounts quantitatively as well as qualitatively for the abnormally high tides which exist at the head of the Bristol Channel.

* These figures are obtained from the Admiralty publication *West Coast of England Pilot*, 6th edition, 1910.

TABLE II. *Comparison between observed and calculated increases in tides at various distances up the Bristol Channel over those at the entrance.*

Station	Distance, x, from Portishead, miles	Rise and fall of tide, η		Increase in tide over that at section A	
		Observed feet	Calculated feet	Observed feet	Calculated feet
Section A	61·7	27	27	0	0
Ilfracombe	58	$27\frac{1}{4}$	27·7	$\frac{1}{4}$	0·7
Mumbles	51·6	$27\frac{1}{4}$	28·6	$\frac{1}{4}$	1·6
Port Talbot	47	29	29·7	2	2·7
Porthcawl	42	$28\frac{1}{2}$	30·7	$1\frac{1}{2}$	3·7
Foreland	42	30	30·7	3	3·7
Minehead	30	$32\frac{1}{4}$	33·1	$5\frac{1}{4}$	6·1
Watchet	25	34	34·2	7	7·2
Bridgewater	$19\frac{1}{2}$	35	35·3	8	8·3
Cardiff	$15\frac{1}{2}$	$36\frac{1}{2}$	36·2	$9\frac{1}{2}$	9·2
Flatholm	$15\frac{1}{2}$	$37\frac{3}{4}$	36·2	$10\frac{3}{4}$	9·2
Newport	8	38	37·9	11	10·9
Portishead	0	42	39·8	15	12·8

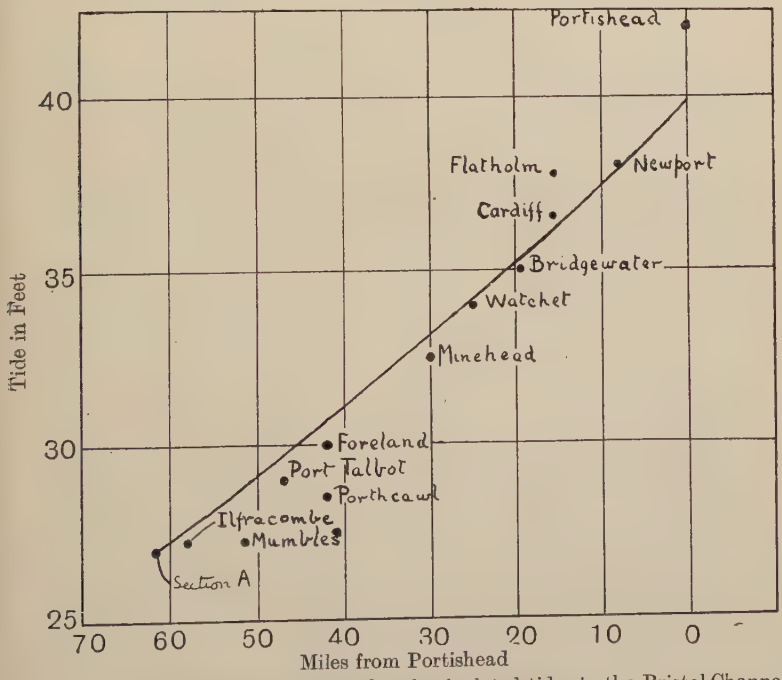


Fig. 4. Comparison between observed and calculated tides in the Bristol Channel.

Experiments with Rotating Fluids. By G. I. Taylor, F.R.S.

[*Read* 24 January 1921.]

The present communication contains a summary of results on three subjects connected with the dynamics of rotating fluids.

1. *Experiments to illustrate the difference between two and three dimensional fluid motion.*

The first experiment to be shown consists in towing a solid circular cylinder of the same density as water through a tank containing water, the whole system rotating at a uniform speed. The tank is a circular glass crystallising dish which is filled three-quarters full with water and floats in another dish of slightly larger diameter. The inner dish is rotated by means of a jet of water which strikes it tangentially. In this way a uniform speed is obtained.

The solid cylinder made of box-wood or wax is placed with its generators vertical, *i.e.* parallel to the axis of rotation. It is held in a position close to the side of the dish by means of a device which releases it at an appropriate moment. It is towed horizontally through the tank by means of two threads which pass through two small rings fixed to the dish on the opposite side to the point to which the cylinder is initially attached. It is found that the cylinder moves straight through the liquid, moving relatively to the rotating system in the direction in which the thread is pulling it, so that it passes through the centre of the dish.

When the same experiment is performed with a solid sphere instead of a solid cylinder, it is found that the motion is very different. The sphere is deflected and moves through the tank in a curved path, leaving the centre of the dish well on its right if the system is rotating in the clockwise direction when seen from above. These experiments confirm a theoretical prediction given by the author*.

Another prediction which is verified with remarkable accuracy is that all small steady motions of a rotating fluid relative to the rotating system must be two dimensional. A consequence of this is that if a spot of coloured water is placed in the rotating water, and if any small motion is communicated to it, the colouring matter is drawn out into sheets which are always parallel to the axis of rotation. If these sheets are observed along the axis of rotation they appear as thin lines. This property is so strikingly

* *Proc. Roy. Soc.* A. 1917, p. 99.

verified that after a time the sheets may become so thin and closely wound round one another that it is only possible to see that the colouring matter is not uniformly diffused through the liquid by placing one eye directly over the rotating basin. The sheets then suddenly reveal themselves as they pass vertically under the eye, and disappear as soon as they get into a part of the basin which is not exactly under the eye.

2. *Motion of a sphere in a rotating fluid.*

The steady motion of a sphere in a rotating fluid along the axis of rotation is discussed mathematically. The velocity of the fluid at any point is expressed by means of Stokes' stream function. So far as the present writer is aware the Stokes' stream function has hitherto only been used in problems where the motion is symmetrical about an axis and is confined to axial planes. It is equally applicable however to cases in which only the first of these conditions holds, and it is used in the present instance. The expression obtained which represents the stream lines when the whole system is given a uniform vertical velocity so as to bring the sphere to rest is

$$\psi = f \sin^2 \theta,$$

where $$f = z^2 + \sqrt{\mu^4 + 3\mu^2} + 9 \left\{ \cos(z + \epsilon) - \frac{\sin(z + \epsilon)}{z} \right\}$$

and $$z = kr,\ k = 2\Omega/v,\ \mu = ka,\ \tan(\mu + \epsilon) = 3\mu/(3 - \mu^2).$$

Ω is the angular velocity of rotation of the fluid.

v is the velocity of the sphere along the axis.

r, θ are the polar coordinates of a point referred to the centre of the sphere as origin, and a is the radius of the sphere.

The components of velocity of the fluid at any point are found from this expression by the formulae

$$u = -2f \cos\theta/r^2,\ v = \frac{1}{r}\frac{df}{dr} \sin\theta,\ w = kf \sin\theta/r.$$

In this expression the axes of reference are not rotating. It is found that although the solution allows slip to take place at the surface of the sphere, the actual solution obtained involves no slip. This is a point of considerable importance because it is the assumption that there is a slip at the surface of solids which vitiates all the ordinary hydrodynamical theories of the motion of fluids. The stream lines due to the motion of a sphere along the axis of a rotating fluid may therefore be expected to be more like the theoretical stream lines than they are in the case when the fluid is not rotating. This is found to be the case.

One consequence of the fact that the velocity of the fluid at the surface of the sphere is zero, relative to fixed axes, is that as a sphere moves up the axis of a rotating fluid the liquid streaming past it will not tend to rotate it. This is found to be true. It is shown experimentally that a light sphere initially rotating with the liquid in a tall rotating jar of water, stops rotating directly it is moved along the axis of the jar, but that it starts rotating again as soon as the motion along the axis ceases.

3. *Stability of fluid motion between two concentric cylinders.*

The late Lord Rayleigh has stated, though without formal proof, that for three dimensional symmetrical disturbances the steady motion of a perfect liquid between two cylinders which rotate with different speeds is stable if the square of the circulation round circular paths concentric with the cylinders increases on passing from the inner cylinder to the outer one. But that it is unstable otherwise. This conclusion is now proved to be correct by calculating the actual motion in a normal disturbance.

All calculations about the stability of liquid between two rotating cylinders have assumed two dimensional motion. In the case of two dimensional motion a rotation of the whole system makes no difference to the type of motion. Its stability or instability are determined only by the relative motion of the two cylinders.

Experiments made by Mallock* and Conette† showed that if the inner cylinder is fixed while the outer one rotates the motion only becomes unstable at a very much higher relative velocity than if the inner one is fixed and the outer one rotates. This evidently suggests that the instability observed is not two dimensional. According to Rayleigh's criterion for the stability of the symmetrical disturbance of an inviscid fluid between rotating cylinders, the case when the outer cylinder is fixed and the inner one rotates should always be unstable. If it is observed to be stable this must be some effect due to viscosity.

In the case when the inner cylinder is fixed and the outer one rotates symmetrical disturbances should be on the limit between stability and instability. The slightest rotation of the inner cylinder in the same direction as the outer one should make the disturbances stable, while the slightest rotation in the opposite direction should make them unstable.

It appears therefore that the method adopted by previous experimenters in which one or other of the cylinders was fixed is unfortunate.

* Mallock. *Phil. Trans.* A, 1896, p. 41.

† Conette. *Annales de Chimie et de Physique*, [6], xxi, p. 433.

Preliminary experiments made with a pair of concentric cylinders which are both of them capable of being rotated, show that the Rayleigh condition appears to be verified for high speed rotation, but that at low speed it is very considerably modified, presumably by viscosity, the motion being stable beyond the limits prescribed by the Rayleigh theory.

Calculations of the effect of viscosity on the stability of symmetrical disturbances are very difficult, but an equation has been obtained which can probably be solved graphically; and the author hopes shortly to be able to say whether viscosity should increase or decrease the stability of symmetrical disturbances.

Experiments on focal lines formed by a zone plate. By G. F. C. Searle, Sc.D., F.R.S., University Lecturer in Experimental Physics.

[*Read* 28 February 1921.]

§ 1. *Introduction**. In the usual theoretical investigation of the properties of a zone plate, the luminous point is taken to be on the axis of the zone plate, and in the practical measurements, such as those which have been made at the Cavendish Laboratory for many years, the incident rays are not inclined at more than small angles to the axis of the zone plate. When, however, the luminous point is not on the axis, the zone plate gives rise to two focal lines, as a thin lens does under similar circumstances; the positions of these focal lines are investigated in the present paper. The theory has been extended to the case in which any non-spherical wave front falls upon the zone plate at any angle of incidence, and the positions and directions of the focal lines of the emergent wave front have been found. The experiments illustrating the theory were made with the kind assistance of Mr G. S. Clark-Maxwell of King's College.

§ 2. *Theory of zone plate.* Let $OG_1G_2, \ldots$ (Fig. 1) be the section by the plane of the figure of an infinitely thin plane opaque screen, and let $X'OX$ be the normal to the screen at O. Let the spaces to the left and right of the screen in Fig. 1 be called the object and image spaces respectively. On the screen take a point G_1 near O and let $OG_1 = \rho_1$. Let G_2, G_3, ... be other points on the screen such that $OG_n^2 = \rho_n^2 = \rho_1^2 + (n-1)k^2$. Let narrow circular slits be cut in the screen with O as centre and passing through $G_1, G_2, \ldots$. This system of screen and slits forms a theoretical zone plate†.

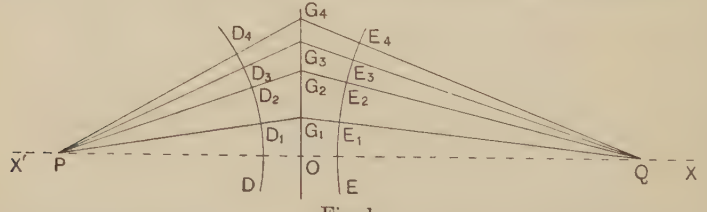


Fig. 1.

* When this paper was read, I did not know that the general theory of the action of a diffraction grating or of a zone plate upon a wave front of any form had been indicated by Sir J. Larmor in "The Dioptrics of Gratings," *Proc. Lond. Math. Soc.* Vol. xxiv. p. 166 (1893).

† To obtain strong images, the widths of the slits through $G_1, G_2, \ldots$ are increased, so that the edges of the nth slit have the radii σ_n, τ_n, where $\sigma_n^2 = \rho_n^2 - z^2$, $\tau_n^2 = \rho_n^2 + z^2$. Zone plates are made by photography from large scale drawings, and the attempt is generally made to make $z^2 = \frac{1}{4}k^2$. If this were *accurately* done, no images of *even* order would be formed. The zone plate used in the experiments is a "phase-reversal" plate made by Prof. R. W. Wood and given by him to the Cavendish Laboratory.

On $X'OX$ take P, Q at distances u, v from O, and let u and v be positive when P and Q are in the object and image spaces respectively. Then, when ρ_n/u, ρ_n/v are small,

$$PG_n + QG_n = (u^2 + \rho_n{}^2)^{\frac{1}{2}} + (v^2 + \rho_n{}^2)^{\frac{1}{2}}$$
$$= u + v + \tfrac{1}{2}\{\rho_1{}^2 + (n-1)\,k^2\}\{1/u + 1/v\}.$$

Hence, for all values of the positive integer n greater than 1,

$$PG_n + QG_n = PG_{n-1} + QG_{n-1} + \tfrac{1}{2}k^2\,(1/u + 1/v).$$

Thus, the paths PG_1Q, PG_2Q, ... increase by equal steps of h, where

$$h = \tfrac{1}{2}k^2\,(1/u + 1/v).$$

Let a train of spherical waves with centre P and wave length λ fall upon the zone plate from the object space, and let D be one of the wave fronts. Let PG_1, PG_2, ... meet D in D_1, D_2, ... Waves will travel out from the slits into the image space. Let E be a sphere about Q as centre, and let G_1Q, G_2Q, ... meet E in $E_1, E_2, \ldots$. Then the disturbances at $E_1, E_2, \ldots$ will have the same phase, if the distances $D_1G_1E_1$, $D_2G_2E_2$, ... increase by steps of $p\lambda$, where p is a positive or negative integer. When the distance of E from O is some thousands of wave lengths, the separate wavelets due to the ring slits will merge into a single wave indistinguishable from E. We may thus speak of E as the emergent wave front. The wave D will thus give rise to the wave E, if $PG_n + QG_n$ exceeds $PG_1 + QG_1$ by $(n-1)\,p\lambda$. An image of P will then be formed at Q.

Hence, Q will be an image of P, if $h = p\lambda$. If f_p be the corresponding focal length, and F_p the corresponding "power," we have

$$1/u + 1/v = F_p = 1/f_p = 2p\lambda/k^2. \qquad \ldots\ldots(1)$$

Thus, the power is proportional to p and is positive or negative with p. We here follow the custom of practical opticians, who treat the power of a thin converging lens as *positive*.

The zone plate thus acts as a lens with a number of positive and negative focal lengths, and, for a given position of a real or virtual luminous point P on $X'OX$, there will be a number of images, some real and some virtual.

If k^2 is found by measuring the rings with a travelling microscope, λ can be found, when f_p and p are known.

§ 3. *Oblique incidence.* Let the luminous point now lie off the axis $X'OX$ at P (Fig. 2) in the plane of the figure. Let $PO = u$, and let u be positive when P is in the object space. Let the acute angle between PO and $X'OX$ be θ. Let Q be a point in the plane of the figure, let $OQ = b$, and let b be positive when Q is in the

image space. Let the acute-angle between OQ and $X'OX$ be ϕ. Then, since $OG_n = \rho_n$,

$$PG_n^2 = u^2 + 2u\rho_n \sin\theta + \rho_n^2, \quad QG_n^2 = b^2 - 2b\rho_n \sin\phi + \rho_n^2.$$

Hence, by expanding,

$$PG_n = u\{1 + \tfrac{1}{2}(2\rho_n \sin\theta/u + \rho_n^2/u^2) - \tfrac{1}{8}(2\rho_n \sin\theta/u + \rho_n^2/u^2)^2 + \ldots\}.$$

Thus, as far as terms in $1/u$,

$$PG_n = u + \rho_n \sin\theta + \tfrac{1}{2}\rho_n^2(1 - \sin^2\theta)/u = u + \rho_n \sin\theta + \tfrac{1}{2}\rho_n^2 \cos^2\theta/u.$$

Similarly, $\quad QG_n = b - \rho_n \sin\phi + \tfrac{1}{2}\rho_n^2 \cos^2\phi/b.$

Hence

$$PG_n + QG_n - (PG_1 + QG_1) = (\rho_n - \rho_1)(\sin\theta - \sin\phi) + \tfrac{1}{2}(\rho_n^2 - \rho_1^2)(\cos^2\theta/u + \cos^2\phi/b).$$

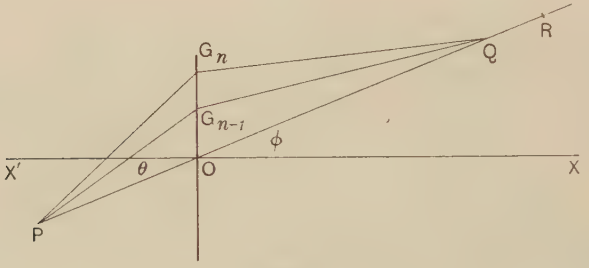


Fig. 2.

If this difference is $(n-1)\,p\lambda$, there will be concentration of light at Q. Now $\rho_n^2 - \rho_1^2 = (n-1)\,k^2$, but $\rho_n - \rho_1$ is not proportional to $n-1$. Hence, if the equation is to hold for all integral values of n, we must have $\sin\phi = \sin\theta$. Thus $\cos^2\phi = \cos^2\theta$, and then,

$$1/u + 1/b = 2p\lambda \sec^2\theta/k^2 = \sec^2\theta/f_p = F_p \sec^2\theta. \quad \ldots(2)$$

Let H_1, H_2, ... lie on a straight line through O perpendicular to the plane of Fig. 2, and let $OH_n = \rho_n$. Then, if R is a point on OP, the path PH_nR is, for given values of $OP = u$ and $OR = c$, equal to the path PH_nR when P and R lie on the axis. Hence, if c be positive when R lies in the image space, there will, by (1), be a concentration of light at R, when

$$1/u + 1/c = 1/f_p = F_p. \quad \ldots\ldots(3)$$

As in § 2, we can speak of the emergent wave front, but in this case the front will not be spherical. By symmetry, the plane of Fig. 2 and a plane through PO and perpendicular to the latter plane are the principal planes of the emergent wave front. The radii of curvature of the principal sections of this front are b and c, as given by (2) and (3).

The emergent rays of order p, i.e. the normals to the emergent wave front of order p, do not pass through a single point but through two focal lines. The primary line passes through Q and is perpendicular to the plane of Fig. 2, and the secondary line through R is in that plane.

If a slit illuminated with sodium light is placed at P, and if the slit is perpendicular to the plane of Fig. 2, there will be a focal line image of the slit at Q, and the image will be perpendicular to that plane. If the slit is in the plane of Fig. 2 and perpendicular to OP, there will be a focal line image at R, the image lying in the plane of Fig. 2.

§ 4. *General case.* Let OX (Fig. 3) be the axis of the zone plate and OP_1 the forward direction of the chief ray of the incident beam. Take OY perpendicular to OX in the plane P_1OX, and OZ perpendicular to that plane. Then the zone plate lies in the plane YOZ. Let $P_1OX = \theta_1$.

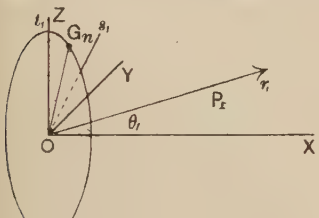


Fig. 3.

Let the refractive indices of the object and image spaces be μ_1, μ_2. Let v_0 be the velocity of light, and λ_0 the wave length, in a vacuum, and let τ be the periodic time of the vibration. Then $\tau v_0 = \lambda_0$.

Take OP_1 as the axis of r_1 in a new set of axes Or_1, Os_1, Ot_1, such that Os_1 is in the plane XOY and Ot_1 coincides with OZ. Let the equation, referred to these axes, of the incident wave front, when passing through O, be

$$r_1 = \tfrac{1}{2}S_1 s_1^2 + W_1 s_1 t_1 + \tfrac{1}{2}T_1 t_1^2. \qquad \ldots\ldots(4)$$

Let G_n be a point on the zone plate on the circle of radius ρ_n, and let $G_nOY = \omega$. Then the x, y, z coordinates of G_n are 0, $\rho_n \cos\omega$, $\rho_n \sin\omega$, and its r_1, s_1, t_1 coordinates are

$$r_1 = \rho_n \cos\omega \sin\theta_1, \quad s_1 = \rho_n \cos\omega \cos\theta_1, \quad t_1 = \rho_n \sin\omega.$$

If a straight line through G_n parallel to OP_1 cuts the incident wave front $OD_1D_2 \ldots$ in D_n, the second and third coordinates of D_n, referred to the axes of r_1, s_1, t_1, are $\rho_n \cos\omega \cos\theta_1$ and $\rho_n \sin\omega$. Hence, by (4), the distance of D_n from the plane $r_1 = 0$, which touches the wave front at O, is

$$\rho_n^2 (\tfrac{1}{2}S_1 \cos^2\omega \cos^2\theta_1 + W_1 \cos\omega \sin\omega \cos\theta_1 + \tfrac{1}{2}T_1 \sin^2\omega),$$

and the distance of G_n from the same plane is $\rho_n \cos\omega \sin\theta_1$. Hence

$$D_nG_n = \rho_n \cos\omega \sin\theta_1 - \rho_n^2 (\tfrac{1}{2}S_1 \cos^2\omega \cos^2\theta_1 + W_1 \cos\omega \sin\omega \cos\theta_1 + \tfrac{1}{2}T_1 \sin^2\omega).$$

When n diminishes, D_nG_n becomes more and more nearly the normal to the wave front, and, for a small aperture, may be treated as the normal in the estimation of *distances*. Thus, when n is not great, D_nG_n may be taken as the ray distance from the wave front to G_n.

Let OP_2 be the forward direction of the chief ray of the emergent beam, which, by symmetry, must lie in the plane XOY. Let $P_2OX = \theta_2$. When referred to axes Or_2, Os_2, Ot_2, chosen similarly to Or_1, Os_1, Ot_1, let the equation to the emergent wave front, as it passes through O, be

$$r_2 = \tfrac{1}{2}S_2s_2^2 + W_2s_2t_2 + \tfrac{1}{2}T_2t_2^2. \qquad \text{.....(5)}$$

If a line through G_n parallel to OP_2 cuts the emergent wave front OE_1E_2 ... in E_n, the distance E_nG_n is ultimately the ray distance from E_n to G_n. We then have

$$E_nG_n = \rho_n \cos\omega \sin\theta_2 - \rho_n^2 (\tfrac{1}{2}S_2 \cos^2\omega \cos^2\theta_2 + W_2 \cos\omega \sin\omega \cos\theta_2 + \tfrac{1}{2}T_2 \sin^2\omega).$$

The optical condition is that the time of passage of light over the distance $E_nG_n - E_1G_1$ in the second medium is less than the time of passage over the distance $D_nG_n - D_1G_1$ in the first medium by $(n-1)\,p\tau$. Hence

$$\mu_2 (E_nG_n - E_1G_1)/v_0 - \mu_1 (D_nG_n - D_1G_1)/v_0 = -(n-1)\,p\tau.$$

Thus, since $v_0\tau = \lambda_0$, we have

$$\begin{aligned}(\rho_n - \rho_1) &\cos\omega (\mu_2 \sin\theta_2 - \mu_1 \sin\theta_1) \\ &- (\rho_n^2 - \rho_1^2) [\tfrac{1}{2} (\mu_2S_2 \cos^2\theta_2 - \mu_1S_1 \cos^2\theta_1) \cos^2\omega \\ &+ (\mu_2W_2 \cos\theta_2 - \mu_1W_1 \cos\theta_1) \cos\omega \sin\omega \\ &+ \tfrac{1}{2} (\mu_2T_2 - \mu_1T_1) \sin^2\omega] = -(n-1)\,p\lambda_0.\end{aligned}$$

Now $\rho_n^2 - \rho_1^2 = (n-1)\,k^2$, but $\rho_n - \rho_1$ is not proportional to $n-1$. Hence, if the equation is to hold for all integral values of n greater than 1, we must have

$$\mu_2 \sin\theta_2 = \mu_1 \sin\theta_1. \qquad \text{.....(6)}$$

Hence, the chief ray obeys the ordinary law of refraction, and θ_2 is known when θ_1 is given. Since $\rho_n^2 - \rho_1^2 = (n-1)\,k^2$, we have

$$\begin{aligned}\tfrac{1}{2} (\mu_2S_2 \cos^2\theta_2 - \mu_1S_1 \cos^2\theta_1) \cos^2\omega \\ + (\mu_2W_2 \cos\theta_2 - \mu_1W_1 \cos\theta_1) \cos\omega \sin\omega \\ + \tfrac{1}{2} (\mu_2T_2 - \mu_1T_1) \sin^2\omega = p\lambda_0/k^2. \qquad \text{.....(7)}\end{aligned}$$

Equation (7) holds for all values of ω. Putting $\omega = 0$, we have

$$\mu_2S_2 \cos^2\theta_2 - \mu_1S_1 \cos^2\theta_1 = 2p\lambda_0/k^2. \qquad \text{.....(8)}$$

Putting $\omega = \tfrac{1}{2}\pi$, we have

$$\mu_2T_2 - \mu_1T_1 = 2p\lambda_0/k^2. \qquad \text{.....(9)}$$

Hence, since $\cos^2\omega + \sin^2\omega = 1$,

$$\tfrac{1}{2}(\mu_2 S_2 \cos^2\theta_2 - \mu_1 S_1 \cos^2\theta_1)\cos^2\omega + \tfrac{1}{2}(\mu_2 T_2 - \mu_1 T_1)\sin^2\omega = p\lambda_0/k^2,$$

and thus, by (7), $\mu_2 W_2 \cos\theta_2 - \mu_1 W_1 \cos\theta_1 = 0$. .....(10)

Since θ_2 is known by (6), the last three equations determine S_2, W_2 and T_2.

In the ordinary use of the zone plate, the medium on either side of the zone plate is air of refractive index μ, and thus $\mu_2 = \mu_1 = \mu$, and $\theta_2 = \theta_1 = \theta$. If the wave length of the light in air is λ, we have $\mu\lambda = \lambda_0$. If the "power" of the zone plate in air is F_p, $2p\lambda/k^2 = F_p$, and then the equations giving S_2, W_2, T_2 become

$$S_2 = F_p \sec^2\theta + S_1, \quad W_2 = W_1, \quad T_2 = F_p + T_1. \quad ...(11)$$

If the incident beam is due to a luminous point at a distance u from O in the object space, $S_1 = T_1 = -1/u$, $W_1 = 0$. Then

$$S_2 = \sec^2\theta/f_p - 1/u, \quad W_2 = 0, \quad T_2 = 1/f_p - 1/u. \quad ...(12)$$

Since $W_2 = 0$, the focal lines of the emergent beam are in and perpendicular to the plane XOY. If their distances from O are c and b respectively, where b, c are positive when the focal lines are in the image space, we have $S_2 = 1/b$, $T_2 = 1/c$, and then

$$1/u + 1/b = \sec^2\theta/f_p, \quad 1/u + 1/c = 1/f_p, \quad .....(13)$$

as was found in § 3.

§ 5. *The principal curvatures of the emergent wave front.* Let the principal planes of the incident front at O (Fig. 4) intersect the tangent plane at O in $O\eta_1$, $O\zeta_1$. Take these lines, with $O\xi_1$ along OP_1, as axes for the front. Let the radii of curvature of the sections of the front by $O\xi_1\eta_1$, $O\xi_1\zeta_1$ be B_1^{-1} and C_1^{-1}, counted positive when the sections are concave towards P_1. The equation to the incident front referred to these axes is then

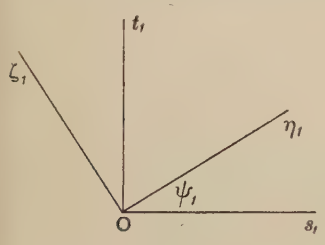


Fig. 4.

$$\xi_1 = \tfrac{1}{2}B_1\eta_1^2 + \tfrac{1}{2}C_1\zeta_1^2. \quad ...(14)$$

Let $O\eta_1$ make an angle ψ_1 with Os_1, as in Fig. 4. Then

$$\xi_1 = r_1, \quad \eta_1 = s_1\cos\psi_1 + t_1\sin\psi_1, \quad \zeta_1 = -s_1\sin\psi_1 + t_1\cos\psi_1,$$

and hence (14) is equivalent to

$$r_1 = \tfrac{1}{2}B_1(s_1\cos\psi_1 + t_1\sin\psi_1)^2 + \tfrac{1}{2}C_1(-s_1\sin\psi_1 + t_1\cos\psi_1)^2. \quad ...(15)$$

Comparing (15) with (4), we find

$$S_1 = \tfrac{1}{2}(B_1 + C_1) + \tfrac{1}{2}(B_1 - C_1)\cos 2\psi_1, \quad \text{.....(16)}$$

$$W_1 = \tfrac{1}{2}(B_1 - C_1)\sin 2\psi_1, \quad \text{.....(17)}$$

$$T_1 = \tfrac{1}{2}(B_1 + C_1) - \tfrac{1}{2}(B_1 - C_1)\cos 2\psi_1. \quad \text{.....(18)}$$

If the equation to the emergent front, referred to its own principal axes, is

$$\xi_2 = \tfrac{1}{2}B_2\eta_2^2 + \tfrac{1}{2}C_2\zeta_2^2, \quad \text{.....(19)}$$

then B_2, C_2 are the principal curvatures. If $\eta_2 O s_2 = \psi_2$, then B_2, C_2, ψ_2 are related to S_2, W_2, T_2 by

$$S_2 = \tfrac{1}{2}(B_2 + C_2) + \tfrac{1}{2}(B_2 - C_2)\cos 2\psi_2, \quad \text{.....(20)}$$

$$W_2 = \tfrac{1}{2}(B_2 - C_2)\sin 2\psi_2, \quad \text{.....(21)}$$

$$T_2 = \tfrac{1}{2}(B_2 + C_2) - \tfrac{1}{2}(B_2 - C_2)\cos 2\psi_2. \quad \text{.....(22)}$$

Solving for B_2, C_2 and using (11), we have

$$B_2 = X + Y, \qquad C_2 = X - Y, \quad \text{.....(23)}$$

where

$$X = \tfrac{1}{2}(S_2 + T_2) = \tfrac{1}{2}[F_p(1 + \sec^2\theta) + B_1 + C_1], \quad \text{.....(24)}$$

$$Y = \tfrac{1}{2}[(S_2 - T_2)^2 + 4W_2^2]^{\frac{1}{2}}$$
$$= \tfrac{1}{2}[F_p^2\tan^4\theta + 2F_p\tan^2\theta\,(B_1 - C_1)\cos 2\psi_1 + (B_1 - C_1)^2]^{\frac{1}{2}}. \quad \text{.....(25)}$$

These equations, with (23), determine B_2 and C_2. But, as either sign can be given to the square root, we are left in doubt as to which of the two focal lines corresponds to B_2 and which to C_2. To avoid confusion we must write

$$Y^0 = \tfrac{1}{2}(F_p\tan^2\theta + B_1 - C_1), \quad Y^{\pi/2} = \tfrac{1}{2}(F_p\tan^2\theta - B_1 + C_1), \quad \text{.....(26)}$$

where Y^0, $Y^{\pi/2}$ are the values to be assigned to Y for $\psi_1 = 0$ and for $\psi_1 = \frac{1}{2}\pi$. For intermediate values of ψ_1, we take Y intermediate between Y^0 and $Y^{\pi/2}$. Since $4Y^2$ is a sum of squares, Y cannot vanish unless $W_2 = 0$, or, what is the same thing, unless $W_1 = 0$. When $B_1 - C_1$ is not zero, W_1 vanishes only when $\psi_1 = 0$ or $\psi_1 = \frac{1}{2}\pi$. Hence Y does not change from positive to negative or *vice versa* as ψ_1 goes from 0 to $\frac{1}{2}\pi$.

In practical work it is convenient to determine the quantities appearing in X and Y by observing B_2^0, C_2^0, $B_2^{\pi/2}$, $C_2^{\pi/2}$, the values of B_2 and C_2 for $\psi_1 = 0$ and $\psi_1 = \frac{1}{2}\pi$. We have, in accordance with (26),

$$\left.\begin{aligned} B_2^0 &= F_p\sec^2\theta + B_1, \quad C_2^0 = F_p + C_1, \\ B_2^{\pi/2} &= F_p\sec^2\theta + C_1, \quad C_2^{\pi/2} = F_p + B_1. \end{aligned}\right\} \quad \text{...(27)}$$

Hence

$$\left.\begin{aligned} X &= \tfrac{1}{2}(B_2^0 + C_2^0) = \tfrac{1}{2}(B_2^{\pi/2} + C_2^{\pi/2}), \\ F_p \tan^2\theta &= B_2^0 - C_2^{\pi/2} = B_2^{\pi/2} - C_2^0, \\ B_1 - C_1 &= B_2^0 - B_2^{\pi/2} = C_2^{\pi/2} - C_2^0 . \end{aligned}\right\} \quad \text{.....(28)}$$

The equation $\tan 2\psi_2 = 2W_2/(S_2 - T_2)$ leads, by (11), to

$$\tan 2\psi_2 = \frac{2W_1}{S_1 - T_1 + F_p \tan^2\theta} = \frac{(B_1 - C_1)\sin 2\psi_1}{(B_1 - C_1)\cos 2\psi_1 + F_p \tan^2\theta},$$

which gives two values of ψ_2 corresponding to the two focal lines. In practical work it is more convenient to use $\sin 2\psi_2$. We have

$$\sin 2\psi_2 = \frac{2W_2}{B_2 - C_2} = \frac{2W_1}{2Y} = \frac{B_1 - C_1}{2Y}\sin 2\psi_1, \quad \text{.....(29)}$$

where, as we may restrict ψ_1 to range from 0 to $\frac{1}{2}\pi$, we may specify that $|\psi_2| \leqslant \frac{1}{2}\pi$. Equation (29) gives the value of ψ_2 corresponding to B_2; the value corresponding to C_2 differs by $\frac{1}{2}\pi$ from that given by (29).

When $Y = 0$, the two focal lines coalesce into a point image of the luminous point. Unless $B_1 = C_1$, this can only occur when $\psi_1 = 0$ or $\psi_1 = \frac{1}{2}\pi$. When this is satisfied, we must further have $S_2 - T_2 = 0$, or, by (11), $F_p \tan^2\theta = -(B_1 - C_1)\cos 2\psi_1$, where $\psi_1 = 0$ or $\psi_1 = \frac{1}{2}\pi$. If θ^0 and $\theta^{\pi/2}$ are the required values of θ, we have

$$\tan^2\theta^0 = -(B_1 - C_1)/F_p, \quad \tan^2\theta^{\pi/2} = (B_1 - C_1)/F_p.$$

For real values of θ, p must have the values 1, 2, ... in one case and $-1, -2, \ldots$ in the other. If $B_1 - C_1$ is positive, p is $-1, -2, \ldots$ for θ^0, and 1, 2, ... for $\theta^{\pi/2}$. If, however, $B_2 - C_2$ is negative, p is 1, 2, ... for θ^0, and $-1, -2, \ldots$ for $\theta^{\pi/2}$. The value of $\tan^2\theta$ will be the same in each case, viz.

$$\tan^2\theta = |(B_1 - C_1)/F_p| = |(B_1 - C_1)/p| . F_1^{-1}. \quad \text{...(30)}$$

The image will be real or virtual according as the values of θ and p make X positive or negative.

§ 6. *Experimental details.* In the experiments the apparatus shown diagrammatically in plan in Fig. 5 was used. The zone plate G is fixed, with its axis ON horizontal, to a table turning about a vertical axis and carrying a horizontal divided circle H, which is read by the pointers J, J. The adjustment of the zone plate so that its centre O lies on the vertical axis of the table may be effected by aid of a long focus microscope or by simple mechanical devices. It is convenient to mount the base of the revolving table

H upon a carriage sliding on a graduated track R at right angles to the optical bench S. The zone plate can then be moved aside and be replaced at will. The focal lines are observed by aid of a low-power eye-piece or of a telescope T mounted on a carriage sliding along the optical bench S. The length of the telescope, which must remain unchanged during any set of observations, should be adjusted so that an object about one metre from the objective may be seen in focus. A cross-wire which has been properly focussed will enable the observer to secure more definite settings of the telescope or eye-piece. If a telescope is used, the positions of virtual as well as of real images and focal lines can be observed.

In testing the results of § 3, a small circular hole P in a sheet of metal, illuminated by a sodium flame Z, may form the object. The hole P lies on the line OX passing through O and parallel to the length of the bench. The distance PO may be about $2f_1$.

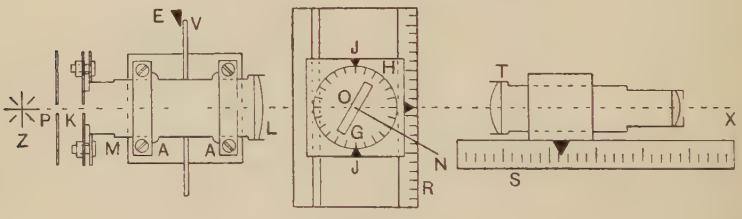


Fig. 5.

A string, stretched parallel to the length of the bench, may be used in setting P on to the line OX and in making the axis of the telescope coincide with OX. The zero reading of the circle H is found by holding a set square against the zone plate and adjusting the table so that the appropriate edge of the square is parallel to OX; suitable optical appliances would give greater accuracy of adjustment. The angle NOX is θ. When $\theta = 0$, the images of P will be found on moving the telescope along the bench. By comparing the bench reading of the telescope for any image with its reading when focussed on the zone plate, the distance of the image from O is found. The distance is called positive when the image is real.

If θ is now increased from zero, the circular image of P will be drawn out into a horizontal *focal line* at a point R on OX, and the position of R will coincide with that of the *image* of P for $\theta = 0$. There will also be a vertical focal line at Q on OX, but this will move towards O as θ is increased, the "power" of the zone plate changing from F_p to $F_p \sec^2 \theta$ for this line. If $PO = u$, and if, for any value of θ, and for any value of p, the distances QO, RO are

b, c respectively, while v is the distance from O of the image when $\theta = 0$, we have

$$\frac{1}{u}+\frac{1}{v}=F_p, \quad \frac{1}{u}+\frac{1}{b}=F_p\sec^2\theta, \quad \frac{1}{u}+\frac{1}{c}=F_p. \quad \ldots(31)$$

The distances u, v or c, and b are measured. The accuracy of the second formula is tested by finding the value of $(1/u + 1/b)\cos^2\theta$ for various values of θ.

The experiment is improved by substituting for the small hole a pair of slits cut in a metal plate and intersecting accurately at right angles. To obtain sharp "images" of the slits formed by focal lines, one slit must be vertical, the other horizontal, since the focal line at P due to a luminous point at Q is vertical, and the focal line at P due to a luminous point at R is horizontal.

When $\theta = 0$, the zone plate forms images of the crossed slits. We can use the multiplicity of focal lengths to produce an *apparent* image of the crossed slits when θ is not zero. With $\theta = 0$, the telescope is set on the *image* of the second order ($p = 2$). Then $1/u + 1/v = F_2$. If θ is now made $\frac{1}{4}\pi$, the horizontal focal line of the second order is given by $1/u + 1/c = F_2$. The vertical focal line of the *first* order for $\theta = \frac{1}{4}\pi$ is given by

$$1/u + 1/b = F_1 \sec^2 \tfrac{1}{4}\pi = 2F_1.$$

But $F_2 = 2F_1$, and hence b for the first order equals c for the second order. The two focal lines will thus be at the same distance from O, and the observer will have the impression that he sees a true image of the slits. If, however, a small hole is used in place of the crossed slits, two focal lines will be seen.

For testing the results of § 5, additional apparatus is required. A horizontal tube M, arranged with a draw tube for adjustment of length, turns in bearings A, A (Fig. 5) and carries a vertical divided circle V, which is read by the index E. A lens L is fixed to the end of the tube nearer O. At the other end is a plate K pierced by a small circular hole, by one slit or by a pair of slits crossed at right angles. This plate can be turned in its own plane about the axis of M and thus a slit can be given any desired direction. The plate K is held against a flange at the end of the tube by three nuts working on three studs carried by a plate on the other side of the flange. The tube passes through a hole in the latter plate with an easy fit. (The plate with the hole P is removed.)

If L has spherical faces, the length of M can be adjusted so that K is in the focal plane of L. The system then forms an ordinary collimator. If L is an astigmatic lens, spherical on one face and cylindrical on the other, and if M is adjusted so that KL equals one of the two focal lengths of L, one of the focal lines formed when a small hole is used in K will be at infinity, and the arrangement

may be called an astigmatic collimator. The emergent wave front in this case will be cylindrical.

By rotating M, the principal planes of the emergent wave front emerging from an astigmatic lens L can be turned about the axis of M*.

The principal planes of the emergent wave front due to a luminous point at K are perpendicular and parallel to the axis of the cylindrical surface of L. We will denote by ψ_1 the inclination of the first of these planes to XON, the horizontal plane of incidence of the chief ray upon the zone plate; then the inclination of the other principal plane is $\psi_1 + \frac{1}{2}\pi$. If, in place of a small hole, a slit is used, its inclination to the horizontal plane through OX must be ψ_1 or $\psi_1 + \frac{1}{2}\pi$, when $\theta = 0$. When θ is not zero and ψ_1 is not zero, no sharp focal line will be formed unless the slit is inclined to XON at one of the two angles δ and $\delta + \frac{1}{2}\pi$. To find the angle δ, first find the focal lines formed at Q and R on OX by the system of lens and zone plate, when a luminous point is placed at K. Next place a luminous point at Q; the direction of that focal line which lies in the plane of K gives one of the required directions of the slit. When the slit is placed in this position and is illuminated, there will be a long sharp focal line at Q. If the slit is turned through $\frac{1}{2}\pi$, there will be a long sharp focal line at R. If we carry out this process mathematically we can calculate δ. In practice it is easy to adjust the direction of the slit so that the images formed by focal lines are sharp.

§ 7. *First practical example.* In this experiment the results of § 3 were tested.

A vertical slit was used in the collimator M (Fig. 5), and the instrument was adjusted so that the slit was accurately at the focus of a converging lens of 3·5 dioptre power. The rays falling on the zone plate came, in effect, from an infinitely distant slit, and thus $1/u = 0$. The real images for $p = +1$ and the virtual images for $p = -1$ were observed by aid of a telescope. The real and virtual horizontal focal lines did not change their positions when θ was increased from zero. Since b, the distance of the vertical focal line from the zone plate is positive or negative with p, the product bp is always positive. The values of bp given for $\theta = 15°$ are the means of those found for $\theta = 15°$ and $\theta = -15°$, and similarly for the other angles. By (2), since $1/u = 0$, $bp \sec^2 \theta$ is constant.

* The appliance (Fig. 5) for rotating the wave front may be used with advantage in the experiment described in § 10 of "Experiments with a plane diffraction grating" (*Proc. Camb. Phil. Soc.* Vol. xx. p. 105). In the method there described, the angle between the vertical and the *cross-wire* is erroneously taken to be ψ_1, whereas, in the theory, the angle between the vertical and the generators of the cylindrical surface of the lens is ψ_1. These two angles are not generally identical. In the actual example given, the difference was small. When the experiment was done, I had not the appliance shown in Fig. 5 above.

θ	Real image bp	Virtual image bp	Mean	$bp \sec^2 \theta$
0°	65·96 cm.	65·84 cm.	65·90 cm.	65·90 cm.
15°	62·02	61·26	61·64	66·06
30°	50·38	48·94	49·66	66·21
45°	33·80	32·69	33·24	66·48
60°	16·71	16·18	16·44	65·76

The mean value of $bp \sec^2 \theta$ is 66·08 cm., and the corresponding "power" of the zone plate for images of the first order is

$$F_1 = 1/66{\cdot}08 \text{ cm.}^{-1} = {\cdot}015133 \text{ cm.}^{-1} = 1{\cdot}5133 \text{ dioptre.}$$

§ 8. *Second practical example.* The results of § 5 were tested in these experiments.

The lens L (Fig. 5) was built up of a spherical lens of 2·5 dioptres in power (2·5 D) and a plano-cylindrical lens of powers 0 and 1 D. The axis of the cylindrical surface will be called the axis of L. When this axis is vertical, $\psi_1 = 0$. In each set of observations, θ was made 45° and $-45°$, and the mean results for the two positions are given. In the first set, the slit was vertical when $\psi_1 = 0$. Then ψ_1 was gradually changed from 0 to $\frac{1}{2}\pi$, and at the same time the direction of the slit and the position of the telescope were changed so as to keep the slit in view throughout. The image of the slit, formed by focal lines, was very sharp if the direction of the slit was very carefully adjusted. The table records B_2, the *reciprocal* of the distance of the focal line from the zone plate, expressed in dioptres. Thus the reciprocal of 40 cm. would appear as 2·5 D.

The axis of L was again made vertical and the slit was made horizontal. Then ψ_1 was changed step by step from 0 to $\frac{1}{2}\pi$. The reciprocal of the distance of this focal line from the zone plate is C_2.

In each case the virtual focal line of the first order was used, and thus B_2 and C_2 were both negative and $p = -1$. From the values given in the table we have, in dioptres,

$$B_2^{\,0} = -2{\cdot}9976, \quad B_2^{\,\pi/2} = -3{\cdot}9002, \quad C_2^{\,0} = -2{\cdot}4201, \quad C_2^{\,\pi/2} = -1{\cdot}5006.$$

Since (28) gives two expressions for X, $F_p \tan^2 \theta$ and $B_1 - C_1$, we take the mean in each case. Thus

$$X = -\tfrac{1}{4}(5{\cdot}4177 + 5{\cdot}4008) = -2{\cdot}7046,$$

$$F_p \tan^2 \theta = -\tfrac{1}{2}(1{\cdot}4970 + 1{\cdot}4801) = -1{\cdot}4886,$$

$$B_1 - C_1 = \tfrac{1}{2}({\cdot}9026 + {\cdot}9195) = {\cdot}9110.$$

Hence $Y^0 = -0{\cdot}2888$, $Y^{\pi/2} = -1{\cdot}1998$, and Y is, therefore, taken as negative for all values of ψ_1. Using the values of $F_p \tan^2 \theta$ and of $B_1 - C_1$ just found, Y was calculated for $\psi_1 = 15°$, ... by (25). There is good agreement between $X + Y$ and the observed value of B_2 and between $X - Y$ and the observed value of C_2.

The distance OL was approximately 10 cm.

$X = -2{\cdot}7046\ D.$

ψ_1	B_2 obsd.	C_2 obsd.	Y	$X+Y$	$X-Y$
°	D	D	D	D	D
0	−2·9976	−2·4201	− ·2888	−2·9934	−2·4158
15	−3·1397	−2·2594	− ·4174	−3·1220	−2·2872
30	−3·3580	−2·0325	− ·6499	−3·3545	−2·0547
45	−3·6127	−1·8037	− ·8726	−3·5772	−1·8320
60	−3·7821	−1·6375	−1·0490	−3·7536	−1·6556
75	−3·8790	−1·5347	−1·1613	−3·8659	−1·5433
90	−3·9002	−1·5006	−1·1998	−3·9044	−1·5048

§ 9. *Third practical example.* In this experiment the directions of the focal lines were observed.

The apparatus was the same as in § 8, and θ was 45°. For each value of ψ_1, the direction of the slit was adjusted so that the image formed by focal lines was as sharp as possible, and the cross-wire of the telescope was set parallel to the edges of the image. The setting of the cross-wire thus depends upon the accuracy of adjustment of the slit, for a change of direction of the slit changes the direction of the "image," in addition to changing the sharpness of its edges. The zone plate carriage was then moved aside on its slide R (Fig. 5) and the telescope was moved back, until a fine wire stretched across the face of L was in focus. The reading of the circle V, when this wire was vertical, was known. The tube M was turned so that this wire was parallel to the cross-wire of the telescope; the direction of the cross-wire was then given by the reading of V. In each case the focal line corresponding to $p = -1$ was used; this focal line was virtual.

Two sets of measurements were made. In the first set, when $\psi_1 = 0$, the slit was vertical. Then ψ_1 was changed by steps of 15° to 90°, and the image, which corresponds to B_2, was followed up. The inclination of this image to the vertical is ψ_2. In the second set, when $\psi_1 = 0$, the slit was horizontal, and this image corresponds to C_2. The inclination of this image to the horizontal is also ψ_2. The table below gives the two values found for ψ_2 and their mean. In calculating ψ_2 from (29), we take $B_1 - C_1 = 0{\cdot}9110$, as found in § 8, and use Y as given in the table in § 8. Since Y is negative when $p = -1$, and since $B_1 - C_1$ is positive, it follows that ψ_2 is negative when ψ_1 is positive.

ψ_1	ψ_2 B_2 image	ψ_2 C_2 image	ψ_2 Mean obsd.	ψ_2 Calcd.
0°	0° 0′	0° 0′	0° 0′	0° 0′
15	−14 30	−16 15	−15 22	−16 32
30	−18 30	−18 45	−18 38	−18 51
45	−12 45	−15 30	−14 8	−15 44
60	−10 0	−11 30	−10 45	−11 3
75	− 5 23	− 5 18	− 5 20	− 5 39
90	0 0	0 0	0 0	0 0

Since $F_p \tan^2\theta = 1{\cdot}4886$, when $p = +1$ and $\theta = 45°$, and since $B_1 - C_1 = {\cdot}9110$, Y_0^0 and $Y^{\pi/2}$ are positive, and thus ψ_2 has the same sign as ψ_1. Using the *real* focal lines corresponding to $p = 1$, we verified that ψ_2/ψ_1 is *positive*.

§ 10. *Fourth practical example.* In these experiments a true image of the object at K (Fig. 5) was produced; the direction of the axis of the zone plate was so adjusted that the astigmatism due to its obliquity balanced the astigmatism due to the lens.

The mean value found for F_1 in § 7 is 1·5133 dioptre. Hence $F_1(1+\sec^2\theta)$ is at least equal to $3{\cdot}0266D$ and $F_2(1+\sec^2\theta)$ and $F_3(1+\sec^2\theta)$ are still larger. Since $B_1 - C_1 = {\cdot}9110$, X and p are positive or negative together.

In the first set of observations, $\psi_1 = 0$. The slit was made horizontal and, with $\theta = 0$, the telescope was focussed on an image for which $p = -1$, -2 or -3; each image was virtual. This image is in the same place as the image which is formed by horizontal focal lines when θ is changed from zero. The slit is then made approximately vertical and the zone plate is turned until θ is such that the image is again in focus. The image is then in focus for all directions of the slit.

In the second set of observations, $\psi_1 = \frac{1}{2}\pi$, and the images for which $p = 1$, 2 or 3 were used; these images were real.

The calculated values of θ were found from (30), using $F_1 = 1{\cdot}5133$ and $B_1 - C_1 = {\cdot}9110$ dioptre.

Real images		Virtual images		Mean obsd.	Calcd.
Order	Obsd.	Order	Obsd.		
p	θ	p	θ	θ	θ
1	37° 45′	−1	37° 30′	37° 38′	37° 48′
2	30 30	−2	28 30	29 30	28 45
3	23 45	−3	22 30	23 8	24 8

Corrections in Papers by Dr G. F. C. Searle.

(i) "A bifilar method of measuring the rigidity of wires" (*Proc. Camb. Phil. Soc.* Vol. xx. p. 61). Equation (10) on p. 67 should read

$$\theta = \frac{C}{1+C}(\phi+\eta) + \frac{\epsilon}{1+C} = D(\phi+\eta) + \frac{\epsilon}{1+C}.$$

When this correction is made, the quantity C remains unchanged, but the values given for P and Q in (18) and (19) must be multiplied by $1+C$. In the practical example (p. 69) P is changed from $-{\cdot}0028$ to $-{\cdot}0029$ and Q from $-{\cdot}0045$ to $-{\cdot}0046$. There will be corresponding small changes in the last column of the table of results on p. 68.

(ii) "Experiments with a plane diffraction grating" (*Proc. Camb. Phil. Soc.* Vol. xx. p. 88). An error affecting §§ 10, 12 of this paper is noticed in a footnote to § 6 of "Experiments on focal lines formed by a zone plate" (*Proc. Camb. Phil. Soc.* Vol. xx. p. 340).

The Tensor Form of the Equations of Viscous Motion. By E. A. Milne, B.A., Trinity College.

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In the general theory of relativity the Principle of Equivalence asserts that all laws relating to phenomena in a geometrical field of force which depend on the g's and their first derivatives only will also hold in a permanent gravitational field. Eddington comments on this* that "it would be quite consistent with the general idea of relativity if the true expression of such laws involved the Riemann-Christoffel tensor, which vanishes in the artificial field and would have to be replaced before the equations were applied to the gravitational field. But were we to admit that, the principle of equivalence would become absolutely useless." The following example from three dimensions illustrates the significance of this point by analogy.

The equations of motion of a viscous fluid in terms of the velocities may be obtained in tensor form either by generalising the corresponding Cartesian equations, or by first generalising the equations of motion involving the pressures and then substituting for the pressures in terms of the velocities. The two forms are found to differ by a term involving $G_{\mu\iota}$, which is of course zero in Galilean space, so that the two forms are in fact equivalent. The explicit emergence of $G_{\mu\nu}$ in such a simple case is however interesting; although there would in any case be no field for the application of an analogue of the principle of equivalence since the second derivatives of the g's are elsewhere involved in both forms.

The stress system (p_{xx}, p_{xy}, ...) in rectangular three-dimensional co-ordinates is a symmetrical tensor of the second rank. The precise generalised definition of p_{xx}, etc., in general co-ordinates is to some extent arbitrary; let us assume they are defined so as to constitute a contravariant tensor $P^{\mu\nu}$. It is to be noted that in this case the contravariant vector expressing the force across the element of surface dS is

$$\tfrac{1}{6}\epsilon_{\nu\rho\sigma} P^{\mu\nu}\, dS^{\rho\sigma}; \qquad \text{.....(1)}$$

here the element of surface is represented by the antisymmetrical contravariant tensor $dS^{\rho\sigma}$ (such that $dS^{\rho\sigma} = 0$ when $\rho = \sigma$ and $dS^{\rho\sigma} = -\,dS^{\sigma\rho}$ when $\rho \neq \sigma$), and $\epsilon_{\nu\rho\sigma}$ denotes the covariant tensor†

* "Report on the Relativity Theory of Gravitation," *Phys. Soc. Lond.* (1918), p. 43.

† For the properties of the ϵ-tensors see J. E. Wright, "Invariants of quadratic differential forms," *Camb. Math. Tracts*, No. 9, p. 21. They may be used for converting a tensor of rank p into one of rank $|n-p|$, n being the number of dimensions; for example, by the use of them it is easy to show how it is that antisymmetrical tensors of the second rank in three dimensions (such as the vector product) degenerate into vectors.

of the third rank whose components are zero when any two of the suffixes are equal, and equal to $\pm\sqrt{g}$ when all three are unequal, the sign being given by the parity of the number of inversions in ν, ρ, σ. If x_1, x_2, x_3 are the co-ordinates, and if the element of surface is denoted in the more usual way by dS_1, dS_2, dS_3, then the x_1-component of (1) reduces to

$$\sqrt{g}\,(P^{11}dS_1 + P^{12}dS_2 + P^{13}dS_3).$$

The "mean pressure" p is the invariant $\frac{1}{3}g_{\mu\nu}P^{\mu\nu}$.

If k is the coefficient of viscosity, the usual equations* for the pressures

$$\left.\begin{aligned} p_{xx} &= -p - \tfrac{2}{3}k\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2k\frac{\partial u}{\partial x} \\ p_{xy} &= p_{yx} = k\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \end{aligned}\right\} \quad \ldots\ldots(2)$$

generalise into

$$P^{\mu\nu} = g^{\mu\nu}\left[-p - \tfrac{2}{3}k\,(u^\tau)_\sigma\right] + k\left[g^{\nu\tau}\,(u^\mu)_\sigma + g^{\mu\tau}\,(u^\nu)_\sigma\right] \ldots(3)$$

the notation $(u^\mu)_\sigma$ denoting the covariant derivative; and the equations of motion in terms of the pressures†

$$\rho\frac{Du}{Dt} = \rho X + \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z} \quad \ldots\ldots(4)$$

generalise into

$$\rho\left(\frac{Du^\mu}{Dt} - X^\mu\right) = (P^{\mu\nu})_\nu \quad \ldots\ldots(5)$$

where the generalisation of "differentiation following the motion" is given by

$$\frac{Du^\mu}{Dt} = \frac{\partial u^\mu}{\partial t} + u^\rho\,(u^\mu)_\rho. \quad \ldots\ldots(6)$$

Substituting for $P^{\mu\nu}$ in (5) and remembering that the covariant derivatives of the g's are identically zero we find

$$\begin{aligned} \rho\left(\frac{Du^\mu}{Dt} - X^\mu\right) &= -g^{\mu\nu}\frac{\partial}{\partial x_\nu}(p + \tfrac{2}{3}k\,(u^\tau)_\sigma) + k\left[g^{\nu\tau}\,(u^\mu)_{\sigma\nu} + g^{\mu\sigma}\,(u^\nu)_{\sigma\nu}\right] \\ &= -g^{\mu\nu}\frac{\partial}{\partial x_\nu}(p - \tfrac{1}{3}k\,(u^\tau)_\sigma) + kg^{\nu\tau}\,(u^\mu)_{\sigma\nu} + kg^{\mu\sigma}\left[(u^\nu)_{\sigma\nu} - (u^\nu)_{\nu\sigma}\right], \end{aligned} \quad \ldots\ldots(7)$$

where in the last term ν and σ have been interchanged as being dummies. These are the differential equations satisfied by the velocities.

On the other hand the usual equations for the velocities obtained by combining (2) and (4) directly, namely,

$$\rho\frac{Du}{Dt} = \rho X - \frac{\partial p}{\partial x} + \tfrac{1}{3}k\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + k\nabla^2 u \quad \ldots(8)$$

* Lamb, *Hydrodynamics*, fourth edition, p. 570. † *Loc. cit.* p. 572.

generalise into

$$\rho\left(\frac{Du^\mu}{Dt} - X\right) = -g^{\mu\nu}\frac{\partial}{\partial x_\nu}(p - \tfrac{1}{3}k(u^\tau)_\sigma) + kg^{\tau\nu}(u^\mu)_{\sigma\nu} \ldots (9)$$

which differ from (7) by the absence of the term

$$kg^{\mu\sigma}[(u^\nu)_{\sigma\nu} - (u^\nu)_{\nu\sigma}]. \qquad \ldots\ldots(10)$$

Now the Riemann-Christoffel tensor $B^\rho{}_{\mu\nu\sigma}$ is conveniently defined by the identity

$$(A_\mu)_{\nu\sigma} - (A_\mu)_{\sigma\nu} = A_\rho B^\rho{}_{\mu\nu\sigma},$$

where A_μ is any covariant vector; but it is easy to prove also that if A^μ is any contravariant vector, then

$$(A^\mu)_{\nu\sigma} - (A^\mu)_{\sigma\nu} = -A^\rho B^\mu{}_{\rho\nu\sigma}.$$

Contracting this by putting $\sigma = \mu$ and summing, we have

$$(A^\mu)_{\nu\mu} - (A^\mu)_{\mu\nu} = -A^\rho G_{\rho\nu}.$$

It follows that the term in question, (10), is simply

$$-kg^{\mu\sigma}G_{\rho\sigma}w^\rho,$$

which vanishes, the space being Galilean. Were one attempting, however, to discuss viscous motion in non-Galilean space, with the generalisations of (2) and (4) as a dynamical basis, one would be led to an incorrect result by hastily generalising (8), although this is merely a combination of (2) and (4); and the interest lies in the circumstance that it is precisely the contracted Riemann-Christoffel tensor that appears as an error term.

Insect Oases. By C. G. Lamb, M.A.

[*Read* 7 March 1921.]

Cases of extremely limited distribution are familiar to all collectors, the limitation being sometimes so great as to amount to a single tree or a few square yards of ground, but it is nearly always possible to correlate the distribution with the presence of the necessary pabulum or with the environment. The establishment of a satisfactory case of a persistent isolated colony requires not only a careful search of the locality in respect to space, but observation over a considerable period of time in order to eliminate possible secular disturbances. It happens that the author has visited a particular locality at the same period of the year for many years and has investigated it with much care, so that the required time and space conditions may be taken to be well satisfied. Further, the part of the district to be dealt with has the additional advantage of being singularly homogeneous in its flora. It consists of a tract of "towans" or sandy waste in the parish of St Merryn, N. Cornwall, known as "Constantine Commons": this waste is of fair extent and is characterised by great uniformity in its flora which includes an exceptional number of the Boraginaceae, *Echium* at times forming a perfectly astonishing spectacle: it also bears the spotted hemlock, the henbane and the opium poppy in fair plenty. The subsoil is clay, so that there is always permanent water in parts, and it is intersected in places with ancient slate slab walls bearing very old tamarisk bushes which indeed form the only shelter against the gales. The district is full of archaeological interest and would repay investigation, and the neighbourhood has yielded several other insects of much interest, though not exhibiting the localised habit of those to be considered. In this perfectly homogeneous area, and in places which careful investigation shows to be in no way different from the surroundings, certain species of Diptera appear to inhabit quite definite oases or islands. One may proceed to such a spot at the proper season with the certainty that the insect will be found, and that in fair or even great numbers, while the rest of the area may yield one or two specimens at most, and that with extreme rarity. In spite of much investigation the author has not been able to find any circumstances whatever correlated to the distribution in the examples given below: whether it be due to a "herd" instinct or not is of course unanswerable: it is more probable that they are cases of approaching extinction. It would be of much interest to

have similar cases recorded, but unfortunately the observations necessitate a very long and close acquaintance with a given district. All the insects mentioned below are quite capable of flight, but are usually very sedentary in habit, being only obtainable by "sweeping."

Lucina fasciata Meigen. This South European species was first recorded many years ago from Ireland, and was rediscovered in England on the coast near Weston, Somerset, by the author. It occurs very sporadically in the locality under consideration, but in one very restricted area it is quite common, the boundary of its distribution being reasonably definite, though no observable physical boundary is to be seen.

Oxyna flavipennis Loew. This species is apparently extremely rare although it is said to breed in *Achillea millifolium*; the late Mr G. H. Verrall in 40 years of most careful investigation only found two specimens. In the present locality it occurs in great numbers on the sheltered side of one out of many of the tamarisk hedges that intersect the commons, but is practically restricted to some 40 yards of the hedge.

Aphaniosoma quadrinotatum Becker. This is a species first described from the Canary Islands and afterwards found in Spain. It is confined to a similarly restricted spot associated with *Matricaria*, which is, however, quite abundant elsewhere where the insect is absent.

Syntormon mikii Strobl. This is another Spanish species which occurs only in a very small marshy hollow near the commons. That same hollow produces three other species of the genus, namely *pumilus*, *monilis* and the ubiquitous *pallipes*.

A second type of homogeneous locality is afforded by the spots where small streamlets run on to the sands of the various bays that break the coast line: these localities are all practically alike in character and flora. On the banks of one of the streams, and in an area of but a few square feet, occurred our last example.

Ochthera mantispa Loew, an insect first described from Rhodes, and found round the Mediterranean littoral. This singularly isolated colony persisted for some years, but is now probably extinct owing to disturbance of the natural conditions by visitors to the bay. In spite of assiduous search no other colony has been found.

It will be noticed that all the species referred to, with the exception of the *Oxyna*, are of southern distribution. Other southern insects have been found in the district, but are not so striking in their isolation. Two other rare flies are also found

there, and in somewhat similar isolation, namely the dolichopids *Dolichopus signifer* and *Acropsilus niger*, but these are so far only known to be of Central European distribution, though the occurrence of the former in Ireland many years ago makes it possible that it is another of the former class.

The most probable reason for the extremely restricted distribution of these insects is that we are in the presence of the last stage of the extinction of a species. With creatures of such comparatively sedentary dispositions, once an island is formed it has little chance of increase. There must be a minimum population density which is such that when the density approaches that minimum the chance of reproduction is practically zero; as there is no evidence that any of the species have sex attracting powers, this minimum density might well be attained at a comparatively small distance from the centre of the colony.

A Note on the Hydrogen Ion Concentration of some Natural Waters. By J. T. Saunders, M.A., Christ's College.

[*Read* 7 March 1921.]

The natural waters referred to in this preliminary note are all fresh and all occur in districts where the soil or sub-soil contains chalk, gault or lime in some form or other. In such districts the natural waters have a fairly constant hydrogen ion concentration. When the water issues from the ground as in the case of springs and wells the P_H is found to vary only within the limits 7·1–7·2. The following table shows the values of P_H for water issuing from the ground:

Locality	P_H	Date
Well at Cherryhinton, Cambs....	7·1	March 1921
Springs at Shelford ("Nine Wells"), Cambs....	7·2	,, 1921
Siphon spring at Warlingham, Surrey ...	7·2	,, 1919
Cambridge tap water (supplied from wells)...	7·1	various dates

These four cases are all very different yet the P_H of the water is remarkably constant. The well at Cherryhinton is 8 ft. deep with 2 ft. of water at the bottom. The water comes through the gault. At Shelford the "Nine Wells" are a number of springs bubbling up through fissures in the chalk. The siphon spring at Warlingham is intermittent, running freely for eight to ten weeks at intervals varying from two to seven years. Here, as at Shelford, the water comes up through fissures in the chalk. The Cambridge Town supply is derived from deep wells running through the gault to water-bearing strata.

As the water leaves the source and flows along in the stream that arises from the spring the value of P_H increases gradually until it reaches a value varying only within the limits 8·25–8·5, at which value it remains constant. The siphon spring water at Warlingham flowed along a wide grassy ditch. Within half-a-mile of the source the P_H of the water had risen to 8·4 at which value it remained constant for, at any rate, another mile, when I was unable to trace it further. The water flowing from the "Nine Wells" at Shelford behaves in much the same fashion, but here I was able to trace the water further and found that the P_H, after reaching the value 8·3 within half-a-mile of the source, fell again at a mile and a half from the source to 8·05. This lowering of the P_H is undoubtedly due to the stream, which has now become almost sluggish, mixing with its waters the acid products of decomposition from the bottom. At first the stream flows over clean

ground, there is no débris and the bottom is swept clean by the force of running water. But further on the stream expands, its pace slackens and débris accumulates at the bottom. In rivers and streams, however sluggish, there is sufficient stream to mix the waters thoroughly and to bring up the acid waters from the bottom. Hence the P_H of rivers and streams is variable and not constant, depending on the pace of the stream and the amount of disturbance of the bottom.

In ponds and lakes that are large and deep, no disturbance of the bottom will occur and there is no general mixing of the waters. Here again we find the P_H to be constant within the limits 8·25–8·5 as the following table shows:

Locality	P_H	Date
Upton Broad, Norfolk	8·4	March, April 1919
S. Walsham Broad, Norfolk	8·35	„ „
Railway Ballast Pits, Chesterton, Cambridge...	8·4	August 1920
Large Brick Pit, Cambridge	8·5	March 1921
„ „ Madingley	8·25	March 1921

The two Broads in Norfolk are large shallow waters not more than 6–8 ft. deep. The pits at Cambridge and Madingley are 20–30 ft. deep. The area in all cases exceeds an acre. In small shallow ponds no such constancy occurs. Here the influence of the bottom will cause a lowering of the P_H; the presence of masses of vegetation, however, may increase the P_H, for plants, during photosynthesis, extract CO_2 from bicarbonates and render the water alkaline. If, on a sunny day, water be taken from a small pond containing masses of Spirogyra the value of the P_H will be found to be as much as 9·0.

On shaking in a test tube well or spring waters from these districts where lime in some form or other is abundant in the soil or sub-soil, it is found that P_H rises to a value varying from 8·25–8·5 and remains constant within these limits. Bubbling air through the water produces the same effect. If the effects of decomposition and photosynthesis be avoided it will be found that the P_H of waters in these districts when saturated with air in solution has a value that is constant within the limits 8·25–8·5.

The method used for the determination of the value of P_H has been that developed by Clark and Lubs.

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Osterhout and Haas. "The Dynamic of Photosynthesis." *J. Gen. Physiol.* 1918, I. 1.

The Mechanism of Ciliary Movement. By J. Gray, M.A., Balfour Student, and Fellow of King's College, Cambridge.

[*Read* 7 March 1921.]

Ciliary movement is a well-known phenomenon in the animal kingdom, and although there is general agreement concerning the morphological structure of the ciliary apparatus, yet our knowledge of its physiology is small. Most of the theories which have been advanced have been based upon the morphological structure of the cilium, and different authors have attributed different functions to the various structures which become visible in stained preparations. The present paper is a preliminary account of an attempt to throw light on this vexed problem by the application of experimental methods.

The material used has been the gills of *Mytilus edulis*, which, as explained elsewhere*, form an admirable subject for study. The main cilia on the gills can be divided into three classes (*a*) the frontal and terminal cilia, (*b*) the lateral cilia, (*c*) the latero-frontal cilia. Of these groups, the first two resemble each other in the fact that they perform work by creating a current of water or bring about the active transport of food and mucus. The function of the latero-frontal cilia is rather difficult to determine, but it seems certain that they do not perform work as do the other types of cilium. These latero-frontal cilia may either serve to keep the filaments of the gill apart, or to direct the water currents formed by the lateral cilia on to the face of the gill.

There is no doubt that the large latero-frontal cilia are composed of a number of cilia or separate fibres fused together. The frontal cilia also have probably the same constitution. There is, further, a wealth of morphological evidence to show that in a very large number of ciliated cells, the cilium itself is composed of fibres fused together, or that part of essential ciliary apparatus is fibrous. The work-performing cilia are remarkably efficient, and are capable of producing a rapid flow of water or food particles. All the cilia possess a very considerable degree of elasticity, since, when deformed by any external agent, they regain their normal shape when the disturbing element is removed.

A considerable number of workers have taken cognisance of the fact that cilia are essentially elastic bodies, but it is of interest to consider the problem afresh.

* J. Gray, *Quart. Journ. Micros. Science*, vol. 64, 1920, p. 345.

Consider a simple strip of steel wire ABC attached at one end C (Fig. 1). If a stress is applied to the wire so as to distort it to A_1B_1C, then it is clear that in passing from the position of rest ABC to its new position A_1B_1C, a considerable part of the energy used to disturb the wire is stored in the wire, in other words when in the position A_1B_1C the strip of wire possesses a definite amount of potential energy. If we wish to make the wire do work in the direction from A to A_1, it is obvious that such a mechanism would be extremely inefficient since the whole of the potential energy taken up by the wire itself would be unavailable for work. Consider now the same wire distorted to abC. Again potential energy is stored. On releasing the wire *the whole* of this energy is set free and is available for work.

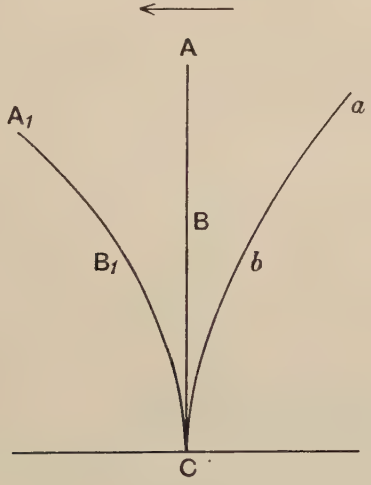


Fig. 1.

Heidenhain and numerous other workers have regarded cilia as comparable to the wire which performs work in moving from A to A_1. We therefore reach the paradoxical conclusion that cilia perform a surprising amount of work*, and yet are exceedingly inefficient machines.

Let us now consider the actual movement of the frontal or terminal cilia on the gills of *Mytilus*.

Under normal conditions the rate of beat of these cilia is so great, that it is only possible to observe the movement in detail when the rate of beat is reduced by the addition of some viscous but non-toxic substance (*e.g.* gum arabic) to the external medium.

During the forward or effective beat (Fig. 2) the cilium behaves as

* Bowditch calculated that each cell of the mucous membrane of the Frog is capable of lifting its own weight to a height of 14 feet in 1 min.

an elastic rod which moves forward from a pivot at its base, and which exposes the maximum amount of surface to the water. During the recovery stroke (Fig. 3), the movement is slower, and the shape of the cilium is different. The cilium is drawn back as a limp string or piece of unstretched rubber in which a stress is set up which starts at the base and is transmitted to the free end. The path followed by the cilium is essentially the same as that of a fishing line during the backward movement of the cast, whereas during the forward effective beat the cilium resembles a stretched spring which is suddenly released. The essential difference is that during the forward effective stroke the cilium is expending energy in the form

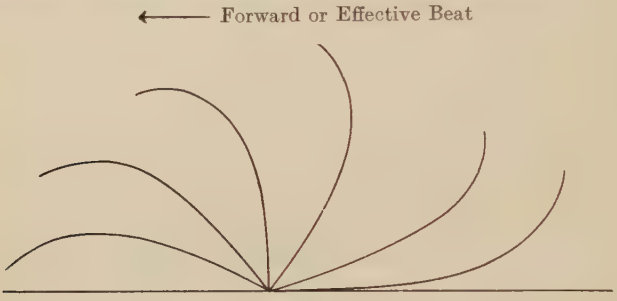


Fig. 2.

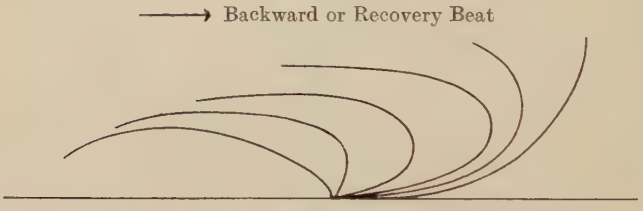


Fig. 3.

of work, whereas during the recovery stroke the cilium is storing potential energy and is performing a minimum of external work during its change of position.

The simplest conception of such a mechanism is to regard the cilium at the end of its effective stroke as a relaxed elastic body. The recovery stroke is brought about by the setting up of a tension in this body: the tension begins at the base of the cilium and passes on to its apex. The direction of the tension is approximately along the line which the cilium occupies at the end of the recovery stroke. In other words the position at the end of the recovery stroke represents the equilibrium between the force which is distorting the cilium and its own elasticity. If the force applied be removed, the potential energy stored in the cilium will be released, and the

cilium will fly forward: in doing so the potential energy will be expended in the form of work done on the water.

There are, however, certain cilia (*e.g.* in Ctenophores) which differ from the cilia of *Mytilus* in that they can be stopped in the position normally occupied at the end of the recovery beat. We must regard such cilia as possessing potential energy when at rest, in the same way as a stretched spring which is held back by a mechanical catch. Such cilia appear to resemble muscle fibres very closely.

Now we may enquire the origin of the potential energy stored in the cilium during the recovery stroke. It must be derived from the chemical energy either in the cilium itself or in some other part of the cell. The problem, therefore, narrows itself to a determination of the possible methods whereby chemical energy can be converted into kinetic energy by means of a fibrous and elastic mechanism. At this point the similarity of the ciliary apparatus to a muscle fibre becomes obvious, and it is convenient to summarize the mechanism of the muscle fibre as analysed by A. V. Hill*.

When at rest a muscle fibre may represent a stretched elastic body which possesses potential energy stored as tension energy in some substance *A* (which may be inactive connective tissue fibres). Within the muscle, however, are certain fibres *B* which are capable of developing a tension when in contact with some chemical substance which is liberated at the time of excitation. We can regard this chemical substance as lactic acid, so that the muscle fibres are capable of developing a tension (just as a piece of catgut develops a tension) when in contact with lactic acid—by the absorption of water. In the resting muscle the substance *A* is kept stretched possibly by the osmotic properties of the liquid in its interstices. When stimulated, lactic acid is set free from some carbohydrate compound, the fibres *B* take up the energy thus set free and develop a tension by taking up water from the interstices of the substance *A*. Consequently the muscle contracts and utilises the energy in both *A* and *B*. After shortening the lactic acid diffuses away, *i.e.*, is removed from the fibres *B*. They therefore give up their water which passes back into the interstices of *A*, and the muscle lengthens. It is clear that there is no *a priori* objection to applying such a hypothesis to the cilia of Ctenophores—but as far as these cilia are concerned it is a hypothesis and nothing more.

In the case of the cilia of *Mytilus* we can analyse the system as follows. At the end of the effective stroke a tension is set up in the fibres by the liberation of some chemical substance (which may be an acid) from the interior of the cell. The equilibrium between this tension and the elasticity of the cilium draws the latter to the position occupied at the end of the recovery stroke, and in doing so

* A. V. Hill and W. Hartree, *Phil. Trans. Roy. Soc.* vol. 210 B, 1920, p. 153.

some of the tension energy of the fibres is stored in the cilium as a whole. The supply of the chemically active substance now ceases, and that portion already located in the fibres diffuses away, so that the cilium flies forward by virtue of its stored potential energy.

In order to test this hypothesis the first obvious line of enquiry is to locate the essential parts of the ciliary mechanism. This problem cannot now be discussed at length: it may suffice to say that the cilium is not of itself automatic; when separated from the cell it does not move. The essential portions of the mechanism lie towards the free edge of the cell. The nucleus does not appear to play an essential rôle.

Owing to the small size of ciliated cells, it is impossible to analyse the movement by such mechanical methods as are applicable to a muscle; ciliated cells have, however, one great advantage in that each individual cell can be observed. Thus, when the amplitude of the contraction of the heart is gradually abolished, it is impossible to say whether this is due to the partial reduction of the contraction of all the cells, or to the total abolition of contraction in some cells, while the amplitude of the others remains unaltered. In the case of cilia this difficulty does not exist, as the beat of a single cilium can be observed throughout the whole experiment.

During the present work an attempt has been made to analyse ciliary movement by a determination of possible means whereby the movement can be influenced or abolished in a reversible manner, *i.e.* without serious derangement of the mechanism itself.

1. *The effect of acids and alkalis.*

As shown in a previous publication* cilia are extremely sensitive to acids. A certain degree of acidity in the external medium causes a cessation of movement. There is no reduction in the amplitude of the beat: the speed of both the effective and recovery beats becomes gradually slower, often with prolonged pauses at the end of each stroke. Eventually the cilia come to rest at the end of the effective stroke. Since there is no diminution in the actual amount of contraction, we are forced to conclude that the cilia do not stop because the actual contractile or elastic mechanisms are deranged but because the rate at which the transformation of chemical into potential energy is gradually reduced and finally ceases. Further, the rate at which potential energy is converted into kinetic energy is also reduced. This conclusion is confirmed by the study of cells in which we know that the amount of convertible chemical energy is small, *viz.*, spermatozoa. The work of Cohn† has shown that in

* J. Gray, *Quart. Journ. Micros. Science*, vol. 64, 1920, p. 345.
† E. J. Cohn, *Biological Bulletin*, vol. 34, 1918, p. 167.

such cells the cessation of movement in an acid solution involves no wastage of energy: the acid simply prevents the chemical energy being converted into kinetic energy.

It is important to notice that the efficiency of acids to cause cessation of movement depends upon the ease with which they penetrate the cell surface, so we may conclude that the liberation of chemical energy into potential energy takes place inside the cell, and not at its surface.

The effect of all moderate strengths of acid (*i.e.* those strengths which rapidly stop movement but do not kill the cell) is entirely reversible by means of alkalis. Again, we find that the efficiency of alkalis depends upon the ease with which they penetrate into the cell, *e.g.* ammonia is much more efficient than the strong alkalis. Within fairly wide limits the rate of ciliary movement depends upon the alkalinity of some area with the cell—from about P_H5 to P_H10 there is a progressive increase in the rate of beat.

The reduction in the rate of transformation of chemical energy into potential energy, by an increase in the acidity of the cell interior, is in accord with Kondo's* investigations on the rate of production of lactic acid from muscle extract. The production of lactic acid from muscle extract is a self-limited reaction which is checked by the formation of the lactic acid—or by another acid in the medium. The fact that the effective beat of a cilium is slowed when the cell interior is more acid than normal is clearly explicable if we assume that the liberation of the potential energy into kinetic energy is dependent upon the rate at which an acid, like lactic acid, diffuses away from some special structure or fibre —the more acid the cell or medium the slower will the potential energy be liberated.

It is clear that the observed facts so far described, place no obstacle in the way of accepting our provisional hypothesis. There is, however, one point which should be mentioned. If potential energy can be stored in the cilium by the liberation of lactic acid at the surface of certain fibres, then one would expect that if lactic acid or any other acid is used as an experimental means of stopping ciliary action, the fibres should remain in the shortened condition, and not in the relaxed state. If the concentration of the experimental acid is raised above the minimum value to cause stoppage of movement, it is true that the cilia move away from the fully relaxed position, but they never approach the end of the recovery beat. The same phenomenon occurs with muscle fibres which are rendered inexcitable by acid. As pointed out by Mines†, however, the shortening of the muscle fibre (or the distortion of a cilium) is brought about by a *local* concentration of acid at the surface of

* K. Kondo, *Biochem. Zeit.* vol. 45, 1912, p. 63.
† G. R. Mines, *Journ. of Phys.* vol. 46, 1913, p. 188.

the fibre, and so the degree of tension set up in the fibres may be due to the *difference* in the concentration of hydrogen ions at the surface of the fibres and in some other part of the cell.

On the other hand, it is possible that if the acidity of the whole cell is raised to the value which normally only exists at the surface of the fibres, there may be a breakdown in the whole colloidal structure of the cell.

2. *The effect of metallic ions.*

It has already been shown* that an artificial solution containing NaCl, KCl, $MgCl_2$, and $CaCl_2$ whose P_H is about 7·8 forms a satisfactory medium for ciliary activity. If a similar solution be prepared in which KCl is omitted the different cilia on the gill react in different ways. The frontal cilia and terminal cilia are practically unaffected within two or three hours; the lateral cilia, however, quickly stop. On adding KCl to the solution they rapidly recover. Recovery can also be brought about by making the solution slightly more alkaline. It is interesting to note that the lateral cilia beat in a definite rhythm, and it is possible that potassium is necessary for this rhythm. If the concentration of potassium be increased above the normal value, the beat of these cilia is well maintained even in solutions in which the whole of the NaCl is replaced by KCl; the frontal and terminal cilia are not appreciably affected; the latero-frontal cilia however go into a state of prolonged contraction. After some time they gradually recover. The effect of potassium on the latero-frontals is antagonised by alkali. Further work is required for an elucidation of these facts, but it is interesting to note the different effects of potassium on different types of cilia.

If an artificial solution be prepared which contains all the normal constituents with the exception of $Ca^{\cdot\cdot}$, all the cilia come to rest within $1\frac{1}{2}$–2 hours. If $Ca^{\cdot\cdot}$ is added as soon as the cilia have ceased to beat, complete recovery takes place: if, however, the addition of $Ca^{\cdot\cdot}$ is delayed for some time, the recovery is much less perfect, the rate of beat is slow and there is often a marked pause at the end of the recovery stroke. After stoppage in the absence of $Ca^{\cdot\cdot}$, rapid and complete recovery takes place on the addition of alkali, even when $Ca^{\cdot\cdot}$ continues to be absent. It is noticeable that the effect of the absence of $Ca^{\cdot\cdot}$ upon cilia is similar to the action of such solutions upon the heart. It is not clear whether the effect of alkali mobilises further stores of $Ca^{\cdot\cdot}$, or whether the absence of $Ca^{\cdot\cdot}$ causes a reduction in the alkalinity of the cell interior. Mines† concluded that in the absence of $Ca^{\cdot\cdot}$ the heart ceases to beat because the actual contractile mechanism is deranged, while the

* J. Gray, *Quart. Journ. Micros. Science*, vol. 64, 1920, p. 345.
† G. R. Mines, *Journ. of Phys.* vol. 46, 1913, p. 188.

supply of potential energy from chemical energy is fully maintained*. It is hoped that further work will throw more light on this problem.

The action of $Mg^{\cdot\cdot}$ is interesting, although the details of the experiments cannot be given here. It seems probable that the presence of $Mg^{\cdot\cdot}$ stabilises the cell—probably the surface of the cell—to the other ions in the medium. It regulates the rate at which other ions can enter the cell, and the rate at which intracellular ions leave the cell. In this respect it can usually be replaced by $Ca^{\cdot\cdot}$.

During the course of these experiments a fairly close parallel is visible between the effect of ions—both anions and kations—upon muscle fibres and the ciliary mechanism. There is, however, one respect in which the cilia of *Mytilus* differ from cardiac muscle—*viz.* they are remarkably insensitive to the salts of the rare earths. On the other hand spermatozoa and the cilia on the blastulae of *Echinus* are just as sensitive to these salts as is cardiac muscle. Possibly the difference depends upon the position of the sensitive surface within the cell: if it lies near the surface, the trivalent ions can reach it, whereas if it lies deeper in the cell they may never penetrate. It is interesting to notice that those cilia which are sensitive to trivalent positive ions are also more sensitive to hydrogen ions than other types of cilia.

3. *Effect of osmotic pressure.*

When cilia are exposed to any solution whose osmotic pressure is above a certain critical value, all movement ceases: the cilia remain in a position between the beginning and end of the effective beat, and are consequently very obvious. On reducing the osmotic pressure normal movement is at once resumed.

Although the details in connection with osmotic stoppage of cilia require further investigation, it may be pointed out that the known facts fit in with our initial hypothesis. The tension set up in a fibre by exposure to an acid depends upon the uptake of water: if the amount of water in the cell is reduced below a certain critical amount, it is obvious that this will affect the tension set up in the fibre, and consequently the beat is affected and at a critical point will be abolished altogether.

The general conclusion which may now be drawn is that the mechanism of ciliary movement and muscular activity may be of essentially the same nature.

* The work of other authors does not appear to agree with this conclusion.

A note on the biology of the 'Crown-Gall' fungus of Lucerne. By J. Line, M.A., Emmanuel College.

[*Read* 7 March 1921.]

'Crown-Gall' of Lucerne has been investigated and described by several workers since the first report upon it (published) in 1898 by von Lagerheim*.

The earlier accounts merely describe the external appearance of the diseased plants, without giving any details of the fungus causing the disease. More recently (1920) two important papers by Wilson†, and Jones and Drechsler‡, have appeared describing the disease and its causative fungus, *Urophlyctis Alfalfae* (Lagerh.), P. Magnus, in great detail.

Before these papers appeared, a detailed investigation of the fungus had been for some time in progress at Cambridge, and was in fact approaching completion at the time of their publication. In view of this it has been thought desirable to publish a brief account of the work, which is in the main confirmatory of the paper by Jones and Drechsler, and like that, is in considerable disagreement with that of Wilson.

External features of the disease.

Diseased plants are found to bear wart-like masses of tissue at about the level of the soil (fig. 1). In advanced cases these may be as much as six inches across, but they are rarely found to extend more than an inch or so below the surface of the ground.

When these masses are cut across, they show characteristic dark brown areas, the spore cavities in section, among the white tissue composing the gall, giving a marbled appearance. The name 'marbled gall' has been suggested to distinguish this type of gall from true bacterial crown-gall. The disease is reported to be fairly common on Lucerne in certain areas west of the Rocky Mountains in the United States; in this country it has so far been reported from three areas only: in 1906 Salmon§ observed it in Kent; it was reported from Bedfordshire by Mr Amos (University Lecturer in Agriculture) in 1917, and was found in two fields near Cambridge in 1919 by the writer. It has again been found in Kent (1920), and in another field, adjoining the first, near Cambridge. It is probable that it is much more common than these reports would suggest,

* Lagerheim, G. (1898). *Bihang K. Svenska Vet. Akad. Handl.* Bd. 24, Afd. 3, No. 4. † Wilson, O. T. (1920), *Bot. Gaz.* v. 70, No. 1, pp. 51–68.

‡ Jones, F. R. and Drechsler, C. (1920). *Journ. Agric. Res.* (U.S.A.), vol. 20, No. 4, p. 295.

§ Salmon, E. S. (1907). *Journ. S.E. Agric. Coll.* Wye, No. 16, p. 296.

as the galls are not easily observed until the plant is removed from the soil. A number of normal, leafy shoots may develop on the plant close to the galls, and in hot weather it is often noticed that these shoots become yellow and show signs of wilting: in this way it is possible to pick out infected plants in a field. The most convenient starting point for a description of the parasite is the resting spore and its germination. Mature spores are globular, flattened at one pole, the average dimensions being 30μ by 45μ, with an extremely brittle wall nearly 2μ thick, of a rich golden brown colour, lined with a thin colourless membrane. Only a very small percentage of the spores examined were induced to germinate; hanging drop cultures were started with different liquid media containing

Fig. 1.

spores from galls of various ages, some of which were more or less completely rotted. Some spores were treated previously with lactic acid, pepsin and other reagents which have been found to induce germination in other cases; others again were exposed to low temperatures, but consistent results were not obtained. It was however found that very slight pressure was often sufficient to start germination in certain cases, and that the most easily germinated spores were obtained from galls which had become rotted owing to the action of mould fungi (often *Fusarium* sp.) and bacteria. The development of external zoosporangia as described by C. E. Scott* was never observed; the first sign of germination being a vibratory motion of the spore contents visible through the wall. This may continue for half an hour, but as a rule the escape of zoospores begins almost at once. Irregular cracks appear in the spore wall; a portion

* Scott, C. E. (1920). *Science*, N.S., No. 1340, pp. 225, 226.

of the inner membrane is sometimes extruded, and through the cracks the zoospores, accompanied by and often entangled in, drops of oil, are seen to escape; they were found to remain active for several hours: no fusion of the zoospores was ever observed. The rapid growth of bacteria in almost all the drop cultures greatly hindered the observations on the behaviour of the zoospores.

Reaction of the host plant to infection.

The zoospores can apparently only penetrate the host at points where the tissues are relatively unprotected by either cuticle or cork. By far the most common starting points of naturally occurring galls are the numerous adventitious buds arising in continuous succession from the woody rootstock of the Lucerne plant. Many of these buds are developed some distance below the ground: they consist of a small axis and a number of leaf rudiments. The zoospores appear to penetrate between the outer scale leaves, and to enter the cells of the young leaves and of the growing point itself. Wherever penetration is effected the host is stimulated to locally increased cell division, the mass of tissue resulting bearing greater or less resemblance to the normal bud, according to the degree of infection. An extensive branching vascular system develops with the gall in direct communication with the vascular system of the host stem; the galls are thus hypertrophied buds or parts of buds. When first observable they appear as minute, white, shining projections from the rootstock or from a bud.

The cell originally entered by the zoospore could often be traced, although it is as a rule rapidly covered in by the division of the surrounding cells. The actual penetration of the host cell by the zoospore has not yet been observed.

From each point of infection the fungus spreads out radially into the host tissues, invading particularly the thin walled cells which have been developed as a result of the presence of the fungus: their contents are absorbed, and their cavities linked up by the absorption of certain of the walls to form an irregularly branching central cavity inhabited by the fungus.

Active living hyphae of the fungus are only found in the peripheral regions, the older portions of the gall being occupied by developing resting spores and degenerated mycelium. The persistent walls of all cells entered become thickened and often curiously pitted; the inner side, in contact with the fungus, appearing somewhat mucilaginous.

In sections of growing galls it is noticeable that part of the walls in the path of the hyphae disappear before the hyphae come in contact with them, but rim-like projections of these walls persist long after the cell is incorporated into the main fungal cavity.

Growth of the fungus is by no means regular in all directions,

and the branching cavity formed by the invaded cells may be of any shape. Once the wall is thickened the fungus never grows through into the adjoining cells, and cell division ceases in the tissues surrounding the older parts of the cavity. It is sometimes observed that a portion of the host tissue is completely isolated by the fungus, and slowly dissolved without actual invasion of the cells composing it. In sectioned material it is not at all easy to make out the method of growth of the hyphae or of the resting spores, but if young and actively growing galls are dissected out and the tissues stained after fixing in bulk, it is possible to reconstruct the fungus thallus with great certainty.

Development of the fungus.

Figs. 2 to 7 are camera-lucida sketches from preparations made in this way. The youngest hyphae (*Hy*, figs. 3 and 5) are seen to possess a very narrow lumen and thin wall, their diameter being about ·5 μ. They are terminated by a swollen portion (*C*, figs. 2, 5 and 6) containing rather dense protoplasm, and at first one nucleus only. This swollen portion will be referred to as the 'collecting cell,' since similar terms have been employed by other writers for analogous structures.

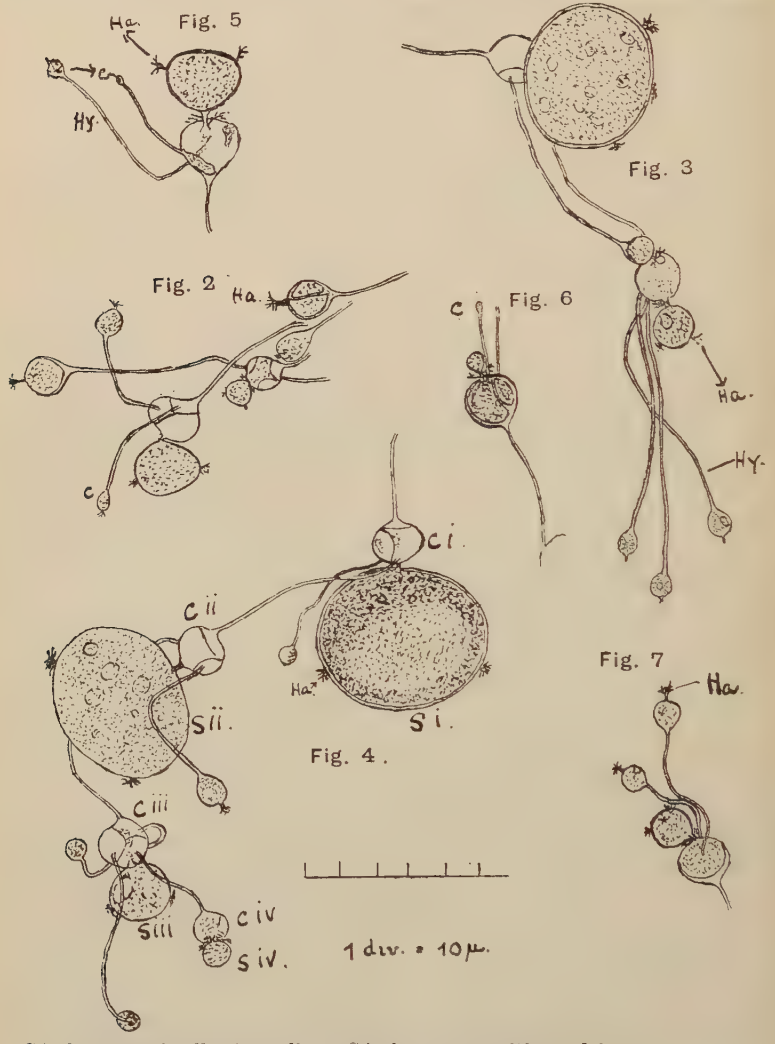
The extreme end of each of these collecting cells develops a short, very delicate and much branched process, which is considered by Jones and Drechsler to have an absorptive function (*Ha*, figs. 2 and 7). It could never be determined with certainty whether this process was a branching hypha or an outgrowth of the wall only: a similar structure is described by Büsgen* for *Cladochytrium Butomi*, and by Schroeter† for *Physoderma* (= *Urophlyctis*) *pulposa*.

The collecting cell increases in size, developing from 10 to 15 nuclei, until it is about 10 μ in thickness. Details of nuclear division have not been made out, the resting nucleus shows one deeply staining mass of chromatinic material, but very little normal reticulum. Fine cross walls are then laid down, oblique to the axis of the collecting cell, cutting off 2 to 4 uninucleate masses of protoplasm, peripheral in position, from a central, multinucleate portion. The former give rise to branch hyphae, the latter to one resting spore; in each case by a process of proliferation. From each of the peripheral cells a papilla arises, the end of which enlarges; into this the single nucleus and contents of the cell are passed; it rapidly elongates to form another hypha of limited growth exactly like the one first considered. The resting spore arises from the centre of the apical haustorial process, as a rounded cell on a short stalk, simultaneously with the branch hyphae. Into this cell the

* Büsgen, M. (1887). Beitrag zur Kenntniss der Cladochytrien. Cohn's *Beitr. Biol. Pflanzen*, Bd 4, pl. 15.

† Schroeter, J. (1882). *Bot. Centbl.* Bd 11, Nos. 31, 32, p. 219.

entire contents of the central portion of the collecting cell are passed; it rapidly increases in size, the absorption of food material being probably assisted by a zone of haustorial processes exactly similar



C i, the original collecting cell. *S i*, the spore proliferated from it.
C ii, *S ii*; *C iii*, *S iii*; *C iv*, *S iv*; similar structures proliferated in succession from *C i*. *Ha*, haustorial processes.

to those developed apically on the young collecting cells, but in this case arising about midway between the equatorial plane and the pole of the spore remote from the collecting cell (*Ha*, figs. 3, 4 and 5).

More than 100 nuclei are formed in the spore as it matures; they are on the whole larger, and show a reticulum better than those in the collecting cells.

Figs. 3 and 4 show this method of proliferation, spores of four different ages developed in succession on one portion of a thallus being shown in fig. 4.

The branched (haustorial) processes are not found to persist on the ripe spores, but as a rule the small depressions from which they arise can be made out.

From this description of the fungus it would appear that it can no longer be regarded as forming its resting spores as a result of the conjugation of two hyphae, in the manner described by Magnus* and Schroeter†; this is the conclusion also reached by Jones and Drechsler‡; their description of the spore formation agrees exactly with that in this paper. It should be noted that the fungus even at the earliest stage in the host plant is bounded by a thin wall, forming a definite mycelium. No trace of a plasmodial stage as described by Wilson§ has been observed.

Host Plants

Lucerne (*Medicago sativa*) has been the only host plant observed associated with the fungus in this country; *M. falcata* is reported to be about equally attacked in the United States with *M. sativa*, under the same conditions, and *M. denticulata* to be immune. Several attempts to infect *M. falcata* and *M. lupulina* have failed, but are being repeated.

Unsuccessful attempts to bring about infection have also been made with all commonly cultivated leguminous crops, and a number of common leguminous pasture plants and weeds.

Infection has been induced at all seasons of the year and with Lucerne plants of all ages from six months old upwards. It is found however that under normal conditions actual infection of the host tissue does not take place during the summer months, the most favourable time being from September to February. From observations made in the field it seems probable that a very wet condition of the soil is favourable for infection, though actual flooding is not necessary. All attempts to cause infection of the youngest seedling stages have so far failed.

I should like to take this opportunity of expressing my thanks to Professor Biffen for suggesting the work, and for supplying some material for investigation, to Professor Seward for laboratory accommodation and to Mr F. T. Brooks for directing the work and for much stimulating criticism.

* Magnus, P. *Ber. Deut. Bot. Gesell.* Bd 20, Heft 5, pp. 291–296.

† Schroeter, J. (1882). *Bot. Centbl.* Bd 11, Nos. 31, 32. p. 219.

‡ Jones, F. R. and Drechsler, C. (1920). *Journ. Agric. Res.* (U.S.A.), vol. 20, No. 4, p. 295.

§ Wilson, O. T. (1920). *Bot. Gaz.* v. 70, No. 1, pp. 51–68.

On some Alcyonaria in the Cambridge Museum. By Sydney J. Hickson, M.A., F.R.S., Professor of Zoology in the University of Manchester.

[*Read* 7 March 1921.]

Clavularia dura n.sp.

A very small specimen of a creeping Clavularia was found in the collection made by Dr J. C. Verco in 20–30 fathoms off Adelaide, S. Australia. The stolon consists of a few flat strands about 5 mm. in width attached to the horny tubes of a Gymnoblastic hydrozoon. On this stolon there stand at considerable distances apart five calices in the shape of inverted cones 1·5 mm. in height with a diameter of ·8 mm. at the distal end and 0·3 mm. at the end where it is attached to the stolon. At the free base of each cone (*i.e.* the distal end of the calyx) there are eight grooves radiating from the centre but the circumference of the cone is quite smooth.

The body wall of the stolon and of the calices is rendered perfectly rigid by a dense amalgamation of calcareous spicules, as in Tubipora, *Telesto rubra* and Sarcodictyon. So hard are the calices that I was unable to break them open with a pair of needles and it required a sharp blow on the cover-glass to crush them. When a calyx had been thus crushed the tentacles were seen to be armed with numerous spicules in the form of curved rods 0·1 mm. in length with a few small tubercles. The whole colony was pure white in colour. In form and habit this new species approaches very closely the *Clavularia ramosa* from the coast of Victoria* but differs from it in the hardness of the walls and the restriction of the eight grooves to the distal end of the calyx. Moreover in the new species I have not been able to find any of the double club spicules which are so characteristic of *C. ramosa*.

Sarcodictyon catenata Forbes.

An Alcyonarian having very much the same appearance as the British *Sarcodictyon catenata* of Forbes was sent to the Museum by Dr J. C. Verco from 20–35 fathoms off Adelaide, S. Australia, in 1904.

The interest of this specimen lies in the fact that the genus Sarcodictyon has been found hitherto only within the British sea area.

In a former paper† I expressed agreement with Sars in suggesting that the genus should be merged with Clavularia and I

* Hickson, *Trans. Zool. Soc.* Lond. xiii, 1894, p. 340.
† *Op. cit.* p. 332.

said that the genus had been only imperfectly described. In doing so I did not do justice to the excellent description of *Sarcodictyon catenata* by Herdman* and I wish to make a sincere although belated apology. The submergence of the genus in Clavularia has unfortunately been accepted by May, Kükenthal and other writers, but since 1894 I have examined several species of Clavularia and by the kindness of Prof. Herdman several specimens of *Sarcodictyon catenata*, and I have come to the conclusion that it is desirable to retain the generic name Sarcodictyon. Sarcodictyon differs from all the species of Clavularia I have examined—except *Clavularia dura*—in having a stolon protected by hard inflexible walls of fused calcareous spicules and in having long retractile anthocodiae which can be withdrawn into shallow convex calices situated on the strands of the stolon. The stolon consists of flattened strands about 1·5–2 mm. in width, forming a network slightly expanded at the nodes and in the places where the zooids are situated. The colour of the stolon seems to be almost invariably red, pink or yellow.

Clavularia dura appears to be a connecting link between the two genera in having hard inflexible walls but differs from Sarcodictyon in the large conical calices and in the absence of colour.

The specimen from Australia is growing on a dead cockle shell about 35 mm. in length and breadth. The stolon is of a pale pink colour and forms a very irregular network of strands with meshes 5 or 6 mm. across, and the strands are about 1 mm. in width. As the zooids are all completely retracted it is very difficult to determine the exact distribution of the spicules in the anthocodiae but in these parts of the colonies free spicules can be found up to 0·2 mm. in length which have the form of irregular spindles provided with numerous irregular tubercles. The wall of the calices and of the stolon can be seen to be formed of spicules of the same form which in further growth have become jammed together to form a solid structure.

On comparing the specimen with specimens from the British sea area I can find no characters to separate it from *S. catenata*. It is true that the pale pink colour distinguishes it from all the other specimens I have seen; but as *S. catenata* is known to vary from red to yellow the colour character is obviously not reliable. There seems to me, therefore, to be no other course to adopt than to name this specimen *S. catenata* notwithstanding the enormous distance that separates the Australian from the British habitat.

* *Roy. Phys. Soc. Edin.* VIII, 1883, p. 31.

Pseudocladochonus hicksoni Versluys.

This very interesting Alcyonarian was first described by Versluys from the collections of the Siboga Expedition near Ceram and Halmaheira*. It exhibits a curious similarity in its method of gemmation and general form to the fossils belonging to the family Auloporidae and particularly to the genus Cladochonus from the carboniferous strata; but according to Versluys, who has made an elaborate study of the genus and of the Auloporidae, it affords us an example of convergence rather than one of genetic affinity with the fossil family.

In a small collection of Alcyonaria from the Uraga Channel near Tokyo, Japan 40–200 fathoms, I found a few small specimens of this species which exhibit all the principal characters described in detail by Versluys.

The size of the retracted zooids 2 mm. × 1 mm. and the diameter of the stems from which they arise 1 mm. are the same as in the type.

The yellow bands where the stem is slightly constricted just above the origin of the zooids can be clearly seen in some of the older fragments but are obscure in the younger branches. Sections through the stem show an arrangement of solenia divided by septa similar to that of the type and the precise arrangement also seems to vary in different parts of the stem. In one of my sections there are four septa meeting in a central column, in another eight septa which do not meet in the centre, but show some fusions of their free borders. There can be little doubt that the 4-septate condition of the first-named section has been derived from an 8-septate condition by an increase in the thickness of the septal walls and fusion at the centre.

The small amount of the material at my disposal has prevented me from making a further investigation of this structure but sufficient has been done to prove that the identification of the genus is correct and that the specimens do not represent an aberrant species of the genus Telesto.

The only difference that I can observe between the Japanese and the Molluccan specimens is in the character of the spicules. In all the preparations I have made which show clearly isolated spicules or spicules which have not yet become firmly joined with their neighbours to form the solid wall or septa of the tubes, the spicules of the Japanese specimens possess more numerous and larger spines and tubercles than they do in the type specimens. The length of the larger spicules (c. 0·12 mm.) does not show any material difference but there is such a wide range of variation both in size and shape of the spicules in any one preparation, that

* Siboga Expedition. Monograph XIII *c*, 1907.

obviously it would be absurd to suggest a specific difference on this single very variable character.

In conclusion it may be said that the comments of Versluys on the relation of this interesting genus to the family Telestidae appear to be sound as also is his conclusion that there is no evidence at present as to any direct relationship with the fossil genus Cladochonus and the Auloporidae.

Telesto trichostemma Dana.

A few branches of a Telesto that must be attributed to the widely distributed species *T. trichostemma* were found in the Uraga Channel off Tokyo, Japan, in 40–200 fathoms. The walls of the main and lateral zooids are so densely crowded with large spindle-shaped and profusely tuberculated spicules that they are quite rigid and they show eight shallow longitudinal ridges as in other specimens of the species. These walls, however, become soft and flexible on prolonged boiling in potash and then the outline of the individual spicules can be clearly distinguished. In this respect the species differs from *T. rubra* in which the walls are also rigid, but do not soften or show the outlines of spicules clearly after prolonged boiling in potash. The colour is pale red. An interesting feature of these specimens is that they support a number of specimens of the rare entoproctous polyzoon *Barentsia discreta*.

Leptogorgia sp.?

The specimen was obtained by Charles Darwin in the Galapagos Islands during the voyage of the "Beagle" in 1835. It is evidently only a small fragment of a much larger specimen.

The branches freely anastomose in one plane forming meshes of 6 sq. mm. or less but of very variable size. The branches are about 2 mm. in diameter and almost cylindrical in shape. They are of a dark red brown colour spotted on both sides with flat yellow calices.

The spicules are spindle-shaped with five or six encircling rows of prominent compound tubercles and have a length of about 0·1 mm. and a breadth, including the tubercles, of 0·04 mm.

The axis is composed of a horny substance without any deposit of calcium carbonate and is perforated longitudinally by a series of chambers filled with a light transparent spongy substance. In a branch of the axis 0·4 mm. in diameter 20 of these chambers can be counted in a millimeter of length.

No attempt has been made to identify the species of this gorgonid because the specimen is only a fragment and because I have not had the opportunity of studying the systematic part of Kükenthal's monograph on the Gorgonaria in the "Valdivia" series of publications. Until this monograph or some other mono-

graph that gives a critical examination of the most unsatisfactory and confusing literature of the family, comes to hand, the attempt to identify the species is little better than a waste of time.

Virgularia mirabilis sp.? O. F. Müller.

In 1889 I recorded the occurrence of specimens of the genus Virgularia from the coast of Victoria under the name *Virgularia löwenii*, but as recent researches on the stages of growth of the northern species of the genus have shown that the type of this species is but a growth form of *V. mirabilis*, it is clear that the name I gave to the Australian specimen must be changed.

The question is whether these specimens, however, are correctly identified with a species that has hitherto been recorded only from the N. Atlantic and Mediterranean waters, and I have therefore re-examined the specimens in order to compare them with examples of *V. mirabilis* from our own coast. The result of this examination has been the failure to discover any satisfactory characters to distinguish them. In the number of the autozooids in the leaves (about 30), in the characters of the calices, and in the number and position of the single row of siphonozooids the Australian specimens resemble *V. mirabilis* and correspond with the description and figures of *V. löwenii* as given by Kölliker.

But before assuming that this identification is sound and that we have in this case an example of a species with bipolar distribution, it is really necessary to examine a large series of growth forms of the Australian Virgularias.

Many years ago specimens of Virgularia from Australia were recorded by Gray and named *V. elegans*, and A. Thomson has recently recorded a Virgularia under the same specific name from the Ceylon seas*. It is difficult to determine from the published descriptions of these specimens what are the specific differences between *V. elegans* and *V. mirabilis*, as no account is given of the position and number of the siphonozooids. It seems possible, however, that they are all representatives of a species that has a very wide distribution where a suitable habitat occurs. For the present, therefore, I am content to leave the name *V. mirabilis* for the Victoria specimens.

Cavernularia chuni Kükenthal and Broch†.

Seven specimens of a species of Cavernularia were found by Dr C. Hose washed ashore after a storm on the beach at Miri, Sarawak, Borneo. As this is, I believe, only the second record of a pennatulid being "washed ashore"—the other being *Cavernu-*

* *Report Pearl Fish.*, Suppl. xxviii, 1905.
† "Valdivia," Pennatulacea, xiii, p. 190.

laria malabarica recorded by Fowler*—the fact is of some interest in the natural history of the sea-pens, and I was surprised to find that the species of the Bornean Cavernularia is not the same as that which is washed ashore in the Bay of Bengal.

The principal measurements of the five perfect specimens were as follows:

	1	2	3	4	5
Length of rachis	25	35	40	48	50 mm.
„ stalk	7	15	8	16	10 mm.

From these figures it is clear that the stalk is relatively short, the rachis-stalk ratio varying from 2·5 : 1 in specimen 2 to 5 : 1 in specimens 3 and 5.

As in the type specimen, there is no sharp distinction in diameter between the stalk and rachis, the stalk passing abruptly into the rachis by the appearance of the zooids only. This feature is in marked contrast to that of *C. malabarica*—in which the passage from the stalk to rachis is marked by a great increase in diameter.

The specimens are so much contracted and distorted that any figures that might be given of the diameter would be untrustworthy. As a guide, however, to the proportions in the species, I may say that I estimated the greatest diameter of the rachis of specimen 5 to be about 15 mm. There is no axis in the two specimens that were dissected.

The spicules of the rachis are needles 0·3–0·5 mm. in length by 0·04 mm. in breadth arranged vertically to the surface and penetrating down almost to the centre of the rachis. There are no spicules that are divided at the extremities.

In the stalk the spicules at the surface are small rods and oval in shape, ·05–0·1 mm. in length, but in the depths there are numerous rod-shaped spicules of the same type as those that occur in the rachis.

The character and arrangement of the spicules are like those described for *Cavernularia chuni* by Kükenthal and Broch, with which species the Bornean specimens also agree in the total absence of an axis.

In the type, however, the stalk is relatively much longer, the stalk-rachis ratio being 1–1·4.

As the species was founded on a single specimen and this stalk-rachis ratio may be variable, it seems probable that the specimens from Borneo should be called *Cavernularia chuni*.

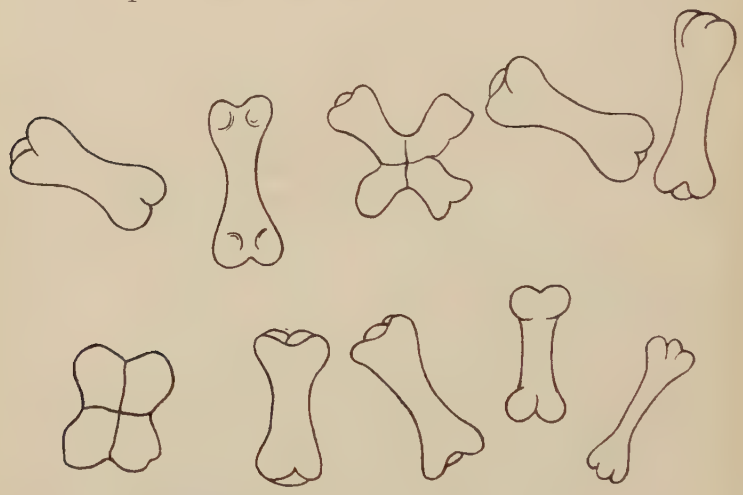
The locality given for the type which is deposited in the Vienna Museum is Coamong (?). (I do not know where Coamong is and cannot find such a place mentioned in Stieler's Atlas.)

* *P.Z.S.* 1894.

Cavernularia darwinii n.sp.

In the stores of the Cambridge Zoological Museum there is a specimen of the genus Cavernularia from C. Darwin's "Beagle" collection labelled Chatham Island, Galapagos Islands, September 1835. As I can find no record of any species of this genus in the Eastern Pacific Ocean and as this specimen is of special interest from its association with the great English naturalist and his memorable voyage in the "Beagle" I decided to examine it carefully with a view to giving it a definite specific name.

The specimen is unfortunately not very well preserved and is strongly contracted and bent, but there are three autozooids killed expanded, one of which has been mounted as a preparation for the microscope.



Spicules of *Cavernularia darwinii*. [× 200.]

Allowing for the contraction and bends the specimen is about 90 mm. in length, the rachis being 38 mm. and the stalk 12 mm. and therefore the stalk-rachis ratio about 1-3.

It is quite impossible to make any accurate statement about the arrangement of the zooids on the rachis or of the relative number of autozooids and siphonozooids owing to the extremely contracted and convoluted condition of the surface. The only statement that can be made without having recourse to a large series of sections is that apparently there are relatively very few autozooids.

The presence of an axis can be proved by probing with a needle and so far as can be judged by this method it extends about half-way up the rachis and about half-way down the stalk. The

general statement may be made therefore that the specimen has an incomplete axis.

The spicules of the rachis, although very variable in shape, are at the same time, very characteristic and can be easily distinguished from the rachis spicules of any other species I have examined. The most prevalent type is that of a short rod 01–0·13 mm. in length, terminating in swollen extremities divided into two, three, four or sometimes five convex facets having an appearance which is extraordinarily like that of a metacarpal bone of a mammal. The variations even in this type of spicule are numerous as the number of the facets varies at each end independently. Thus, there may be one facet at one end, two at the other or two at each end, three at one end and four at the other and so on; but generally speaking if the number of these facets is not the same at each end the excess at one end is not greater than one over that at the other end. This may be represented as follows in figures:

1–1, 1–2, 2–2, 2–3, 3–3, 3–4, 4–4, 4–5.

In addition to the spicules of this type there are some quadruplets which are either simple crosses with rounded ends or crosses with two or three convexities at the end of each of the branches or, in a few cases, simple square plates with very rounded angles. The most noteworthy thing about these rachis spicules, however, is the absence of plain rods of full size or of oval spicules. Apart from a few small spicules, which are probably growth stages, all the spicules are swollen at the extremities and most of them show divided lines of growth.

In the outer layer of the stalk there is a dense armature of spicules of the same type as those prevalent in the rachis; in the inner structures of the stalk there are apparently very few spicules, but several of these which are found lying vertically to the surface in the fleshy septa are longer than those in the rachis, being 0·2 mm. in length.

Cavernularia malabarica Fowler.

Two specimens of this species were obtained by Dr Imms from Puri, Orissa Coast, Bay of Bengal. Their principal measurements are:

Length of rachis 27·25 mm.	Diameter of rachis 35·22 mm.
,, stalk 13·10 mm.	,, stalk 10·8 mm.

The sharp distinction in diameter between rachis and stalk which is a character of this species and of *Cavernularia glans* was well marked in the specimens. The stalk seems to be a little longer in proportion to the rachis than in the type specimens, but it is nevertheless a short stalk.

The species is of some interest as it afforded the first known examples of pennatulids to be "washed ashore."

The Influence of Function on the Conformation of Bones. By A. B. Appleton, M.A., Downing College.

[*Read* 7 March 1921.]

Inspection in a museum of a series of mammalian skeletons is sufficient to indicate some sort of relationship between the osseous details and the locomotor abilities of the animal, whether on plains, in the trees, or in water.

The femur is a bone which, in association with neighbouring bones and muscles, repays detailed study.

We find something in common between the femora of jumpers belonging to quite different animal groups, even though all may not jump in exactly the same way. The same is true of runners (cursorials). It is true in spite of the fact that each mammalian group of living forms tends to exhibit its own characteristic musculature, skeletal features and probably, too, characteristic nervous mechanisms. And it is to be presumed in the first place that tendencies exhibited by cursorials, say, belonging to various groups, may be legitimately regarded as *adaptations*. How far their peculiar musculature is really of advantage to them will be discussed in the sequel.

A short summary will be here given of some muscular peculiarities of specialised cursorials, jumpers and arboreal mammals, and their relationship to peculiarities of the femur discussed.

To what extent such peculiarities are determined by "environmental" influences acting in the individual requires not only a study of ontogeny, but experimental investigation of the effect of modifying the conditions of growth. In fact, a study of human variations demands from us an answer to the question: Are all of these variations of hereditary origin, as Pearson and Lee assert*?

The great plasticity of bone under mechanical influences suggests these influences as possible modifying circumstances during ontogeny. In what manner bone will react, is, however, at present ill-understood; though it seems unlikely that Manouvrier's supposition can be true, *viz.* that under transmitted pressure during youth, a femur will *bend* as a *vital* process or reaction. We only know of bending as a twig is bent when the pressure is too great for a femur softened by rickets or osteomalacia.

Bone, however, does react to stress†. If too little stress falls upon an adult bone, it undergoes partial atrophy; it becomes im-

* Pearson and Lee, 'Long Bones of the English Skeleton,' *Drapers' Company Memoirs* (*Biometric Series*), x and xi, pp. 267, 287.

† Murk Jansen, 'On Bone Formation,' 1920.

possible to draw the line between pathology and physiology. Certain observations from human orthopaedic surgery throw light on this question.

Human bone-grafts grow in thickness under certain conditions of stress. Els* has recently stated that a thick bone-graft diminishes in girth till it resembles the bone it replaces. The repair, again, of a fracture is assisted by the transmitted pressure obtainable when a calliper-splint is used. The *internal* structure of an astragalus changes under gross alterations of mechanical stress as occur in flat-foot. The observations of Dendy and Nicholson† on the spicules of sponges suggest also the continual living reaction of skeletal elements to mechanical factors.

When therefore we recall Hunter's observation that during growth there must be a continual remodelling of the neck of the human femur, we find ample grounds for supposing that one's habit of walking, or peculiarities of musculature, or of the methods of employing it, will probably play an important part in the production of individual variations.

Much valuable information is provided by Pearson and Bell in the vast array of statistics in their recent monograph on the femur of mediaeval Londoners and of Primates. They show that while there is a considerable correlation between dimensions of epiphyses of the femur, and between these and its total (and shaft) length, this is much less evident in the female, in whom also the (upper) epiphysis shows less variation in absolute size. On the other hand, and this seems very important, the correlation of coronal and sagittal diameters of the shaft is quite small, and of the neck diameters smaller still. This appears to me just what one would expect from the incidence of "environmental" factors, affecting diameters in different ways. The assumption that such are hereditary features, evolved by selection, is unnecessary. The different correlations and variabilities on the two sides may be accounted for by muscular peculiarities, asymmetrical locomotion, and unusual habits.

Could we but anticipate the differences of right from left which may appear in the rising generation of to-day with its "scooters" and differentiation in function of lower limbs!

Musculature and Cursorial Specialisation.

By some means many groups have evolved cursorial types with similar muscular characteristics. The **Primitive** type of thigh musculature was doubtless of the pattern found in the living tree-shrew (Tupaia) and the lemurs. Matthew, in fact, suggests an arboreal ancestry for mammalia.

* Els, *Anatom.* Hefte, 1 Abt. 176 Heft. 58 Band, Heft 3, 1920.
† Dendy and Nicholson, *Proc. Royal Soc.* LXXXIX, B, 1917, p. 573.

Certainly, eocene mammalia had femora of the same pattern; and probably also musculature. The M. femorococcygeus (Fig. 1) may be first considered. In primitive types this muscle passes down to be attached to the outer border of the femur below the third trochanter (Fig. 2). From the latter the M. gluteus superficialis passes upwards and forwards. These muscles are often, and were probably originally, continuous, as also with the M. biceps and M. tenuissimus. Tupaia presents this general arrangement with little modification.

A **Cursorial** life leads in all groups to a downward migration of the attachment of the femorococcygeus, while this muscle tends at the same time to gain a new upper attachment to the tuber ischii

Fig. 1. Thigh musculature of hedgehog. A primitive condition. *F.C.* = femorococcygeus continuous with superficial gluteus above. *G.M.* = gluteus medius. *V.E.* = vastus externus. (From a preparation by Dr W. L. H. Duckworth)

of the pelvis. The femorococcygeus may even extend downwards till it reaches the patella and then it gains a new function—knee extension. The Felidae show various degrees of specialisation. Even within the single species *Felis pardalis*, the muscle may be attached to either femur or patella (Parsons). The "long vastus" of veterinary surgeons is none other than this transformed femorococcygeus with (in Artiodactyla) accession of the superficial gluteus which has also lost its femoral attachment. This muscle in the horse plays an important part in bracing up the thigh during a stride (Stillman). The superficial gluteus, if not thus incorporated in the long vastus, undergoes, first, proximal migration, then regression —as in Carnivora. Whether its persistence in Equidae is of some

functional value is not clear. Certainly in the rabbit it may experimentally be deprived of its femoral attachment without noticeable difference in the animal's locomotion; though it is possible that minute differences would be found in its skill at turning rapidly or in the placing of its feet when at full speed.

A hyrax (*H. capensis*), which Prof. Keith kindly placed at my disposal, presented an interesting intermediate stage, foreshadowing the "long vastus." The superficial gluteus was attached not only to the femur, but its anterior fibres were adherent to the femorococcygeus, which passed down to the patella as in the cursorials.

Most jumping animals present very similar changes in the femorococcygeus. Whatever the group of origin, whether a cat, a cow, or a kangaroo, the line of specialisation is similar.

In the Primates the femorococcygeus tends to disappear or become incorporated in the superficial gluteus and to lose its lower femoral attachment by regression or migration of its lower fibres. The superficial gluteus (*i.e.* gluteus maximus) is attached just below the great trochanter to a more or less rudimentary third trochanter. The lemurs, however, and the giant apes retain the femorococcygeus attachment along the femoral shaft; they are not specialised, as are the monkeys, for a modified arboreal life, and man for a new mode of progression. That the gibbon presents in this respect to some extent a *parallel* evolution with monkeys and man is supported by the fossil Paidopithex whose femoral shaft is more like that seen in lemurs and the gorilla (*vide infra*).

In man traces of the femorococcygeus are still to be found in the external intermuscular septum as low down as the condyle—just in the position which the muscle occupies in the gorilla. The primitive condition found in the gorilla, chimpanzee, and orang, is probably truly primitive. There is no reason to suppose that this condition is ever acquired secondarily after the specialised features found in cursorials, jumpers, and monkeys have arisen. There is in fact reason to believe that *Irreversibility* of evolution of a muscle like the femorococcygeus is as true as that of many other organs, such as teeth. Various kinds of mammalian specialisation alter its primitive arrangement; to this it will not revert, but would more probably merely atrophy, with lethargic habits.

To prove the condition in the gorilla to be secondary to that in catarrhine monkeys requires proof that the "primitive" femorococcygeus is more suited to him than, say, the catarrhine arrangement with a biceps which has extended its fascial attachment upwards to the middle of the thigh, and with complete absence (as an independent muscle) of the femorococcygeus.

The plasticity of muscle and of bone is of the same order; they both show phylogenetic as well as ontogenetic response to function.

But reversion to slow-moving habits will no more resuscitate an absent muscle than a lost third trochanter.

The adductor group of musculature (with which must be considered the MM. semimembranosus and caudofemoralis [Leche] from their frequent synergism) shows important modifications characteristic of various specialisations.

Attached in the primitive type along the whole femoral shaft and down to the head of the tibia, the bulk of this musculature in cursorials comes to be concentrated in the region of the knee. The value of this will be pointed out subsequently. At the same time it shows great increase in size.

Carnivora deviate far less from the primitive type than do Artiodactyla and Perissodactyla, though the specialised types, such as Canidae, show great increase in size of the adductors.

TABLE I. Weight* of Adductor Group Musculature† (including Semimembranosus).

	Choloepus didactylus	Civet	Nasua	Ursus	Felis (cat)	Canis (dog)	Ovis	Rhinoceros	Equus	Macropus	Lepus cuniculus	Gorilla	Homo (at birth)	Cyno-
A Attached to shaft of femur	2·7	2·8 ?	3·0	2·3	1·6	3·0	1·3	1·2	0·5	1·3	1·5	5·3	3·0	0
B Attached to region of knee	1·2	1·2 ?	1·5	0·8	1·7	2·6	2·8	1·5	5·4	1·0	2·1	1·6	2·1‡	1
Total weight $A+B$	3·9	4·0	4·6	3·1	3·3	5·6	4·1	2·7	5·9	2·4	3·6§	6·9	5·0	2
Ratio $\frac{A}{B}$	2·2	2·3[1]	2·0[1]	3·1	1·0	1·1	0·5	0·8	0·1	1·2	0·7	3·3	1·4	0
Author providing muscle-weights	Macintosh	Macalister		Haughton	Appleton	Appleton	Appleton	Haughton	Appleton	Haughton	Appleton	Appleton	Appleton	

[1] Approximate.

It is seen that there is a reduction in the adductor musculature attached to the shaft in just those specialised types, cursorials and saltatorials, which exhibit the specialised femorococcygeus; but that the total mass may become very large, as in the horse, dog (and lion). The adductor mass in the gorilla is also of great size. A resemblance of the baboon to the cursorials is noticeable in this as in other respects.

* Unit of weight is M. rectus femoris; the best unit yet found for comparative purposes; little affected by specialisation, and containing a factor depending on femoral length.

† Muscles concerned: all adductors, pectineus, presemimembranosus (ischio-condyloideus), semimembranosus, caudofemoralis.

‡ Includes condylar part of adductor magnus. The figures of Dursy for the adult are similar.

§ The rectus femoris is exceptionally small (partly replaced in function by tensor fasciae femoris).

The knee extensors show in cursorials and saltatorials a considerable development in size, the vastus externus being particularly affected. Its method of attachment to the femur is influenced by its internal structure, the penniform pattern, as in man, being relatively more powerful than the simple form of the hedgehog; it is largely determined by the attachments of the neighbouring femorococcygeus and gluteus superficialis. Where these lose their attachment to the lateral margin of the femur, the vastus externus and crureus are able to spread around the outer aspect (as in the baboon, man, and sheep) of the shaft. In primitive types, however, it is unable so to spread backwards on account of the

Fig. 2. Left femur of *Priodontes giganteus*, showing the descending extensor ridge ("frontal pilaster") and the flange-like third trochanter. M. vastus externus envelops the extensor ridge and occupies the hollow in front of the third trochanter.

femorococcygeus and the third trochanter. Now the vasti muscles become large in other animals beside cursorials and saltatorials. In edentates, which present a primitive type of femorococcygeus, gluteus, and adductor, the vasti are, compared with the rectus femoris, of unusually large size. As will be seen below, a special outgrowth of bone, the "descending extensor ridge" (Fig. 2), appears on the front below the great trochanter for the attachment and accommodation of the vastus externus. This long outgrowth would be unnecessary and does not occur when the superficial gluteus (and femorococcygeus) have no attachment to the shaft as in the specialised runners and jumpers; in them the vastus externus

extends backwards round the shaft. The ridge occurs in some edentates, rodents, and the Lemuroidea (including Tarsius). That it does not appear in gorilla, where the superficial gluteus is attached close up to the great trochanter and the third trochanter has disappeared (as in recent Carnivora), suggests the close relationship to this structure.

TABLE II. Weights* of Knee Extensors.

	Badger	Dog	Sheep	Opossum	Kangaroo	Choloepus	Armadillo	Lemur	Macaque	Gorilla	Tarsius	Man
Total weight of vasti and crureus	2·0	2·8	2·0	2·1	2·7	1·2	—	—	3·0	—	27·4	7·7
Weight of vastus externus only	1·5	1·8	1·2	—	2·3	—	3·3†	4·2†	2·0	3·4	17·6†	2·2

The special tendency to development of the knee-extensors in jumpers is to be seen, notably in Tarsius with its very long femur. That the M. rectus femoris (the ambiens of reptiles) is not included in this development is illustrated in the above figures.

Correlation of changes in the Femur with Muscular Specialisation.

Special attention will be here directed to the form of the shaft in cross-section. Much interest attaches thereto owing to the great contrast in form presented by the gorilla to the human genus.

Unspecialised mammalian types generally present a transversally oval cross-section at mid-shaft, and from the lateral margin somewhere near its middle a third trochanter projects: a common type in early tertiary mammalia (Pantolestes, Meniscotherium, Nesodon). Cursorials and saltatorials tend to exhibit a rounded shaft, and the third trochanter shifts up the femur and becomes reduced or disappears.

There is in these more **active** types an increase in the antero-posterior diameter of the femur. Does this appear in adaptation to the increased bending moment, tending to break the shaft across, or is it merely a means of providing attachment for muscles, *viz.* the vasti and crureus, the knee extensors?

The more active animals do throw an enormous strain on the femur in springing from the limb when at the gallop or jump; far greater than the animal's dead weight. An increase in this sagittal diameter is indeed very necessary, and is the more necessary in that these animals have angulated limbs, unlike the graviportal types (Gregory) with straight pillar-like limbs. The increased

* Weights are given in terms of the M. rectus femoris as the unit.

† A descending extensor ridge is present; it is no test of blood-relationship.

antero-posterior diameter of the condyles, so remarkable a feature of running and jumping types, especially in certain extinct artiodactyla (see Fig. 3), must be a part of the same evolutionary response to requirements. The internal structure of the deep condyles illustrates their true nature. The data of Pearson and Lee* provide an excellent illustration of this effect in the lower Primates, though not so interpreted by them. Tarsius, Indris, and Nyctipithecus, all jumpers, contrast strongly with Apes, Loris, Mycetes and others†. Man again shows the deeper and narrower condyles of active hind-limbs; how precarious must be an argument as to his ancestry from a study of indices based on these measurements,

Fig. 3. Lower end of femur of Anthracotherium, showing deep condyles of a cursorial type. From a specimen in the possession of Mr Forster Cooper, reproduced with his kind permission.

without further knowledge of the knee-joint movements in various animals.

The importance of the shaft of a long bone as a surface for muscular attachment is suggested by an examination of animals in which transmitted propulsive forces are not great. They exhibit flattening of the shaft, the width far exceeding the sagittal diameter; this is conspicuous in the sloth. The great relative width is necessary for provision of muscular attachments; there can hardly be lateral strain sufficient to justify such width. The slow loris and apes are broad in the shaft (Pearson and Lee). And when width of shaft is

* Pearson and Lee, 'Long Bones of the English Skeleton.' *Drapers' Company Memoirs (Biometric Series)*, X and XI, pp. 329, 337, 412.

† The length of the femur is a poor standard for comparison because of its great variability with function.

influenced by the presence of a long flange-like third trochanter the requirements of muscular attachments are obvious. In fact the shaft in the small primitive types may be regarded as consisting of a pillar for strength and an external flange for muscular attachment, as in the eocene Meniscotherium and others. The great width in the extinct Megatherium is probably of a similar nature: X-ray investigation of the primitive femur of Priodontes supports this view of its nature.

In larger and more active animals increase of stress requires a stouter femur; there is now less need for flanges. Also, as Scott has shown, all heavy types, despite their relative want of activity, show a general tendency to reduction of the external flange and third trochanter *in situ* (*e.g.* elephant). Data do not appear available to show whether reduction of the superficial gluteus, or of the vasti, can account for such loss in the elephant.

The descending extensor ridge already mentioned is developed in association with a peculiar muscular combination. That it has also the advantage of withstanding stress from the forwardly projecting great trochanter which in these animals (*e.g.* Tarsius, Priodontes) forms a conspicuous attachment for the vastus exexternus, is suggested by X-ray examination of the bone*.

The actual bulk of the vastus externus is probably closely related to the degree of projection of the third trochanter, from which the fibres of gluteus superficialis pass upwards and forwards across it. A third trochanter placed lower down the shaft will tend to be longer.

The term "frontal pilaster" applied to this ridge by Pearson and Bell suggests a comparison with the pilaster on the back of a human femur: only so long as both are regarded as adaptations for muscular attachment does the term seem justifiable.

The human "pilaster," to which the adductors are attached, was regarded by Manouvrier as being due to the backward extension of the crureus muscle, on the outer side; and to expansion of the front of the bone. How is it that in Tarsius, with far larger knee extensors, this backward encroachment does not occur? It is an example of the influence exerted by the M. femorococcygeus, which in primitive types is attached to the outer margin of the flattened femur. A distinct line is produced which marks also the attachment of the crureus, vastus externus and adductors, the attachment in fact of the external intermuscular septum. A similar line for attachment of the vastus internus is found along the inner margin. These lines may for convenience be termed the External and Internal **septal lines.** While the M. femorococcygeus is attached to the femoral shaft, the external septal line never gets displaced to the back of the femur.

* Bone trabeculae may be considered indicative of lines of Pressure (Murk Jansen).

Hence a "linea aspera," formed by approximation of the two septal lines on the back of the femur cannot be formed.

The rounded shaft with linea aspera, which some artiodactyles possess (*e.g.* sheep), is rendered possible by the specialisation of the femorococcygeus, and by the downward displacement of adductors. The shaft is rounded in Tarsius from considerations of strength; but the septal lines are on the lateral margins of the bone—a femorococcygeus is attached. Since the "interseptal" space provides attachment for adductors, this if broad, as in the elephant, tends to flattening—merely to enable the adductors to pass to their more lateral insertion. Thus the posterior surface of types with septal lines remains flattened, even where the anterior surface may become very convex. Heavy types, such as the elephant and Toxodon, show this peculiarity to advantage.

The obliquity of the posterior surface in Carnivora has a similar origin; the adductors here have a presumably important lateral attachment; the backward external projection permits of the accommodation of the large adductors.

Actual production of a narrow interseptal space, and finally of a "linea aspera," as in man, is associated with expansion of the knee extensors, and downward migration* of the bulk of the adductor musculature: these are characteristic of cursorials and saltatorials. Considerations of accommodation would make it necessary, when vasti muscles are attached behind the femur, that flattening of the inner and outer margins of the bone should occur. For the vasti ultimately gain attachment to the patella. It is a similar problem to the accommodation of the enormous vastus externus of Tarsius. The surface below the great trochanter is hollowed out for its accommodation.

Why is it that carnivora, and some Artiodactyla (*e.g.* Cervus), show such a tendency to a wider interseptal space and a prominent *lateral* position for the external septal line? It is not merely due to the bulk of the adductors attached to the shaft, though large for cursorials; for these are large also in catarrhine monkeys, and still larger in man, but in them there is a "linea aspera." There is probably some functional significance in an extreme lateral attachment for the adductors in the cursorial Carnivora; their external rotatory effect will counterbalance the internal rotatory effect of the synergic gluteus medius. This attachment is not found in the catarrhine monkeys and man; it is sufficient in them for the adductors to be attached largely to intermuscular septa; and the septal lines become placed close together. The structure of the quadriceps extensor appears to be closely associated with the backward displacement of the *internal* septal line in man, and perhaps monkeys too; a penniform arrangement of the fibres of the vastus

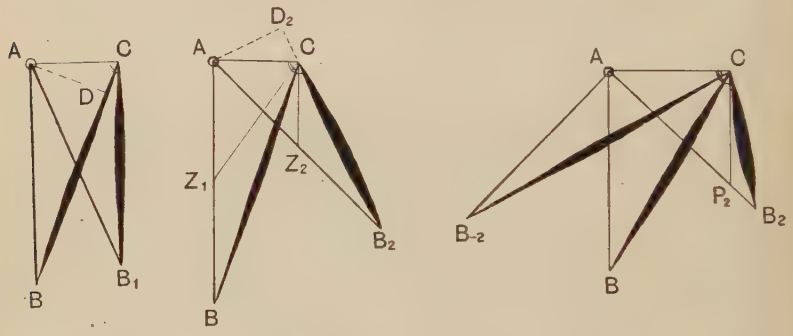
* Except in monkeys and man.

internus is rendered possible, increasing power and diminishing range of movement.

There appears then abundant reason to believe that peculiarities of shape of the femur cannot be properly understood without reference to the muscles attached and adjacent thereto. And further, that the muscles themselves have been evolved in accordance with the habits of an animal and of its forerunners.

Effective Leverage as a factor in Muscular Specialisations.

A consideration of the mode of action of the **Adductor*** group shows that in cursorials and saltatorials their function is largely one of hip-extension. The adductor function proper is more characteristic of arboreal types with a wide range of abduction.



A Cavia. B Lepus. C Bos.

Fig. 4. Lengths AB, AC, are proportional to lengths of femur and postacetabular pelvis in various types. (Used merely as a diagram)

A. (Cavia porcellus) illustrates in AB_1 position of thigh in which semimembranosus obtains maximum effective leverage, measured by perpendicular AC to joint A. AC is greater than AD, which measures leverage when thigh occupies position AB.

B. (Lepus cuniculus) when extension proceeds further till thigh is in position AB_2, leverage is again less, AD_2 being less than AC. At this time, a shaft adductor CZ_2 is acting to the best advantage.

C (Guernsey cow) all positions of flexion in front of the vertical AB require use of the lowest, viz. the knee adductors, which only attain their maximum power at AB_1 (ACB_1 being a right angle). At same point of extension AB_2 at which rabbit employs shaft conductor CZ_2, the cow is still employing CP_2 adductors attached close to the knee.

Considering extension alone in the first place, the maximum effective leverage can be easily shown (*e.g.* for a semimembranosus or ischio-condyloideus) to obtain when the muscle is at right angles (Fig. 4 A) to the line joining acetabulum with tuber ischii. If the thigh is at right angles to the pelvis, the most effective

* Including semimembranosus.

muscles of the whole adductor group will be those attached in the region of the knee; so also in flexed positions of the knee. It is thus that the perpendicular from *muscle* to *joint* is the longest—and this is a measure of the leverage. The femur is of course practically always longer than the postacetabular pelvis; if shorter, the muscle would have to be placed at right angles to the femur for optimum leverage effect.

In more extended positions of the thigh (Fig. 5) such "adductor" muscles as pass to the knee-joint will be at some disadvantage; the most effective muscles will now be attached further up the shaft; and the greater the extension, the higher up the shaft will hip extensors be required.

A relatively longer femur will after a smaller movement of extension involve the employment of these shaft "adductors," rather than those of the knee. Application of these principles to the cursorial, saltatorial, and arboreal animals shows some correspondence with actual life.

Arboreal animals require a greater range of extension at the hip than do cursorials; and the mass of their adductors is attached to the femoral shaft (Fig. 5). Ungulata present a much smaller range of extension (Fig. 4 C). Carnivora are intermediate and so is their adductor musculature (*vide* Table I).

Again, cursorials present a shortened femur, not only as compared with other limb-segments (Gregory), but as compared with the postacetabular pelvis (Fig. 4 C, cow). This will in itself involve the continued use of the semimembranosus at a range of extension at which in longer-limbed animals shaft adductors might be employed. Hence the concentration of "adductors" at the knee in Artiodactyla and other cursorials and jumpers; and, along with this, changes in the form of the femur.

That shortening of the femur, alone, will not result in the cursorial pattern of adductors, *viz.* with attachment largely to the knee-region, is shown in the gorilla (Fig. 5); in him, a large part of the adductors is attached to the middle of the shaft, in association with the habitual employment of the thigh in the almost fully-extended position. In man, the ratio of shaft to knee adductors is large also, larger than the figure in Table I would suggest; his semimembranosus is in fact reduced far below its size in the gorilla; extension from full flexion is not a frequent or habitual movement.

The conditions governing adductor attachment, in so far as true adduction is concerned, are probably similar; in man the extensor function of the group is largely taken over by the enlarged gluteus maximus (*seu* superficialis) owing to the unsatisfactory position of the adductors; and the adductor longus in particular has become an adductor *par excellence*. The relationship of the form of the preacetabular part of the pelvis to the functions of the

adductors should prove full of interest. The postacetabular part of the pelvis shows in its modifications in various groups an attempt to provide the semimembranosus, etc., with more effective leverage (notably in Artiodactyla), as judged on these principles. Lowering (or lengthening) the postacetabular pelvis vastly increases the effective leverage of hamstring muscles. The changes in the femorococcygeus, its transference to the tuber ischii and knee in cursorials, the migration upwards of the biceps in the catarrhines, are all explainable on these lines.

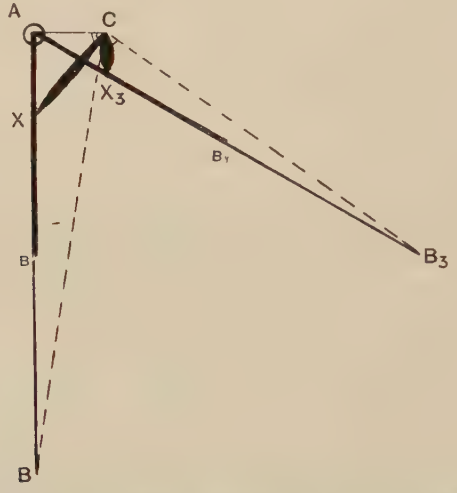


Fig. 5. AB and AB_3 = femur of Semnopithecus. A_B and A_{B_3} = femur of Gorilla, reduced to same AC. AC = postacetabular pelvis. In positions of great extension, AB_3, A_{B_3}, semimembranosus to knee is almost useless as an extensor in any but flexed positions or positions near AB. The most effective position for an extending adductor lies up near head of femur, represented by a muscle CX_3 (relatively lower on the shaft of the gorilla). The muscle CX has to shorten itself CX_3 to 0·4 of its length at CX, but muscle CB is CB_3 only shortened 0·1 of its length.

Shortening of the femur in cursorials is explained by Gregory* as an adaptation which allows of more "open angles of insertion" of hip-extensors and other muscles. As a matter of fact, in ordinary locomotion, the femoral attachments are the fixed points or origins of the muscles; and their effective leverage on the hip-joint will be *lessened* by shortening of the femur. The shortening appears to me, considered only from this point of view, to be a disadvantage. Gregory is in difficulties over explaining the lengthening of the femur in the recent greyhound and racehorse. Perhaps they lengthen to increase the effective leverage.

* Gregory, *Annals of the New York Academy of Sciences*, vol. XXII, p. 291.

What, then, has brought about the shortening of the typical cursorial femur (*e.g.*, Artiodactyla [Gregory])? I think it is to be found in the enormous strain thrown upon the knee extensors in the active animals with long femora. The longer femur when the knee is flexed puts an almost impossible burden on the vasti during a violent thrust from the tibia. In large animals it is inadmissible; the femur shortens. Confirmation of this is found in the enormous size of the vasti in Tarsius (see Table II), a small-sized jumper with long limb-segments. The larger jumpers, the kangaroos, exhibit distinct shortening of the femur. It is better to develop the necessarily powerful hip muscles than to depend on powerful knee muscles; the centre of gravity is, in fact, thus placed so much the higher up the limb.

Considerations of the actual shortening (Fig. 5) undergone by muscles passing to the higher and lower parts of the femur respectively in fully extended positions, would suggest that muscles do not all present the same percentage shortening over the complete range of hip movement. If all shortened by the same fraction of their length they would effect different ranges of hip movement, distally attached muscles causing extensive angular movements, proximally attached muscles small ones. Comparative study of the internal structure of the Great Adductor will, it is hoped, prove enlightening.

Animal Oecology in Deserts. By P. A. Buxton, M.A., Fellow of Trinity College, Cambridge.

[*Read* 7 March, 1921.]

The following notes are the outcome of a somewhat extended sojourn in Mesopotamia and N.W. Persia during the war. Conditions were very unfavourable for consistent investigations. I take this opportunity of publishing my unfinished observations, in the hope that they will furnish raw material for others. Aspects of the subject which are already well known have been entirely omitted from this short paper in which comprehensiveness is not aimed at.

Climatic conditions.

It is of course well known that deserts are dry, and many of them hot as well. It must be clearly recognized that well-nigh all the hottest places in the world are in desert or semi-desert country, and that cold is almost as characteristic of deserts as is heat. Wind too must be considered as a factor with which the desert fauna has to contend. At Menjil in N.W. Persia the wind sweeps through a pass in the Elburz mountains with such velocity that one can barely stand against it. This wind blows with great regularity from about 9 a.m. till sundown throughout the warm weather. It is caused by the daily heating of the N. Persian plateau under the sun's rays; the heated air rises and is replaced by an inrush from the sides: at Menjil, this inrush is concentrated in a rocky defile. Such a terrific but steady wind must be an important spreader of small organisms. A large camp at Ruz in the valley of the Diyala in E. Mesopotamia was smitten by an exaggerated dust-devil early in November, 1918. The wind carved a lane through the camp ripping every tent that lay in its path: heavy articles of kit were blown through the air and deposited on the opposite bank of the Ruz canal: an officer of my acquaintance was pulled out of his tent with all his camp furniture and dropped twice, with such violence that three of his ribs were broken. Such winds are rare, but in most desert countries small dust-devils are common; they wander about the desert in a somewhat aimless manner, sweep débris, bushes, etc. from the ground, and drop them later in a different place. The fact that so many desert animals live beneath rocks and excavate burrows is possibly to be explained as an attempt to reach an equable temperature, and to avoid wind and dust: the generally accepted explanation is, I believe, that the

scanty vegetation affords no cover and necessitates an underground life. Some of the meteorological conditions of deserts have received, I believe, no attention so far as their effect on the fauna is concerned. The extremely low relative humidity, the sharp spell of heat by day and of cold by night, the great heat of the stones and soil on which some insects crouch and the effect of direct sunlight on animals are among the problems which call for study. We know indeed that they must have their influence on the desert fauna, but we have no accurate knowledge of what that influence is. Much might be learnt by exposing animals to one or other of these conditions, using an adequate number of controls.

Colouration of animals.

We have all of us known from childhood that in the desert the animals, the birds and the insects are coloured like the desert: any desert fauna furnishes countless examples of this. There are, however, a few very disturbing facts, which, I believe, are not widely known. The Gerbilles (*Gerbillus*, *Meriones*, *Dipodillus*, *Tatera*, etc.) and the Jerboas (*Jaculus*, *Alactaga*) all appear to be perfect examples of protective coloration, but they are strictly nocturnal. This is also true, I believe, of the Cape Jumping Hare (*Pedetes*) and the jumping mice of the deserts of southern North America (*Perodipus*, *Zapus*) and the Spiny Mice (*Acomys*) of Sinai, etc. It is difficult to explain the coloration of these animals unless we suppose that it is of some protective value by moonlight. One does not know from what foe the Gerbilles need to escape, unless it is foxes: certainly in Mesopotamia and Persia owls were almost non-existent in the desert, except the resident Little Owl (*Athene noctua* subsp.), and in winter the Short-eared Owl; both of these owls feed almost entirely by day. As I have said, the Gerbilles and Jerboas are strictly nocturnal, not appearing at dusk or at dawn: at Qazvin in N.W. Persia I lived for four months close to colonies of *Meriones erythrourus* Gray and *M. blackleri lycaon* Thos. Every night I was out in the desert between sundown and 9 p.m., and I was often about just when the dawn was breaking, but I never once saw a Gerbille except in the hours of darkness, when they were extremely abundant, crossing the light cast by my lantern. Among the birds the Cream-coloured Courser (*Cursorius gallicus*) is efficiently protected by its colour and disruptive pattern so long as it crouches; but when it runs the long legs raise the body from the ground and cause it to cast an extremely conspicuous black shadow. The See See Partridge (*Ammoperdix griseogularis*) [*bonhami*] is another example of imperfect protection by coloration: the bird is crepuscular, hiding by day in holes under rocks. In the evening its movements are betrayed by the long shadow it casts as the suns rays decline, and this in spite of its short legs. It is a curious fact that most

desert birds are protectively coloured as chicks and as adults, but their eggs are in no way specially coloured and in general resemble the eggs of related birds which do not breed in deserts.

It appears to have been overlooked that black animals form a definite element in the fauna of the great palaearctic desert belt which stretches from Marocco to the Gobi. Examples are a number of the Wheatears (*Saxicola melanoleuca*, *S. lugens*, *S. leucopyga*, *S. monacha*, *S. morio*, etc., all of them predominantly black, with a greater or lesser amount of white). One might also mention the ravens *Corvus umbrinus* and *C. rhipidurus* (*affinis*). Among the insects the coleopterous family Tenebrionidae is characteristic of deserts in many parts of the world. Black tenebrionids, belonging to no less than five sub-families (Erodiinae, Zophosinae, Tentyrinae, Adesmiinae and Pimeliinae) are a very conspicuous feature of the deserts of the palaearctic region from Marocco eastwards to Turkestan, and to Sind. The great majority of these black forms are diurnal: in these same deserts there occur a number of Tenebrionidae which are not black: the majority of these are buff, or grey, or brown, and these species are mainly nocturnal. Other sub-families of Tenebrionidae occur in the deserts of Australia and America, and many of these insects are black, but I do not know whether they are diurnal. In the Orthoptera there is an example in the Phasgonuridae (Locustidae): in Algeria I took that remarkable insect *Eugaster guyoni*; this is a large stout locust, black and highly polished, with some red prominences on the thorax. It is unable to leap, for its hind legs are barely stronger than its forelegs, an unusual condition in this family. This species also is diurnal and of course extremely conspicuous: it is probably protected by its copious secretion of blood, or at least one is tempted to suppose that this is the case. It is probably the case that none of these black animals are preyed upon by larger animals: wheatears are extremely wary birds, which perch on the very summit of some upstanding rock and are always ready to dive under a stone at the approach of danger: ravens, so far as one knows, act more often on the offensive than the defensive: the tenebrionids are covered with an intensely hard exoskeleton, and are probably often attacked by birds or lizards without suffering harm: I say this because I have frequently taken specimens of *Adesmia* and *Pimelia* with several legs or antennae missing, or with dinted but unpierced elytrae. Granted that the black creatures are in some way protected, and therefore not in need of protective coloration, we do not in the least know why they are black, a colour which must render them extremely hot in the desert while the sun shines. To sum up, of course I admit that the majority of animals which live in deserts are coloured protectively, but find the protection much less efficient than I had supposed. There is

a small element in the fauna which is not protectively coloured, and these animals are all black.

Concentration.

One is justified in saying that concentration in point of time and of place is a characteristic of desert animals. The rain falls, the plants blossom, the animals appear and breed and are no more seen. All that is most apparent in the life of the desert is concentrated in a couple of months. Concentration in place is just as noticeable. The great leaf-bases of the date palms in Lower Mesopotamia give cover, especially after rain, to a host of animals, grass-hoppers, beetles, bugs, ants, termites, centipedes, millipedes, woodlice, scorpions, spiders, snails, oligochaets, lizards and others. In deserts generally plants do not cover the ground and every bush or patch of scrub is an oasis in itself, full of specialized forms of animal life. In some places in the Algerian deserts for instance, nearly every large stone shelters a collection of insects, myriapods, arachnids, isopods, and often also lizards, snakes, and even small mammals and birds. The hordes of migratory birds which suddenly appear and pack every bush in an oasis are yet another example of the concentration of life in the desert.

Winter and summer.

In the palaearctic desert region the butterflies for the most part are on the wing in spring, and a few appear also in the autumn. In cold weather they are not seen and presumably they pass the winter as they do in more northerly climates: that is to say some of them as eggs, some as hibernating larvae, some as pupae. In the hot weather also the majority of species are not in the imago stage, and we do not yet know what they are doing. One might suppose that the rise of temperature would cause the stages to be passed through more and more rapidly, but this is not the case. In Lower Mesopotamia (at Amara) *Colias croceus* (*edusa*) is on the wing from March to May, and in November and December; *Pieris rapae* February to May and October to December, and as a great rarity in summer. Both these species certainly produce a succession of broods in spring, and probably in autumn also. They are in fact continuous-brooded while the temperature is neither very hot nor very cold. The three lycaenids, *Zizera lysimmon*, *Tarucus mediterraneae* and *T. balcanicus*, have, I think, similar periods of emergence, but I have not a sufficient number of records for each species to enable me to state definitely that this is the case. We do not know in what stage these five species pass the summer, nor what is the factor which retards their development during the hot season. It is not lack of food for the pabula of *C. croceus* and *P. rapae* are

grown as crops and irrigated all through the summer, and the Zizyphus, the food-plant of the two Taruci, is evergreen. Possibly development is retarded or inhibited when the relative humidity falls below a certain figure. The lycaenid *Chilades galba* is in quite a different biological group: at Amara it is on the wing from June to August abundantly, and rarely during early September; I have examined 80 specimens, the earliest taken 8 June. It appeared that a large number of pyralid moths were only on the wing in the heat of summer in Mesopotamia.

Belenois mesentina is a vigorous species able to maintain itself in deserts, but found in other terrains also. Grosvenor informs me that it breeds in great numbers continuously through the summer at Tank and Bojikhel, Waziristan; it is the only butterfly on the wing at that time of year. The food-plant (Capparis) is green and succulent all the summer. Possibly the butterfly's ability to continue breeding through a period of great heat is one of the factors which make it an abundant species over a very wide extent of country.

Venational Abnormalities in the Diptera. By C. G. Lamb, M.A.

[*Read* 7 March 1921.]

The published records on this subject are very scanty, only two or three notes having appeared for many years past. The most remarkable case on record is described and figured by F. W. Edwards (*Entomologist's Monthly Magazine*, 1914, p. 59). It is that of a tachydromid in which the simple venation natural to that genus suddenly flourished out into a highly complex and irregular network of cross- and accessory-veins, which show nearly every form of abnormality that the wing can be afflicted with; this physiological explosion remains unique. Dr Keilin records a case of true malformation, in which the distal part of both wings is greatly abbreviated and deformed, in the *Bull. Soc. Ent. France*, 1917, p. 194, and suggests that the condition has been brought about by pressure. The abnormalities which are now referred to are usually smaller and more regular, and are scarcely striking enough to be called teratological, but as there is apparently no definite boundary between the two sets of cases, that term will be used for convenience.

The only recently published matter bearing on the abnormalities to be considered is contained in three short papers by Kröber published in the *Zeitschrift für Insekten Biologie* in 1910, where several cases similar to the following were figured, these will be referred to in passing. Apparently the above include all that has been published on the subject of wing teratology though there are several scattered notes on antennal teratologies.

Abnormal venation is naturally more probable in those flies which have an approach to the generalized Panorpa-like venation, both from the greater number of the veins and from the presence of two possible stress systems at right angles, and consequently it is not surprising that Mr F. W. Edwards of the British Museum, who has had an exceptionally intimate knowledge of the Nematocera, informs me that he has frequently seen aberrations in the venation of that section, and that they are particularly common in the tipulid genera Eriocera and Tricyphona. On the other hand Mr J. E. Collin considers flies to be remarkably free from wing abnormalities, and that they occur only in certain species such as the one to be referred to later on. The author's experience is practically confined to the non-nematocerous families and agrees with that of Mr Collin. Neglecting one exceptional species, among the many thousands of individuals that have passed before him, there

were less than 20 cases of teratological conditions, most of which are here figured. The families the author has principally dealt with are all characterised by simple venation, roughly of the syrphid or muscid type, in which the cross-veins are very few in number.

These teratological conditions are either those of deficit or of excess. The first is shown in Figs. 1 and 2; the former is the tip of the wing of *Platypeza dorsalis* showing the absence of the lower part of the fork to the fourth vein; such a break is sometimes definitely specific as in the genus Cryptophleps (Dolichopids), and on the other hand the presence or absence of such a fork may be a matter of indifference; thus in the tachinid *Rhacodineura antiqua*, the upward bend so characteristic of the family is normally absent or represented only by a small fragment of the upper portion: very rarely, as in one specimen in the Cambridge series, it is fully developed on both sides. Mr Colbran Wainwright drew the author's attention to the fact that this bend is also normally absent in the tachinids *Actia frontalis* and *Phytomyptera nitidiventris*. Fig. 2 shows the wing of *Oscinis nana*, in which the second vein definitely falls short of the costa, a very rare condition in flies. Such a non-attainment of the wing margin is not uncommon for the parts other than the costa, and may even be such a constant character as to be helpful in determination of the family. Teratology by deficit is however rare except as a definite character. In the above cases it is normally bilateral, and one of the remarkable things is that bilateral occurrence is quite usual and occurs in the next series of cases, especially in those shown in Figs. 9 to 12.

Teratology by excess takes three principal forms; the commonest is that of little hang-veins ("anhangs") which occur in many places, for example on the second long vein as in Fig. 10 or on the discal cell as in Fig. 3 which shows one on the discal cell of a specimen of *Ocydromia glabricula* (Empidae) in which family the only other recorded case of abnormal veins is that of the Tachydromia mentioned above. These hang-veins are often present normally as for example in *Ernoneura argus* (Cordylurids) where the second vein has a regular row of them hanging from it, each carrying a spot of pigment. It is also always present in some species of Parhydra (Ephydridae) at the end of the second vein, where it forms a little hook which is often quite constant for a given species, though in others it is rather variable, or, in rare instances, absent. The size of the hang-veins differs greatly; in Fig. 4 is shown one in an agromyzid which is doubled at the base, though simple on the other wing; here it is long; that in Fig. 3 is smaller, while those in Fig. 14, the case to be referred to later of *Ptilonota guttata* (Ortalidae), show various forms; the limit is when the hang-vein degenerates into a little hard spot on the vein, a sort of local breakdown in its continuity.

The second class consists in an extra cross-vein between the third and the fourth long veins as shown in Fig. 5, *Sapromyza pallidicornis* (Sapromyzidae), Fig. 6, *Minettia plumicornis* (Sapromyzidae) and Fig. 7, *Ditaenia grisescens* (Sciomyzidae). The only known case in which the cross-vein at the end of the discal cell is duplicated is shown in Fig. 8, *Orthochile nigrocerulea* (Dolichopodidae).

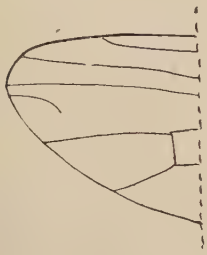


Fig. 1. *Platypeza dorsalis.*

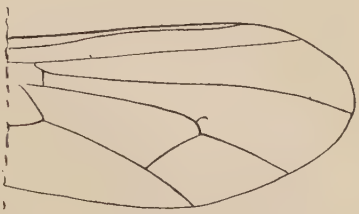


Fig. 3. *Ocydromia glabricula.*

Fig. 2. *Oscinis nana.*

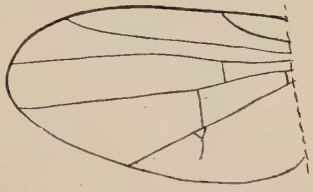


Fig. 4. *Agromyza* sp.

Fig. 5. *Sapromyza pallidicornis.*

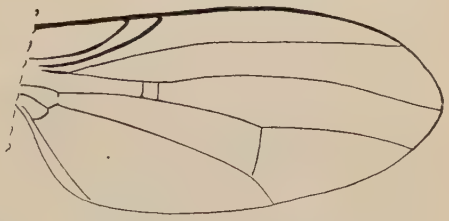


Fig. 6. *Sapromyza plumicornis.*

Fig. 8. *Orthochile nigrocerulea.*

Fig. 7. *Ditaenia grisescens.*

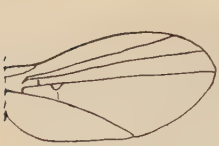


Fig. 9. *Agromyza* sp.

Fig. 10. *Cyrtoneura stabulans.*

Fig. 11. *Sapromyza dedecor.*

Fig. 12. *Noterophila glabra.*

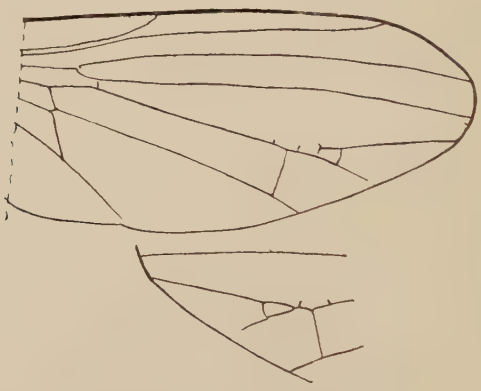


Fig. 13. *Platypeza modesta.*

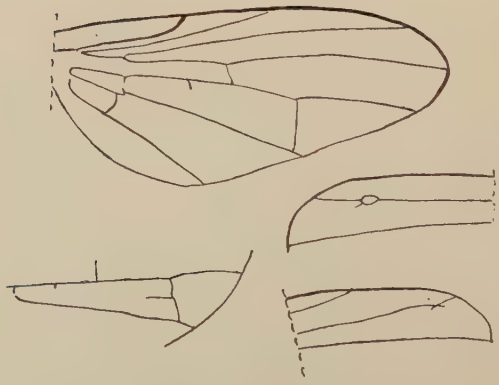


Fig. 14. *Ptilonota guttata.*

The third class is the most interesting and is shown in Fig. 9, a species of Agromyza, Fig. 10 *Cyrtoneura stabulans* (Muscidae), Fig. 11, *Sapromyza dedecor* (Sapromyzidae), and Fig. 12, *Noterophila glabra* (Drosophilidae). Kröber also figures two similar cases, in *Mydaea urbana* and *Polietes lardaria*, both in the Anthomyiidae. The small triangle formed looks just as if the cross-vein in its development had torn the fourth vein into its two constituent parts M_1 and M_2: this condition is often bilateral.

A similar formation of a triangle, but taking the form of a cross-vein between the two branches M_1 and M_2 beyond the discal cell, is shown in Fig. 13, *Platypeza modesta* (Platypezidae), this is accompanied in both wings by small hang veins. A noticeable feature is that the teratologies very frequently occur in the neighbourhood of the discal cell, that is to say, in that part of the wing where one would expect stress phenomena to occur owing to the presence of veins at right angles.

The above examples constitute almost all those that the author has met with, and they occur in single specimens of various species spread over many families: in some of those species hundreds of specimens have been seen. This remarkable degree of uniformity breaks down in a species of ortalid, *Ptilonota guttata* (Fig. 14). Here the teratological diathesis is extraordinary, taking as a rule the form of hang-veins, which vary from dots near the second vein to simple veinlets or complex forms as shown in the figure: these hang-veins are also common in the discal cell itself. Out of the 40 specimens in the Cambridge collection no fewer than 15 exhibit one at least of these abnormalities, many of them more than one. Mr Edwards has kindly informed the author that in the British Museum set, about 5 out of 21 show small defects of the nature of hang-veins or dots, and Mr J. E. Collin tells him that 9 out of his 23 specimens exhibit the same phenomenon. The species also shows much inconstancy in the acrostichal bristles; further, the author figured in the *E.M.M.* 1911, p. 216 a unique case of true Batesonian teratology occurring in the same species, the specimen having on one of the antennae a small accessory third joint bearing two extra aristas. The species seems to be endowed with extreme natural instability, and it would be of interest to hear of similar occurrences in other families of insects.

PROCEEDINGS
OF THE
Cambridge Philosophical Society.

The Cooling of a Solid Sphere with a Concentric Core of a Different Material. By Professor H. S. Carslaw.

[*Read* 2 May 1921.]

1. Fourier himself remarked* that the present temperature gradient near the surface might be used to obtain an estimate of the time that has elapsed since the earth began to cool from its molten state. And in a paper† which attracted much attention at the time of its publication, and to which, even towards the end of his life, Kelvin attached considerable weight, he based his estimate of the Age of the Earth upon the simple mathematical problem of the distribution of temperature in a solid bounded by the plane $x = 0$ and extending to infinity in the direction of x positive; the initial temperature is constant and the boundary $x = 0$ is kept at zero.

In this case, with the usual notation, the temperature v at the time t is given by

$$v = \frac{2v_0}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{(\kappa t)}}} e^{-\alpha^2} d\alpha,$$

and

$$\frac{\partial v}{\partial x} = \frac{v_0}{\sqrt{(\pi\kappa t)}} e^{-\frac{x^2}{4\kappa t}},$$

the initial temperature being v_0.

In 1895 Perry reopened the question in a series of papers‡ the aim of which was to show that other possible internal conditions would give greater ages than Kelvin's estimate of 10^8 years, which was regarded by the geologists as quite inadequate. Heaviside§ made important contributions to this discussion and the problem solved in this paper is one to which he refers. He mentions that he had obtained its solution by his "operational method," but his work has not been published.

* Cf. *Œuvres de Fourier* (Darboux's Edition), T. 2, p. 284.
† *Trans. R. Soc., Edinburgh*, 23, p. 157, 1864.
‡ *Nature*, 51, pp. 224, 341 and 582, 1895.
§ Heaviside, *Electromagnetic Theory*, vol. 2, ch. v. 1899.

Though the discovery of radioactivity has definitely closed the controversy as to the reliability or otherwise of the results obtained by Kelvin's method, or similar methods, the mathematical problem treated in this paper seems of sufficient interest to justify the publication of a solution on lines which I have found useful in dealing with other questions of the conduction of heat.

2. It is more convenient to start with the case of a sphere whose surface is kept at a constant temperature, the initial temperature of the whole being zero.

Let the sphere be of radius b, the core being of radius a. The surface $r = b$ is kept at a constant temperature v_0.

Let the temperature, conductivity, specific heat and density of the core from $r = 0$ to $r = a$ be v_1, K_1, c_1 and ρ_1: and the corresponding quantities from $r = a$ to $r = b$ be v_2, K_2, c_2 and ρ_2.

Also let $\kappa_1 = K_1/c_1\rho_1$ and $\kappa_2 = K_2/c_2\rho_2$.

Then if we write $u_1 = v_1 r$ and $u_2 = v_2 r$, we have the equations:

$$\frac{\partial u_1}{\partial t} = \kappa_1 \frac{\partial^2 u_1}{\partial r^2}, \quad 0 < r < a \ldots(1), \qquad \frac{\partial u_2}{\partial t} = \kappa_2 \frac{\partial^2 u_2}{\partial r^2}, \quad a < r < b \ldots(1'),$$

$$u_1 = 0, \text{ when } r = 0 \ldots\ldots(2), \qquad u_2 = bv_0, \text{ when } r = b \ \ldots(2'),$$

$$u_1 = 0, \text{ when } t = 0 \ldots\ldots(3), \qquad u_2 = 0, \text{ when } t = 0 \ \ldots(3'),$$

$$u_1 = u_2, \text{ when } r = a \ \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$K_1\left(a\frac{\partial u_1}{\partial r} - u_1\right) = K_2\left(a\frac{\partial u_2}{\partial r} - u_2\right), \text{ when } r = a \ldots(5).$$

It is clear that

$$u_1 = A_1 \sin \alpha r \, e^{-\kappa_1 \alpha^2 t},$$

$$u_2 = (A_2 \sin \mu\alpha\,(r - a) + B_2 \sin \mu\alpha\,(b - r))\, e^{-\kappa_1 \alpha^2 t}$$

satisfy (1) and (1′), when $\mu = \sqrt{(\kappa_1/\kappa_2)}$.

They also satisfy (4) and (5), provided that

$$A_1 \sin \alpha a = B_2 \sin \mu\alpha\,(b - a),$$

$$K_1 A_1 [\alpha a \cos \alpha a - \sin \alpha a] = K_2 [\mu\alpha a\,(A_2 - B_2 \cos \mu\alpha\,(b - a)) - B_2 \sin \mu\alpha\,(b - a)].$$

Thus we have

$$A_2 = \frac{\sigma \cos \alpha a \sin \mu\alpha\,(b-a) + \sin \alpha a \cos \mu\alpha\,(b-a) + \dfrac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha\,(b-a)}{\sin \mu\alpha\,(b-a)} A_1,$$

$$B_2 = \frac{\sin \alpha a}{\sin \mu\alpha\,(b-a)} A_1,$$

where $$K_1 = \mu\sigma K_2.$$

Now introduce the path (P) of Fig. 1*. In this standard path the argument of α lies between 0 and $\frac{\pi}{4}$ at infinity on the right, and between $\frac{3\pi}{4}$ and π on the left. The condition (2′) at $r = b$ suggests the suitable value for A_1. Then we are led to the solutions

$$u_1 = \frac{bv_0}{\iota\pi}\int \frac{\sin \alpha r}{F(\alpha)} \frac{e^{-\kappa_1\alpha^2 t}}{\alpha} d\alpha \quad \text{.....(6),}$$

$$u_2 = \frac{bv_0}{\iota\pi}\int \left\{ \frac{\sin \mu\alpha (r-a)}{\sin \mu\alpha (b-a)} + \frac{\sin \alpha a \sin \mu\alpha (b-r)}{F(\alpha) \sin \mu\alpha (b-a)} \right\} \frac{e^{-\kappa_1\alpha^2 t}}{\alpha} d\alpha \text{...(7),}$$

where

$$F(\alpha) = \sigma \cos \alpha a \sin \mu\alpha (b-a) + \sin \alpha a \cos \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha (b-a),$$

and the integrals are taken over the path (P) of Fig. 1 in the α-plane.

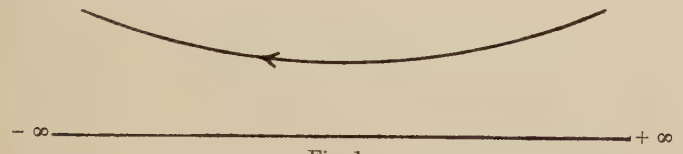


Fig. 1.

The value of u_2 given by (7) reduces to

$$u_2 = \frac{bv_0}{\iota\pi}\int \frac{G(\alpha)}{F(\alpha)} \frac{e^{-\kappa_1\alpha^2 t}}{\alpha} d\alpha \quad \text{.....(8),}$$

where

$$G(\alpha) = \sigma \cos \alpha a \sin \mu\alpha (r-a) + \sin \alpha a \cos \mu\alpha (r-a) + \frac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha (r-a),$$

and the integrals are taken over the path (P).

The values of u_1 and u_2 given in (6) and (8), from the way in which they have been obtained, satisfy the differential equations (1) and (1′), and the conditions (2), (4) and (5), which hold when $r = 0$ and $r = a$.

Putting $r = b$ in (8), we have $\frac{bv_0}{\iota\pi}\int \frac{e^{-\kappa_1\alpha^2 t}}{\alpha} d\alpha$, over the path (P). Introduce the path (Q) of Fig. 2 formed by the path (P), the image of this path in the real axis, and the circular arcs dotted in the

* A similar path was used by me in Chapter XVIII of my book on *Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat*, 1906. See also, for the method of this paper, *Phil. Mag., London* (Ser. 6), **39**, p. 603, 1920.

diagram, joining the ends of these two curves. Then use Cauchy's Theorem and we see that $\dot{u}_2 = bv_0$ when $r = b$.

Thus the condition (2′) is satisfied.

There remains only the initial condition, u_1 and u_2 are to vanish when $t = 0$.

Now the equation

$$F(\alpha) \equiv \sigma \cos \alpha a \sin \mu\alpha (b-a) + \sin \alpha a \cos \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha (b-a) = 0 \quad \ldots(9)$$

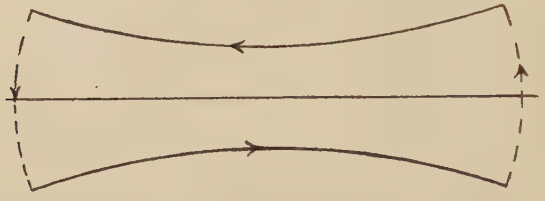


Fig. 2.

has no imaginary roots, or repeated roots, and it has an infinite number of real roots symmetrically placed with regard to the origin*.

Putting $t = 0$ in (6) and (8), we have the integrals

$$\frac{bv_0}{\iota\pi} \int \frac{\sin \alpha r}{F(\alpha)} \frac{d\alpha}{\alpha} \quad \text{and} \quad \frac{bv_0}{\iota\pi} \int \frac{G(\alpha)}{F(\alpha)} \frac{d\alpha}{\alpha}.$$

Consider the closed circuit of Fig. 3, consisting of the path (P) and the part of a circle, centre at the origin, lying above the path (P). There are no poles of these integrands within this circuit and, when the radius of the circle tends to infinity, the integral over the circular arc vanishes.

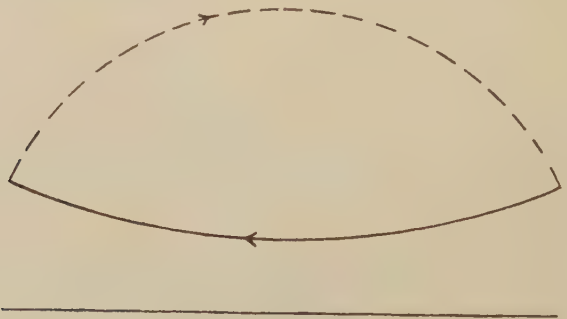


Fig. 3.

It follows that both integrals vanish over the path (P) and the initial conditions (3) and (3′) are satisfied.

* See below § 5.

Finally the solutions in (6) and (8) are obtained as infinite series. For we have

$$u_1 = \frac{bv_0}{2\iota\pi}\int \frac{\sin \alpha r}{F(\alpha)}\frac{e^{-\kappa_1\alpha^2 t}}{\alpha}\,d\alpha,$$

and

$$u_2 = \frac{bv_0}{2\iota\pi}\int \frac{G(\alpha)}{F(\alpha)}\frac{e^{-\kappa_1\alpha^2 t}}{\alpha}\,d\alpha,$$

over the path (Q) of Fig. 2.

Then, by Cauchy's Theorem, we have

$$u_1 = rv_0 + 2bv_0 \sum_1^\infty \frac{\sin \alpha_n r}{F'(\alpha_n)}\frac{e^{-\kappa_1\alpha_n^2 t}}{\alpha_n} \quad \text{.....(10),}$$

$$rv_0 + 2bv_0 \sum_1^\infty \frac{\sigma\cos\alpha_n a \sin\mu\alpha_n(r-a) + \sin\alpha_n a\cos\mu\alpha_n(r-a) + \dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin\alpha_n a\sin\mu\alpha_n(r-a)}{F'(\alpha_n)}\frac{e^{-\kappa_1\alpha_n^2 t}}{\alpha_n}$$

.....(11),

the summation being taken over the positive roots of (9).

3. In the discussion of § 2, put $V_1 = v_0 - v_1$ and $V_2 = v_0 - v_2$. Then V_1 and V_2 are the temperatures in the core and surrounding sphere, when the surface $r = b$ is kept at temperature zero and the initial temperature of the whole is v_0.

Also the gradient, when $r = b$, namely $\dfrac{\partial V_2}{\partial r}$, is given by the equation

$$\left(\frac{\partial v}{\partial r}\right)_{r=b} = -2\mu v_0 \sum_1^\infty \frac{\sigma\cos\alpha_n a\cos\mu\alpha_n(b-a) + \dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin\alpha_n a\cos\mu\alpha_n(b-a) - \sin\alpha_n a\sin\mu\alpha_n(b-a)}{F'(\alpha_n)}e^{-\kappa_1\alpha_n^2 t}$$

.....(12),

the summation extending over the positive roots of (9).

4. With the constants which Perry and Heaviside adopted,

$$a = 6{\cdot}38 \times 10^8, \quad b - a = 4 \times 10^5, \quad v_0 = 4 \times 10^3,$$

$$K_1 = {\cdot}47, \qquad K_2 = {\cdot}00595,$$

$$c_1\rho_1 = 2{\cdot}86, \qquad c_2\rho_2 = {\cdot}507,$$

$$\kappa_1 = K_1/c_1\rho_1 = {\cdot}1643, \qquad \kappa_2 = K_2/c_2\rho_2 = {\cdot}0117^*,$$

$$\mu = \sqrt{(\kappa_1/\kappa_2)} = 3{\cdot}742, \qquad \sigma = \sqrt{(K_1c_1\rho_1/K_2c_2\rho_2)} = 21{\cdot}1.$$

Thus $\mu\sigma = 79$ and $\mu\dfrac{(b-a)}{a} = 2{\cdot}35 \times 10^{-3}$.

Also the gradient of 1° in 50 ft. is 1/2743 degrees per cm.

* This corresponds to Kelvin's value of 400 for κ in foot-year units (*loc. cit.* § 15 and *Mathematical and Physical Papers*, vol. 3, p. 302).

The equation (9) is

$$F(\alpha) \equiv \sigma \cos \alpha a \sin \mu\alpha (b-a) + \sin \alpha a \cos \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha (b-a) = 0.$$

The roots of this equation will be the common roots, if any, of

$$\left.\begin{aligned} \sin \alpha a &= 0 \\ \sin \mu\alpha (b-a) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

and the roots of

$$\sigma \cot \alpha a + \cot \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} = 0 \quad \ldots\ldots (14).$$

Since $(b-a)/a$ is small, the values of $a\alpha$, if any, given by (13) will be large. Thus for our solution we require only the earlier roots of (14), which, with the above constants, reduces to

$$21{\cdot}1 \cot x + \cot (2{\cdot}35 \times 10^{-3} x) = \frac{20{\cdot}8}{x} \quad \ldots\ldots\ldots (15),$$

where $x = \alpha a$.

It will be found* that the first two roots of (15) are

$$x_1 = 2{\cdot}9871 \text{ or } 180^\circ - 8^\circ\, 51',$$

$$x_2 = 5{\cdot}980 \text{ or } 360^\circ - 17^\circ\, 22',$$

and that x_n lies between $n\pi - \frac{1}{2}\pi$ and $n\pi$.

Taking the first term only, the value of t is required for which

$$\frac{1}{2743} = -2\mu v_0 \left(\frac{\sigma \cot \alpha_1 a \cot \mu\alpha_1 (b-a) + \frac{1-\mu\sigma}{\mu\alpha_1 a} \cot \mu\alpha_1 (b-a) - 1}{\sigma a \operatorname{cosec}^2 \alpha_1 a + \mu (b-a) \operatorname{cosec}^2 \mu\alpha_1 (b-a) + \frac{1-\mu\sigma}{\mu\alpha_1^2 a}} \right) e^{-\kappa_1 \alpha_1^2 t} \quad \ldots (16).$$

This gives the equation

$$\frac{1}{2743} = 8 \times 3{\cdot}742 \times 10^{-4} \times \frac{2{\cdot}0298}{5{\cdot}9763} \times e^{-{}^{\cdot}1643 \times \left(\frac{2{\cdot}9871}{6{\cdot}38 \times 10^8}\right)^2 t},$$

since the numerator and denominator of the fraction on the right of (16) are, respectively, $-2{\cdot}0298 \times 10^4$ and $5{\cdot}9763 \times 10^{11}$.

Reducing the answer to years, this leads to $9{\cdot}02 \times 10^9$ years†.

* I am indebted to Mr R. J. Lyons for the solution of this equation.

† Heaviside (*loc. cit.* p. 19) gives $9{\cdot}03 \times 10^9$ but adds that he has not taken special pains to get the third figure right.

As Heaviside compares his result with Perry's for the problem when the capacity of the skin is neglected, it may be worth while to point out that some arithmetical errors have crept into Perry's solution. (Cf. *Nature*, **51**, p. 225.)

The first two roots of the equation $19 \tan \alpha + \alpha = 0$ are

$$\alpha_1 = 2{\cdot}985676 \text{ or } 180^\circ - 8^\circ\, 56',$$

$$\alpha_2 = 5{\cdot}9783345 \text{ or } 360^\circ - 17^\circ\, 28'.$$

The first term in the series for the temperature is 138·13 instead of 142·7, and the second term is 4·82 instead of 5·65 given by Perry.

5. In § 2 it has been assumed that the equation (9) has no imaginary roots or repeated roots, and that it has an infinite number of real roots, symmetrically placed with regard to the origin.

There is no difficulty at all in proving that the equation has no pure imaginary roots, and that its real roots are not repeated; and, since $F(\alpha)$ is an odd function of α, to every positive root α, there corresponds a negative root $-\alpha$.

It is harder to prove that there is no root of the form $\xi \pm \iota\eta$, where ξ and η do not vanish. To prove this, we need to show that the equation

$$\sigma \cot \alpha a + \cot \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} = 0$$

has no root of this form, when μ, σ, a and b are real and positive and $b > a$.

Let $U_1 = \sin \alpha r, \ 0 < r < a$

and $U_2 = \dfrac{\sin \mu\alpha (b-r)}{\sin \mu\alpha (b-a)} \sin \alpha a, \ a < r < b$ .....(17).

Then $\dfrac{d^2U_1}{dr^2} + \alpha^2 U_1 = 0$ and $\dfrac{d^2U_2}{dr^2} + \mu^2\alpha^2 U_2 = 0$ .....(18).

Also

$U_1 = 0$, when $r = 0$: $U_1 = U_2$, when $r = a$: $U_2 = 0$, when $r = b$ .....(19).

Further, if α is a root of our equation (14),

$$a\frac{dU_2}{dr} - U_2 = \mu\sigma\left(a\frac{dU_1}{dr} - U_1\right), \text{ when } r = a \quad ...(20).$$

Now let α, β be two roots of equation (14).

Also let U_1, U_2 be as above, and let V_1, V_2 be the corresponding expressions, when β is substituted for α.

Then

$$(\alpha^2 - \beta^2)\left(\int_0^a U_1V_1\,dr + \frac{\mu}{\sigma}\int_a^b U_2V_2\,dr\right)$$

$$= \int_0^a (U_1V_1'' - V_1U_1'')\,dr + \frac{1}{\mu\sigma}\int_a^b (U_2V_2'' - V_2U_2'')\,dr, \text{ by (18)}$$

$$= \left[U_1V_1' - V_1U_1'\right]_0^a + \frac{1}{\mu\sigma}\left[U_2V_2' - V_2U_2'\right]_a^b$$

$$= (U_1V_1' - V_1U_1') - \frac{1}{\mu\sigma}(U_1V_2' - V_1U_2'), \text{ when } r = a, \text{ by (19)}$$

$= 0$, by (20).

But if α and β are conjugate imaginaries $\xi \pm \iota\eta$, U_1 and V_1 are conjugate imaginaries; also U_2 and V_2 are conjugate.

Thus $\int_0^a U_1 V_1 \, dr$ and $\int_a^b U_2 V_2 \, dr$ are both positive.

But we have just shown that

$$(\alpha^2 - \beta^2)\left(\int_0^a U_1 V_1 dr + \frac{\mu}{\sigma}\int_a^b U_2 V_2 \, dr\right) = 0.$$

If follows that our equation cannot have roots of the form $\xi \pm \iota\eta$, when neither ξ nor η vanish.

[*Added* June 3, 1921].

6. My attention has been called to the fact that the problem with which this paper deals was set in the Mathematical Tripos, Part II, 1904, 4 June, 2–5 p.m. In that paper Question 7 reads as follows:

A solid sphere of conductivity k and diffusivity a^2 and of radius b is enclosed in a spherical shell of conductivity k' and diffusivity a'^2 and of internal and external radii b and c. Initially the whole system is at uniform temperature u_0 and from the epoch $t = 0$ onwards the surface $r = c$ is kept at zero temperature. Prove that at any subsequent time t the temperature at a distance r from the centre is given by equations of the form

$$u = \Sigma_s A_s \frac{\sin \lambda_s r}{\sin \lambda_s b} \frac{1}{r} e^{-\lambda_s^2 a^2 t}, \text{ when } b > r > 0,$$

$$u = \Sigma_s A_s \frac{\sin \lambda_s' (c-r)}{\sin \lambda_s' (c-b)} \frac{1}{r} e^{-\lambda_s^2 a^2 t}, \text{ when } c > r > b,$$

where $\lambda_s' a' = \lambda_s a$, and λ_s is the sth root in order of increasing magnitude of the equation

$$k'a \tan \lambda b + ka' \tan \lambda' (c-b) = 0,$$

further

$$A_s = \frac{2u_0}{\lambda_s^2} \frac{\{(k-k') \sin \lambda_s' (c-b) + k'c\lambda_s'\} \sin^2 \lambda_s b \sin^2 \lambda_s' (c-b)}{kb \sin^2 \lambda_s' (c-b) + (a^2/a'^2) k' (c-b) \sin^2 \lambda_s b}.$$

Explain the bearing of a numerical solution of this problem on the calculation of the age of the Earth.

This solution of our problem is wrong. The equations which the temperatures (u, u') must satisfy at $r = b$ are

$$\left.\begin{aligned} u &= u' \\ k\frac{\partial u}{\partial r} &= k'\frac{\partial u'}{\partial r} \end{aligned}\right\},$$

so that, if we take terms of the type given in the summation, the second of these equations leads to

$$b(k\lambda_s \cot \lambda_s b + k'\lambda_s' \cot \lambda_s' (c-b)) = k - k'.$$

It is clear that the mistake arose from taking

$$k\frac{\partial}{\partial r}(ur) = k'\frac{\partial}{\partial r}(u'r),$$

instead of the proper equation at this surface.

As a matter of fact the equation for λ given in the Tripos question arises in the corresponding problem for linear flow of heat, when the equations for the temperature, with this notation, are as follows:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < b\text{:}\ \frac{\partial u'}{\partial t} = a'^2 \frac{\partial^2 u'}{\partial x^2},\ b < x < c.$$

$$u = 0,\ \text{when } x = 0\text{:}\ u' = 0,\ \text{when } x = c.$$

$$u = u_0,\ \text{when } t = 0\text{:}\ u' = u_0,\ \text{when } t = 0.$$

$$\left.\begin{aligned} u &= u' \\ k\frac{\partial u}{\partial x} &= k'\frac{\partial u'}{\partial x} \end{aligned}\right\},\ \text{when } x = b.$$

And this problem* can be solved by Fourier's method on the same lines as the solution of the sphere problem which is given in the next sections, a solution suggested by this Tripos question.

7. With the notation of § 2, the sphere problem reduces, on substituting $vr = u$, to the solution of the following equations:

$$\frac{\partial u_1}{\partial t} = \kappa_1 \frac{\partial^2 u_1}{\partial r^2},\ 0 < r < a \ldots(21), \qquad \frac{\partial u_2}{\partial t} = \kappa_2 \frac{\partial^2 u_2}{\partial r^2},\ a < r < b \ldots(21'),$$

$$u_1 = 0,\ \text{when } r = 0 \ldots\ldots(22), \qquad u_2 = 0,\ \text{when } r = b \quad \ldots(22'),$$

$$u_1 = v_0 r,\ \text{when } t = 0 \ldots\ldots(23), \qquad u_2 = v_0 r,\ \text{when } t = 0 \quad \ldots(23'),$$

$$u_1 = u_2,\ \text{when } r = a \ldots\ldots\ldots\ldots\ldots\ldots(24),$$

$$K_1\left(a\frac{\partial u_1}{\partial r} - u_1\right) = K_2\left(a\frac{\partial u_2}{\partial r} - u_2\right),\ \text{when } r = a \ldots(25).$$

As before

$$\left.\begin{aligned} u_1 &= \sin \mu\alpha\,(b-a)\sin \alpha r\, e^{-\kappa_1\alpha^2 t} \\ u_2 &= \sin \alpha a \sin \mu\alpha\,(b-r)\, e^{-\kappa_1\alpha^2 t} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(26)$$

satisfy (21), (21′), (22), (22′) and (24), provided that $\mu^2 = \kappa_1/\kappa_2$.

Further (25) is satisfied, if

$$F(\alpha) \equiv \sigma \sin \mu\alpha\,(b-a)\cos \alpha a + \sin \alpha a \cos \mu\alpha\,(b-a) + \frac{1-\mu\sigma}{\mu\alpha a}\sin \alpha a \sin \mu\alpha\,(b-a) = 0 \ldots(27),$$

where $K_1 = K_2\mu\sigma$.

This is the equation in α given in § 2 (9).

Unless $\mu\,(b-a)/a$ is rational, the only roots of $F(\alpha) = 0$ are those of

$$\sigma \cot \alpha a + \cot \mu\alpha\,(b-a) + \frac{1-\mu\sigma}{\mu\alpha a} = 0 \ldots\ldots\ldots(28).$$

* See also §§ 5, 6 of my paper cited above.

In the first place we confine ourselves to the case when $\mu(b-a)/a$ is not rational.

As in § 5, let α, β be two different positive roots of (28).

Also let

$$\left.\begin{aligned} U_1 &= \sin \mu\alpha\,(b-a)\sin \alpha r, \quad 0<r<a \\ U_2 &= \sin \alpha a \sin \mu\alpha\,(b-r), \quad a<r<b \end{aligned}\right\} \ldots\ldots\ldots(29).$$

And let V_1, V_2 be the corresponding expressions when β is substituted for α.

We know from § 5 that

$$\int_0^a U_1 V_1\,dr + \frac{\mu}{\sigma}\int_a^b U_2 V_2\,dr = 0 \quad \ldots\ldots\ldots\ldots(30).$$

Following Fourier's method, our solution is obtained by expanding $v_0 r$ in an infinite series of these terms:

$$\left.\begin{aligned} v_0 r &= \sum_n A_n \sin \mu\alpha_n\,(b-a)\sin \alpha_n r, \quad 0<r<a \\ &= \sum_n A_n \sin \alpha_n a \sin \mu\alpha_n\,(b-r), \quad a<r<b \end{aligned}\right\} \ldots(31).$$

Then with the usual assumptions as to the possibility of this expansion and of term by term integration of the series, we have from (30)

$$A_n\left[\sin^2 \mu\alpha_n\,(b-a)\int_0^a \sin^2 \alpha_n r\,dr + \frac{\mu}{\sigma}\sin^2 \alpha_n a\int_a^b \sin^2 \mu\alpha_n\,(b-r)\,dr\right]$$

$$= v_0\left[\sin \mu\alpha_n\,(b-a)\int_0^a r\sin \alpha_n r\,dr + \frac{\mu}{\sigma}\sin \alpha_n a\int_a^b r\sin \mu\alpha_n\,(b-r)\,dr\right].$$

Evaluating these integrals and using (28), it will be found that

$$A_n = \frac{2bv_0 \sin \alpha_n a}{\alpha_n\left(a\sigma\sin^2 \mu\alpha_n(b-a) + \mu(b-a)\sin^2 \alpha_n a + \dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin^2 \alpha_n a \sin^2 \mu\alpha_n(b-a)\right)} \ldots(32).$$

Also, from (26), it will be seen that the solution of the problem is

$$u_1 = 2bv_0 \Sigma \frac{\sin \alpha_n a \sin \mu\alpha_n\,(b-a)\sin \alpha_n r}{a\sigma\sin^2 \mu\alpha_n(b-a) + \mu(b-a)\sin^2 \alpha_n a + \dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin^2 \alpha_n a \sin^2 \mu\alpha_n(b-a)} \frac{e^{-\kappa_1 \alpha_n^2 t}}{\alpha_n} \ldots(33),$$

$$u_2 = 2bv_0 \Sigma \frac{\sin^2 \alpha_n a \sin \mu\alpha_n\,(b-r)}{a\sigma\sin^2 \mu\alpha_n(b-a) + \mu(b-a)\sin^2 \alpha_n a + \dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin^2 \alpha_n a \sin^2 \mu\alpha_n(b-a)} \frac{e^{-\kappa_1 \alpha_n^2 t}}{\alpha_n} \ldots(34).$$

This agrees with the results given in § 2 (10) and (11), if we remember that u_1 and u_2 now correspond to $(rv_0 - u_1)$ and $(rv_0 - u_2)$ of that section.

8. The case when $\mu\,(b-a)/a$ is rational remains to be discussed. Suppose that it is equal to p/q, a positive fraction in its lowest terms.

Then the equation $F(\alpha) = 0$ [cf. § 7 (27)] is satisfied by

$$a\alpha = q\pi,\ 2q\pi, \text{ etc.}$$

as well as by the roots of (28).

When $a\alpha = sq\pi$, it will be seen that $\mu\alpha\,(b-a) = sp\pi$.

Thus, in addition to the terms of (26),

$$\left.\begin{aligned} u_1 &= \sin\mu\alpha_n\,(b-a)\,\sin\alpha_n r\,e^{-\kappa_1\alpha_n^2 t} \\ u_2 &= \sin\alpha_n a\,\sin\mu\alpha_n\,(b-r)\,e^{-\kappa_1\alpha_n^2 t} \end{aligned}\right\},$$

where α_n is a root of (28), we have

$$\left.\begin{aligned} u_1 &= \mu K_2\cos ps\pi\,\sin qs\pi\,\frac{r}{a}\,e^{-\kappa_1\frac{q^2s^2\pi^2}{a^2}t} \\ u_2 &= -K_1\cos qs\pi\,\sin ps\pi\left(\frac{b-r}{b-a}\right)e^{-\kappa_1\frac{q^2s^2\pi^2}{a^2}t} \end{aligned}\right\} \quad \ldots(35),$$

where s is any positive integer.

The theorem of § 5 [cf. § 7 (30)] applies to all the solutions

$$\left.\begin{aligned} &\sin\mu\alpha_n\,(b-a)\,\sin\alpha_n r \\ &\sin\alpha_n a\,\sin\mu\alpha_n\,(b-r) \end{aligned}\right\},$$

and

$$\left.\begin{aligned} &\mu K_2\cos ps\pi\,\sin qs\pi\,\frac{r}{a} \\ &-K_1\cos qs\pi\,\sin ps\pi\left(\frac{b-r}{b-a}\right) \end{aligned}\right\}.$$

Thus we assume that $v_0 r$ can be expanded in an infinite series with terms of the type

$$\left.\begin{aligned} &A_n\sin\mu\alpha_n\,(b-a)\,\sin\alpha_n r, \quad 0<r<a \\ &A_n\sin\alpha_n a\,\sin\mu\alpha_n\,(b-r), \quad a<r<b \end{aligned}\right\},$$

and

$$\left.\begin{aligned} &A_s\,\mu K_2\cos ps\pi\,\sin qs\pi\,\frac{r}{a}, \quad 0<r<a \\ &-A_s\,K_1\cos qs\pi\,\sin ps\pi\left(\frac{b-r}{b-a}\right), \quad a<r<b \end{aligned}\right\}.$$

The coefficient A_n has been found above in (32). The coefficient A_s is given by

$$\begin{aligned} A_s\Big[&\mu^2K_2^{\,2}\cos^2 ps\pi\int_0^a \sin^2 qs\pi\,\frac{r}{a}\,dr \\ &+\frac{\mu}{\sigma}K_1^{\,2}\cos^2 qs\pi\int_a^b \sin^2 ps\pi\left(\frac{b-r}{b-a}\right)dr\Big] \\ = v_0\Big[&\mu K_2\cos ps\pi\int_0^a r\,\sin qs\pi\,\frac{r}{a}\,dr \\ &-\frac{\mu}{\sigma}K_1\cos qs\pi\int_a^b r\,\sin ps\pi\left(\frac{b-r}{b-a}\right)dr\Big]. \end{aligned}$$

It follows that

$$A_s = -\frac{2bv_0}{ps\pi}\frac{(b-a)\cos qs\pi}{K_2\left(a+\frac{K_1}{K_2}(b-a)\right)}.$$

Thus

$$A_s = -\frac{2bv_0\cos qs\pi}{s\pi\mu K_2(\sigma p+q)}, \text{ since } K_1 = \mu\sigma K_2 \quad \ldots\ldots(36).$$

Also these terms in our solution are as follows:

$$\left.\begin{aligned} &-2bv_0\frac{\cos ps\pi\cos qs\pi\sin qs\pi\frac{r}{a}}{s\pi(\sigma p+q)}e^{-\kappa_1\frac{q^2s^2\pi^2}{a^2}t} \\ \text{and}\quad &2bv_0\frac{\sigma\sin ps\pi\left(\frac{b-r}{b-a}\right)}{s\pi(\sigma p+q)}e^{-\kappa_1\frac{q^2s^2\pi^2}{a^2}t} \end{aligned}\right\} \quad \ldots\ldots(37).$$

On referring to § 2 (10) and (11), it will be seen that this solution agrees with the result obtained in § 2 by the method of contour integrals. And it may be noted that the assumptions involved in Fourier's method were not necessary in the discussion of § 2.

Symbolical methods in the theory of Conduction of Heat. By Dr T. J. I'a. Bromwich, F.R.S.

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In 1914, I communicated a paper "Normal Coordinates in Dynamical Systems" to the London Mathematical Society*; and I explained there (§§ 2, 3, 4) the relation of my methods to the symbolical methods used by Heaviside for various Electrical problems. But I reserved† any general application of the corresponding methods in Conduction of Heat and Diffusion; owing to the pressure of the War and other difficulties I have not had leisure to arrange my results for publication until now.

I have given here only the special points which are suggested in connexion with problems arising out of the question of the "Age of the Earth" as handled by Heaviside‡ and Perry; reference is also made to the paper by Prof. H. S. Carslaw on the same topic, communicated at the same time as this paper.

Not much importance is attached to the estimate made here ($9{\cdot}37 \times 10^9$ years) for the age of the Earth; but the same *data* have been adopted as in the original suggestion of Perry, that the internal conductivity and heat capacity affected the estimate more than the corresponding constants of the skin. It would be easy to estimate new values of κ and k with which the formula of § 2 (for g/v_0) would yield almost any value of t from 10^9 to 10^{10} years§.

The ease of manipulating the constants in the formula of § 2 is in marked contrast to the labour involved in solving the same problem by means of a Fourier-expansion; and mistakes are far more quickly detected in the numerical work. Some comparisons of the work will be found in § 3.

The contents of the paper are as follows:

§ 1. General consideration of a method for solving Conduction of Heat problems.

§ 2. Application of Heaviside's method to the problem of a sphere surrounded by a shell of different material‖.

* *Proc. Lond. Math. Soc.* ser. 2, vol. 15, 1916, p. 401.

† *L.c.* p. 402.

‡ *Electromagnetic Theory*, vol. 2, §§ 227–237.

§ Compare Heaviside's estimates (*l.c.* §§ 232, 236). On radio-active grounds, the most recent estimates appear to be from 3×10^9 to 5×10^9 years (H. N. Russell, *Proc. Roy. Soc.* A, vol. xcix. (1921), p. 84).

‖ This problem appears to have been solved originally by Heaviside himself (see his remarks in *Electromagnetic Theory*, vol. 2, § 230), but his solution was not published. A solution with an erroneous result was set as a question in Part II of the Mathematical Tripos, 1904.

§ 3. Numerical tests of the formulae.

§ 4. Evaluation of certain symbolical expressions required in §§ 2, 3.

§ 1. *General consideration of a method for solving Conduction of Heat problems.*

Using the notation explained in §§ 4, 5 of my paper quoted above, it is evident that the typical solution of a Conduction of Heat problem, with a solid originally at zero temperature, and the surface of the solid maintained at constant temperature v_0, is of the form

$$v = \frac{v_0}{2\pi\iota}\int_{\alpha-\iota\infty}^{\alpha+\iota\infty} e_\lambda{}^t u\,\frac{d\lambda}{\lambda}. \qquad (\alpha > 0)$$

Here u is to reduce to unity at the surface and is to satisfy the differential equation

$$\kappa\Delta_2 u - \lambda u = 0$$

at other points in the solid, where Δ_2 is Laplace's operator and $\kappa = k/c$ is the fundamental constant of the heat-equation in the solid; k is the conductivity and c the heat capacity (per unit *volume*).

To connect this formula with Prof. Carslaw's (*l.c.* § 2) is easy; if we write

$$\lambda = -\kappa\theta^2,$$

u will become U say, where U satisfies the equation

$$\Delta_2 U + \theta^2 U = 0$$

and also reduces to unity at the surface. The integral for v becomes

$$v = \frac{v_0}{\pi\iota}\int e^{-\kappa\theta^2 t}\,U\,\frac{d\theta}{\theta},$$

where the path of integration is given by

$$-\kappa(\xi^2 - \eta^2) = \alpha, \quad \text{or} \quad \eta^2 - \xi^2 = \alpha/\kappa, \quad \text{if} \quad \theta = \xi + \iota\eta.$$

The beginning and end of the path correspond to the two points given by

$$-2\kappa\xi\eta \to -\infty, \quad -2\kappa\xi\eta \to +\infty;$$

thus, choosing the upper half of the rectangular hyperbola

$$\eta^2 - \xi^2 = c/\kappa,$$

the path will start at infinity in the first quadrant and will end at infinity in the second quadrant as sketched.

This path agrees with Carslaw's path (P), and so the integral v is identical with his, when U is written out at length.

To explain the connexion of these complex integrals with Heaviside's symbolical treatment is also easy; Heaviside writes symbolically

$$p = \frac{\partial}{\partial t} = \kappa q^2,$$

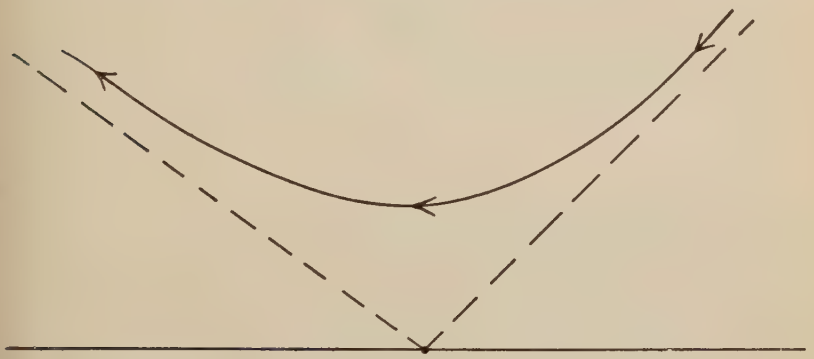
and then solves the equation

$$\Delta_2 V - q^2 V = 0$$

subject to the condition $V = 1$ at the surface.

It is evident that the function V will be equivalent to U on writing $\theta = \iota q$, or $q = -\iota\theta$, and thus Heaviside's forms of the solutions can be translated at once into complex integrals, if desired*; and it has been proved (see § 4 of my L.M.S. paper) that in general Heaviside's standard method of interpreting his symbolical solutions is equivalent to the evaluation of the original complex integral as a sum of residues (taken for all the poles of the function u/λ).

Before leaving these general considerations it will be convenient to note briefly the theorems relating to the special differential equation of Diffusion which follow immediately on the lines of §§ 8, 9 in my paper previously quoted.



The potential energy will be expressible in the form

$$Q = \frac{1}{2}\int k\left\{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\} d\tau$$

and the dissipation-function will be

$$F = \int c\left(\frac{\partial v}{\partial t}\right)^2 d\tau,$$

both integrals being taken through the volume of the solid; and here there is no kinetic energy function.

Thus, if $\lambda = \alpha_1, \alpha_2$ are two distinct poles of the function u, and if ϕ_1, ϕ_2 are the corresponding residues of the function, we find (as in formula (75) in § 9 of my L.M.S. paper)

$$\int c\phi_1\phi_2 d\tau = 0 \quad \text{and} \quad \int k\left(\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial x} + \frac{\partial\phi_1}{\partial y}\frac{\partial\phi_2}{\partial y} + \frac{\partial\phi_1}{\partial z}\frac{\partial\phi_2}{\partial z}\right) d\tau = 0.$$

* Since this relation gives $q = \eta - \iota\xi$, it follows that the path selected above corresponds to one in which *the real part of q is positive*; and this agrees with the convention adopted in § 4 below.

Thus *it follows that the poles* α_1, α_2 *are purely real*; for if α_1, α_2 are supposed to be conjugate complexes, so also are ϕ_1, ϕ_2; and thus the product $c\phi_1\phi_2$ is essentially positive, so that the integral $\int c\phi_1\phi_2\, d\tau$ could not vanish.

Further we have $F = -\dfrac{\partial Q}{\partial t}$, and taking the special value ϕ_1 for v, we find that

$$\alpha_1{}^2\int c\phi_1{}^2 d\tau = -\alpha_1\int k\left\{\left(\frac{\partial\phi_1}{\partial x}\right)^2+\left(\frac{\partial\phi_1}{\partial y}\right)^2+\left(\frac{\partial\phi_1}{\partial z}\right)^2\right\}d\tau,$$

and (since α_1 is not zero) it is evident that α_1 must be *negative**.

The corresponding poles for the functions U will be *real* (since $\kappa\theta^2 = -\lambda$) and will occur in pairs given by $\theta = \pm\sqrt{(-\alpha_1/\kappa)}$.

There is no need for the thermal coefficients c and k to be constants in these theorems, provided that at any surface of discontinuity, the functions v and $k\dfrac{\partial v}{\partial\nu}$ are continuous, where $\partial/\partial\nu$ implies differentiation along the normal to the surface of discontinuity.

It will be noticed that the theorem in § 5 of Prof. Carslaw's paper follows at once, because c has one of two constant values; and the ratio of these constants is equal to

$$\frac{k_2}{\kappa_2}\Big/\frac{k_1}{\kappa_1} = \frac{\kappa_1}{\kappa_2}\Big/\frac{k_1}{k_2} = \frac{\mu^2}{\mu\sigma} = \frac{\mu}{\sigma}.$$

Hence
$$\int_0^a r^2\phi_1\phi_2\, dr + \frac{\mu}{\sigma}\int_a^b r^2\phi_1\phi_2 dr = 0,$$

which is equivalent to the result obtained by Prof. Carslaw.

§ 2. *Application of Heaviside's method.*

We can illustrate Heaviside's method conveniently by solving symbolically the problem of a sphere of radius a and heat-constants k, c, surrounded by a thin shell of thickness $l = b - a$ and heat-constants k_1, c_1. The solid so formed is initially at zero temperature, and at a certain instant ($t = 0$) the outer surface is raised to temperature v_0 and is maintained at this temperature afterwards.

We shall write
$$\kappa = k/c,\quad \kappa_1 = k_1/c_1,$$
and
$$p = \frac{\partial}{\partial t} = \kappa q^2 = \kappa_1 q_1{}^2.$$

Then suppose that V reduces to the value A at $r = a$; the differential equations of the problem reduce to

$$\frac{\partial^2}{\partial r^2}(rV) = q^2(rV),\quad \frac{\partial^2}{\partial r^2}(rV_1) = q_1{}^2(rV_1),$$

* It should be noticed that we cannot draw this conclusion immediately from the corresponding theorem of § 9 of my L.M.S. paper.

which yield the formulae

$$V = \frac{Aa \sinh qr}{r \sinh qa}, \quad V_1 = \frac{Aa \sinh q_1 (b-r)}{r \sinh q_1 l} + \frac{b \sinh q_1 (r-a)}{r \sinh q_1 l},$$

because $V = V_1 = A$ at $r = a$, and $V_1 = 1$ at $r = b$.

Further we are to have

$$k \frac{\partial V}{\partial r} = k_1 \frac{\partial V_1}{\partial r} \text{ at } r = a,$$

and so

$$kA \left(q \coth qa - \frac{1}{a}\right) = -k_1 A \left(q_1 \coth q_1 l + \frac{1}{a}\right) + \frac{k_1 q_1 b}{a \sinh q_1 l},$$

giving* $$A = \frac{q_1 b / \sinh q_1 l}{(k/k_1)(qa \coth qa - 1) + (q_1 a \coth q_1 l + 1)}.$$

If we write $kl/k_1 b = s$, this formula can be written

$$A = \frac{q_1 l / \sinh q_1 l}{s(qa \coth qa - 1) + (a/b)(q_1 l \coth q_1 l + l/a)}.$$

The problem actually amounts to evaluating

$$\frac{g}{v_0} = \frac{\partial V_1}{\partial r} \text{ at } r = b,$$

where g is the gradient of temperature at the outer surface of the shell at time t. It will be found at once that

$$\frac{g}{v_0} = -\frac{Aa}{b} \frac{q_1}{\sinh q_1 l} + q_1 \coth q_1 l - \frac{1}{b}.$$

An expression for g as a Fourier-sum can be found by Heaviside's general process; but as this should be equivalent to the result calculated by Prof. Carslaw, I do not stop to write out the result and proceed to simplify the above formulae by means of approximations suitable to the data of this particular problem. These approximations correspond to (i) treating qa as having a sufficiently large real part to allow $\coth qa$ to be replaced by unity and (ii) treating $q_1 l$ as small.

The above formulae reduce then to the following, on rejecting $(q_1 l)^4$ and higher powers,

$$A = \frac{1 - \frac{1}{6}(q_1 l)^2}{1 - s + sqa + \frac{1}{3}(a/b)(q_1 l)^2},$$

$$\frac{g}{v_0} = -\frac{Aa}{bl}\{1 - \tfrac{1}{6}(q_1 l)^2\} + \frac{1}{l} - \frac{1}{b};$$

* On writing $q = \iota a$, $q_1 = \iota \mu a$, $k/k_1 = \mu\sigma$, it will be found that

$$Aa = b \sin \alpha a / F(\alpha),$$

where $F(\alpha)$ is the function defined in Prof. Carslaw's paper. Then V_1 is easily seen to lead to formula (7) of that paper.

where in the last formula the relation* $q_1^2 = 0$ has been used in simplifying $q_1 \coth q_1 l$.

Hence
$$\frac{g}{v_0} = \frac{a}{bl}\{1 - A + \tfrac{1}{6}A\,(q_1 l)^2\}$$
$$= \frac{a}{bl}\left\{1 - \frac{1 - \frac{1}{3}(q_1 l)^2}{1 - s + sqa + \frac{1}{3}(q_1 l)^2}\right\},$$

where we write 1 for a/b in the small term containing $(q_1 l)^2$ in the denominator.

Write for brevity $\gamma = 1 - s + sqa$ and then
$$\frac{g}{v_0} = \frac{a}{bl}\left\{1 - \frac{1 - \frac{1}{3}(q_1 l)^2}{\gamma} + \frac{\frac{1}{3}(q_1 l)^2}{\gamma^2}\right\}$$
$$= \frac{a}{bl}\left\{1 - \frac{1}{\gamma} + \tfrac{1}{3}(q_1 l)^2\left(\frac{1}{\gamma} + \frac{1}{\gamma^2}\right)\right\}.$$

Following Heaviside, we proceed to expand $1/\gamma$ in powers of $sqa/(1-s)$, and then retaining only terms in q, q^3 (the terms in q^2, q_1^2 being identically zero), we find
$$\frac{g}{v_0} = \frac{a}{bl}\left[-n + \frac{1}{1-s}\left\{nqa + n^3q^3a^3 - \tfrac{1}{3}(q_1 l)^2\left(nqa + \frac{2nqa}{1-s}\right)\right\}\right],$$

where, for brevity, we write $n = s/(1-s)$.

In the actual problem s is small (about $\frac{1}{20}$) and so in the last term of the formula we may put
$$2n/(1-s) = 2n$$
without sensible error, and then the term in l^2 reduces to
$$-\tfrac{1}{3}(q_1 l)^2(3nqa) = -nl^2aq_1^2q.$$

To interpret these formulae we use the results†
$$q = \frac{1}{\sqrt{(\pi\kappa t)}}, \quad q^3 = -\frac{1}{2\kappa t}\frac{1}{\sqrt{(\pi\kappa t)}};$$
then
$$q_1^2 q = \frac{\kappa}{\kappa_1}q^3 = -\frac{1}{2\kappa_1 t}\frac{1}{\sqrt{(\pi\kappa t)}}.$$

Hence we find that
$$\frac{g}{v_0} = \frac{na}{bl}\left[-1 + \frac{a}{(1-s)\sqrt{(\pi\kappa t)}}\left(1 - \frac{n^2a^2}{2\kappa t} + \frac{l^2}{2\kappa_1 t}\right)\right],$$

which is a slight extension of Heaviside's formula, given for the case when the thickness of the shell is negligible‡.

* See § 4 (iv) below.

† § 4 below (i) and (ii).

‡ *Electromagnetic Theory*, vol. 2, § 236, formula (39). The extension consists merely in the addition of the last two terms in the bracket: that terms of this type would be present could be foreseen from Heaviside's formula (27) in § 229 (for the corresponding *plane* problem). There is also the external factor a/b in the present formula.

To estimate the numerical values of these terms is easy; take the *data* adopted in Prof. Carslaw's paper (following Perry and Heaviside),

$$l = 4 \times 10^5, \quad a = 6{\cdot}38 \times 10^8, \quad \kappa = {\cdot}1643, \quad \kappa_1 = {\cdot}0117.$$

Then $$s = {\cdot}04953, \quad n = {\cdot}05211.$$

After a little trial it was found that a value of t about 3×10^{17} (that is, about $9{\cdot}36 \times 10^9$ years) would fit the *data* for g and v_0 used by Lord Kelvin (see below): and then it appears that

$$\tfrac{1}{2}l^2/\kappa_1 t = 2{\cdot}3 \times 10^{-5},$$

so that this term will be negligible in our calculations (and naturally the same inference can be made with reference to the terms already neglected in $l^4, l^6, \ldots$).

On the other hand it is found that

$$\tfrac{1}{2}n^2a^2/\kappa t = {\cdot}0114;$$

and accordingly this term, and other terms in n^4a^4/κ^2t^2, will probably affect the conclusion.

We shall accordingly complete the formula for g, by including higher powers of nqa; on rejecting $q^2, q^4, q^6, \ldots$, the result is

$$\frac{g}{v_0} = \frac{a}{bl}\left\{-n + \frac{1}{1-s}(nqa + n^3q^3a^3 + n^5q^5a^5 + \ldots)\right\}$$

$$= \frac{na}{bl}\left\{-1 + \frac{a}{(1-s)\sqrt{(\pi\kappa t)}}\left(1 - \frac{1}{2}\frac{n^2a^2}{\kappa t} + \frac{1.3}{2^2}\frac{n^4a^4}{\kappa^2t^2} - \frac{1.3.5}{2^3}\frac{n^6a^6}{\kappa^3t^3} + \ldots\right)\right\}.$$

As already explained, we have rejected the terms in $l^2, l^4, \ldots$; and the interpretation of $q, q^3, q^5, \ldots$ follows from § 4 (i)-(iii) below.

The series now obtained is not convergent; but it possesses the asymptotic property that the error in stopping at any stage in the series is less than the following term of the series (see § 4 (ix) below).

Inserting the values given above for a, κ, t, the series in the bracket ( ) becomes

$$1 - {\cdot}0114 + {\cdot}0004 = {\cdot}9890,$$

the error being less than the following term (roughly ·00002).

Thus our formula becomes (approximately)

$$\frac{g}{v_0} = \frac{na}{bl}\left\{-1 + \frac{a\,({\cdot}9890)}{(1-s)\sqrt{(\pi\kappa t)}}\right\}.$$

If we now assume the values assigned by Lord Kelvin

$$g = 1/2743, \quad v_0 = 4000,$$

the corresponding value of t can be estimated by writing our formula in the shape

$$\frac{a}{\sqrt{(\pi\kappa t)}}({\cdot}9890) = (1-s)\left(1 + \frac{g}{v_0}\frac{bl}{na}\right) = {\cdot}9505 + {\cdot}6653 = 1{\cdot}6158.$$

On reduction this formula gives

$$t = 2{\cdot}95 \times 10^{17}$$

or in *years** $$t = 9{\cdot}36 \times 10^{9}.$$

Finally we shall estimate the error which may be due to replacing $\coth qa$ by 1 in the formula for A and g used above. It is easy to see that

$$1 - \frac{1}{1-s+sqa\coth qa} = 1 - \frac{1}{1-s}\left\{1 + nqa\left(\frac{1+\rho}{1-\rho}\right)\right\}^{-1}$$

where $$\rho = e^{-2qa}.$$

Thus on expansion we obtain

$$\begin{aligned}
1 - \frac{1}{1-s}&\left\{1 - nqa\left(\frac{1+\rho}{1-\rho}\right) + n^2q^2a^2\left(\frac{1+\rho}{1-\rho}\right)^2 - \ldots\right\} \\
&= -\frac{s}{1-s} + \frac{nqa}{1-s}(1 + 2\rho + 2\rho^2 + \ldots) - \frac{n^2q^2a^2}{1-s}(1 + 4\rho + \ldots) \\
&\qquad + \frac{n^3q^3a^3}{1-s}(1 + 6\rho + \ldots) - \ldots \\
&= -\frac{s}{1-s} + \frac{1}{1-s}(nqa - n^2q^2a^2 + n^3q^3a^3 \ldots) \\
&\qquad + \frac{2}{1-s}(nqa - 2n^2q^2a^2 + 3n^3q^3a^3 - \ldots)\rho + \ldots \\
&= P_0 + P_1\rho + P_2\rho^2 + P_3\rho^3 + \ldots
\end{aligned}$$

where the first term in P_1 is $2nqa/(1-s)$.

The series P_0 leads to the asymptotic series already used; and similarly we can obtain series for $P_1\rho$, $P_2\rho^2$, etc. Clearly the most important term (with the numerical values under consideration) is the first term in $P_1\rho$, which is

$$\frac{2nqa}{1-s}e^{-2qa} = \frac{2n}{1-s}\frac{a}{\sqrt{(\pi\kappa t)}}e^{-a^2/\kappa t}$$

by § 4 (vii) below. Thus in comparison with the term $nqa/(1-s)$ in P_0, the relative order of this term is $2e^{-a^2/\kappa t}$.

* On comparing this result with the value $9{\cdot}02 \times 10^9$ found by Prof. Carslaw from the first term of the Fourier-expansion, I was led (by comparison with the numbers given in the first example of § 3 below) to the conjecture that the discrepancy must be due to the neglect of the second and higher terms in the Fourier-formula.

Prof. Carslaw has kindly re-calculated his formula for the above value of t, and obtains (using the first and second terms) a gradient

$$g = {\cdot}00035092 + {\cdot}00001348 = {\cdot}0003644.$$

This is equal to 1/2744; and so agrees with the value assumed for g to one part in 3000, which is roughly the same order as the correction on account of neglecting $e^{-a^2/\kappa t}$.

Now here $a^2/\kappa t = 8{\cdot}4$ roughly, and so

$$2e^{-a^2/\kappa t} = 4{\cdot}5 \times 10^{-4} \text{ nearly.}$$

The corresponding correction to the value of t will be of the relative order 9×10^{-4}, and so will change t to about $9{\cdot}37 \times 10^9$ (in years).

The correction on account of the terms in $P_2\rho^2$, $P_3\rho^3$, ... will lead similarly to an estimate of the relative orders $2e^{-4a^2/\kappa t}$, $2e^{-9a^2/\kappa t}$, which are entirely unimportant in the present problems.

§ 3. *Numerical tests of the formulae of* § 2.

In view of the difference in form between the series of § 2 and the corresponding Fourier-expansions, it seemed desirable to compare the results of numerical calculation with values not very different from those of § 2. Prof. J. Perry* had given the results of some calculations in connexion with a problem which may be regarded as the limit of that of § 2, when the thickness of the shell (l) tends to zero, the value of s remaining fixed. However some discrepancies were found (see below) and I decided to recalculate with slightly different constants so as to reduce the labour of calculating the Fourier-expansion; I had not then† the advantage of Prof. Carslaw's results with which to compare my work.

If we make l tend to zero in § 2 we find that

$$A = 1/(1 - s + sqa \coth qa),$$

and then if we replace V by $(v_0 - v)/v_0$ we obtain the formula

$$\frac{v_0 - v}{v_0} = \frac{1}{(1 - s + sqa \coth qa)} \frac{a \sinh qr}{r \sinh qa}.$$

This represents the symbolical solution for the temperature v of a sphere initially at temperature v_0, radiating into a medium at zero temperature; this problem was solved by Fourier‡ in the form

$$\frac{v}{v_0} = \frac{2a}{rs} \Sigma \frac{\sin\theta \sin(\theta r/a)\, e^{-\kappa\theta^2 t/a^2}}{\theta(\theta - \sin\theta\cos\theta)},$$

where the summation refers to the roots of the equation

$$1 - s + s\theta \cot\theta = 0.$$

* *Nature*, vol. 51, 1895, p. 225.

† These calculations were made at intervals in the latter part of 1916 and in 1917.

‡ *Theory of Heat* (Freeman's translation), § 293.

It is easy to confirm the Fourier-expansion from the symbolical formula above, by using Héaviside's general rule*.

However, in the actual problem we want the value of v at $r = a$; and for this purpose, from the considerations already given in § 2, it will be sufficient to replace $\coth qa$ by unity to obtain the series

$$\frac{v}{v_0} = 1 - \frac{1}{1 - s + sqa} = -n + \frac{1}{1-s}(nqa + n^2q^2a^2 + n^3q^3a^3 + \ldots)$$

where $n = s/(1-s)$.

Then (as in § 2) we obtain the asymptotic series

$$\frac{v}{v_0} = n\left\{-1 + \frac{a}{(1-s)\sqrt{(\pi\kappa t)}}\left(1 - \frac{1}{2}\frac{n^2a^2}{\kappa t} + \frac{1 \cdot 3}{2 \cdot 4}\frac{n^4a^4}{\kappa^2t^2} - \ldots\right)\right\}.$$

As explained in § 2, the formula is valid only if $e^{-a^2/\kappa t}$ is negligible; and to obtain four-figure accuracy, this requires $a^2/\kappa t$ to be not less than about 8·5. Hence to obtain good results from the asymptotic series, n must be small, of about the order 1/15 to 1/20.

In the Fourier-expansion, to avoid the labour of the actual calculation of the roots of the equation for θ, I decided to adopt a simple value for θ_1 (which is the root requiring the greatest accuracy), and to deduce the corresponding value of s.

The value $\theta_1 = 170\pi/180 = 2{\cdot}9671$ was selected; and this gave $1/s = 17{\cdot}827$. Then θ_2 was not very different from $2\theta_1$, and the value $\theta_2 = 5{\cdot}944$ was comparatively easy to calculate. A further simplification in the arithmetic was made by taking

$$a^2/\kappa t = \theta_1^2 = 8{\cdot}8 \quad \text{(nearly)}.$$

Then the first and second terms in the Fourier-expansion for v were found to be

$$v_0\,(\cdot 04248 + \cdot 00191) = (\cdot 04439)\,v_0.$$

The corresponding asymptotic series is found to be

$$1 - \cdot 01546 + \cdot 000725 - \cdot 000075 + \cdot 000008 = \cdot 98520.$$

Then on substitution we find

$$v = v_0\,(\cdot 04441);$$

* This rule may be written in the form

$$\frac{F(p)}{\Delta(p)} = \frac{F(0)}{\Delta(0)} + \Sigma\frac{F(\alpha)\,e^{\alpha t}}{\alpha\Delta'(\alpha)},$$

where the summation refers to all the roots $p = \alpha$ of $\Delta(p) = 0$.

Proofs of this equation (from different points of view) have been given in *Phil. Mag.* vol. 37, 1919 (see pp. 417, 418), and *Proc. Lond. Math. Soc.* vol. 15, 1917 (see pp. 419, 420); Heaviside's own discussion will be found in his *Electrical Papers*, vol. 2, pp. 226, 373.

In the present problem, the term $F(0)/\Delta(0)$ reduces to 1 and cancels a term from the other side of the equation; and the values of α are $-\kappa\theta^2/a^2$ (found by writing $qa = \iota\theta$).

so that the two formulae agree as well as could be expected with four-figure accuracy.

In Prof. Perry's calculation, the value $1/s = 20$ was taken, and the value of $\kappa = k/c$ as in § 2 above, while t was taken as 96×10^8 years; then with $v_0 = 4000$ the first and second terms are stated to be equal to

$$142{\cdot}7 \quad + 5{\cdot}65 = 148{\cdot}4.$$

I did not succeed in confirming this value; and Prof. Carslaw has recalculated the Fourier-formula with the above *data*. His result is

$$138{\cdot}13 + 4{\cdot}83 = 142{\cdot}96, \quad \text{or say } 143{\cdot}0.$$

The corresponding value of the asymptotic series is found to be

$$1 - {\cdot}01131 + {\cdot}00038 - {\cdot}00002 = {\cdot}98905.$$

Hence
$$\frac{v}{v_0} = \tfrac{1}{19}(-1 + 1{\cdot}6783) = {\cdot}03570,$$

and so
$$v = 142{\cdot}8,$$

which agrees sufficiently closely with Prof. Carslaw's result*.

Perry has also given numerical results for the same Fourier-expansion when the constants are adjusted so that $s = 1$; the asymptotic series used above will clearly fail under this condition, and a fresh formula becomes necessary.

When $s = 1$, the equation for θ becomes

$$\cot\theta = 0,$$

thus the values of θ are $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, ...; and the Fourier-expansion simplifies to

$$\frac{v}{v_0} = \Sigma \frac{2}{\theta^2} e^{-\kappa\theta^2 t/a^2} = \frac{8}{\pi^2}\left(e^{-\omega} + \tfrac{1}{9}e^{-9\omega} + \tfrac{1}{25}e^{-25\omega} + \ldots\right)$$

where
$$\omega = \pi^2\kappa t/4a^2.$$

In Perry's actual calculation, the value of ω is nearly equal to 1; and so the corresponding value of

$$a^2/\kappa t = \tfrac{1}{4}\pi^2 = 2{\cdot}47 \quad \text{nearly.}$$

Thus the method of approximation adopted in § 2 needs reconsideration; and it turns out that the new formulae are not very convenient for numerical work in this special case (see below). However it is easy to recalculate this simple Fourier-expansion for other values of ω; and to select values which are suitable for purposes of comparison.

* The correction on account of the first term in $e^{-a^2/\kappa t}$ is of the same relative order of magnitude as in the calculation of § 2; and this gives

$$\left(\frac{1{\cdot}6782}{19}\right)(4{\cdot}5 \times 10^{-4})\,(4000) = {\cdot}17,$$

which accounts for the small residual difference.

Putting $s = 1$ the symbolical formula for v at $r = a$ becomes

$$\frac{v}{v_0} = 1 - \frac{\tanh qa}{qa} = 1 - \frac{1}{qa}(1 - 2e^{-2qa} + 2e^{-4qa} - \ldots).$$

Thus the first approximation (analogous to § 2) is given by

$$\frac{v}{v_0} = 1 - \frac{1}{qa} = 1 - 2\sqrt{\frac{\kappa t}{\pi a^2}} = 1 - \frac{4}{\pi}\sqrt{\frac{\omega}{\pi}},$$

where the value of $1/q$ is given by § 4 (vi). The value of the next term in this series is

$$\frac{2}{qa}e^{-2qa} = \frac{4}{\sqrt{\pi}}\int_{a/\sqrt{(\kappa t)}}^{\infty} e^{-w^2}\frac{dw}{w^2},$$

which requires the error-function integral to obtain a formula suitable for actual numerical calculation. However, the numerical value is less than

$$(\kappa t/a^2)\, e^{-a^2/\kappa t} \times 2\sqrt{\frac{\kappa t}{\pi a^2}},$$

or

$$\left(\frac{4\omega}{\pi^2}\, e^{-\pi^2/4\omega}\right) \times \frac{4}{\pi}\sqrt{\frac{\omega}{\pi}}.$$

The following terms in the series will be negligible if $a^2/\kappa t$ exceeds 2.

It is easy to test the accuracy of our results by taking say $\omega = \frac{1}{4}$, with $a^2/\kappa t = \pi^2 = 9{\cdot}87$; so that $(\kappa t/a^2)\, e^{-a^2/\kappa t}$ is of order $\frac{1}{2} \times 10^{-5}$, and so is negligible.

The Fourier-expansion is then

$$\frac{8v_0}{\pi^2}(e^{-\frac{1}{4}} + \tfrac{1}{9}e^{-\frac{9}{4}} + \tfrac{1}{25}e^{-\frac{25}{4}} + \ldots) = \frac{8v_0}{\pi^2}(\cdot 7789 + \cdot 0117 + \cdot 0001)$$

$$= \frac{8v_0}{\pi^2}(\cdot 7907) = v_0(\cdot 6409).$$

The symbolical formula gives (for $\omega = \frac{1}{4}$)

$$v = v_0\left(1 - \frac{2}{\pi\sqrt{\pi}}\right) = v_0(1 - \cdot 3592) = v_0(\cdot 6408)$$

and thus the agreement is as close as could be hoped for.

Even for $\omega = \frac{1}{2}$, when $(\kappa t/a^2)\, e^{-a^2/\kappa t}$ is of the order 1/700, the two formulae agree to three significant figures; the Fourier-expansion is

$$\frac{8v_0}{\pi^2}(e^{-\frac{1}{2}} + \tfrac{1}{9}e^{-\frac{9}{2}} + \ldots) = \frac{8v_0}{\pi^2}(\cdot 6065 + \cdot 0012) = v_0(\cdot 4926)$$

while the other gives

$$v = v_0\left(1 - \frac{2\sqrt{2}}{\pi\sqrt{\pi}}\right) = v_0(1 - \cdot 5079) = v_0(\cdot 4921).$$

It is therefore clear that, under proper conditions, the Fourier-

expansion is numerically equivalent to the simpler formula obtained by the symbolical method*.

§ 4. *Evaluation of certain symbolical expressions used in the foregoing sections.*

As explained already in § 1, the fundamental meanings of the symbolical formulae can be obtained by translation into complex integrals. In fact the meaning of the function $f(q)$, where

$$\kappa q^2 = p = \partial/\partial t,$$

is given by the complex integral

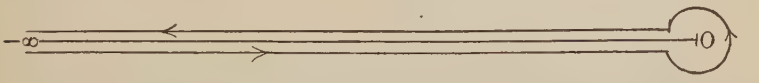
$$\frac{1}{2\pi\iota}\int_{\alpha-\iota\infty}^{\alpha+\iota\infty} f(\nu)\, e^{\lambda t}\,\frac{d\lambda}{\lambda}$$

where

$$\kappa\nu^2 = \lambda.$$

In the first instance we are concerned only with functions of q, which are in reality even functions†; or functions which are (in theory) such that $f(\nu)$ is expressible as a one-valued function of λ. But in the symbolical transformations of $f(q)$, it is generally convenient to manipulate algebraically, without restricting the functions used; and then we must adopt some definite convention as to the interpretations. First we make ν single-valued by means of a cut along the negative real axis in the λ-plane; and we select that value for ν which has its real part positive‡.

Further the functions $f(q)$ are (in their original forms) such as to tend to zero when $|q| \to \infty$; and thus the complex integral above can be replaced by an integral along the path indicated in the λ-plane.



When the path has been modified, we can suppose the algebraic transformations of the function $f(\nu)$ carried out so as to correspond

* For Perry's case ($\omega = 1$) the two formulae work out as v_0 (·2982) and v_0 (·2816); but, as already remarked, the value of $(\kappa t/a^2)\, e^{-a^2/\kappa t}$ is then about $\frac{1}{3\,0}$, so that a discrepancy is to be anticipated unless the definite integral given is evaluated numerically.

† For instance qa coth qa can be written in the form

$$\left\{1+\frac{1}{2!}(qa)^2+\frac{1}{4!}(qa)^4+\ldots\right\} \Big/ \left\{1+\frac{1}{3!}(qa)^2+\frac{1}{5!}(qa)^4+\ldots\right\}.$$

Then the corresponding function of ν is equal to

$$\left(1+\frac{a^2\lambda}{2!\kappa}+\frac{a^4\lambda^2}{4!\kappa^2}+\ldots\right)\Big/\left(1+\frac{a^2\lambda}{3!\kappa}+\frac{a^4\lambda^2}{5!\kappa^2}+\ldots\right).$$

‡ This has practically the same effect as if we regard q as *real and positive*, in our algebraic work.

to the symbolical manipulation of $f(q)$. For, on this path we can write $\nu = +\iota\theta$ (on the upper straight part),

or $-\iota\theta$ (on the lower straight part)

and $\lambda = -\kappa\theta^2$;

thus the convergence is always ensured by the presence of the exponential $e^{-\kappa\theta^2 t}$.

With these general remarks we proceed now to evaluate the special functions used.

(i) *Interpretation of* q.

This becomes

$$\frac{1}{2\pi\iota}\int \nu e^{\lambda t}\frac{d\lambda}{\lambda} = \frac{1}{2\pi\iota}\int_{\infty}^{0}(-\iota\theta)\,e^{-\kappa t\theta^2}\frac{2d\theta}{\theta} + \frac{1}{2\pi\iota}\int_{0}^{\infty}(+\iota\theta)\,e^{-\kappa t\theta^2}\frac{2d\theta}{\theta}$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\kappa t\theta^2}\,d\theta = \frac{2}{\pi}\frac{1}{\sqrt{(\kappa t)}}\frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{(\pi\kappa t)}};$$

the contribution from the small circle there tends to zero with the radius (its value is in fact proportional to the square-root of the radius).

Hence $q = 1/\sqrt{(\pi\kappa t)}$.

(ii) *Interpretation of* q^3.

Repeating the foregoing transformations, we obtain

$$-\frac{2}{\pi}\int_0^\infty \theta^2 e^{-\kappa t\theta^2}\,d\theta = \frac{1}{\kappa}\frac{\partial}{\partial t}\frac{1}{\sqrt{(\pi\kappa t)}},$$

and so $$q^3 = -\frac{1}{2\sqrt{\pi}}\frac{1}{(\kappa t)^{\frac{3}{2}}}.$$

(iii) *Interpretation of* q^{2m+1}.

Similarly

$$q^{2m+1} = \frac{1}{\kappa^m}\left(\frac{\partial}{\partial t}\right)^m\frac{1}{\sqrt{(\pi\kappa t)}} = (-1)^m\frac{1\,.\,3\,\ldots\,(2m-1)}{(2\kappa t)^m\sqrt{(\pi\kappa t)}}.$$

It should be noticed that symbolically the relation

$$q^{2m+1} = \frac{1}{\kappa^m}\left(\frac{\partial}{\partial t}\right)^m q$$

is obvious, since $\kappa q^2 = p$.

(iv) *Interpretation of* $q^2, q^4, \ldots$.

It is here easy to verify that the two integrals from the upper and lower paths cancel; so that $q^2 = 0$, $q^4 = 0$, etc. This again is obvious symbolically, since $q^2 = \frac{1}{\kappa}\frac{\partial}{\partial t}(1) = 0$.

(v) *Interpretation of* $1/q$.

This is $$\frac{1}{2\pi\iota}\int_{a-\iota\infty}^{a+\iota\infty}\left(\frac{\kappa}{\lambda}\right)^{\frac{1}{2}}e^{\lambda t}\frac{d\lambda}{\lambda},$$

and by a known result, due to Cauchy,

$$\frac{1}{2\pi\iota}\int_{a-\iota\infty}^{a+\iota\infty}\frac{e^{\lambda t}d\lambda}{\lambda^{1+m}}=\frac{t^m}{\Gamma(1+m)},$$

if $$1+m>0 \text{ and } t>0.$$

Hence $$\frac{1}{q}=\frac{\sqrt{(\kappa t)}}{\Gamma(\frac{3}{2})}=\frac{\sqrt{(\kappa t)}}{\frac{1}{2}\Gamma(\frac{1}{2})}=2\sqrt{\left(\frac{\kappa t}{\pi}\right)},$$

a result which follows symbolically from (i) by observing that

$$\frac{1}{q}=\frac{\kappa}{p}q=\kappa\int_0\frac{dt}{\sqrt{(\pi\kappa t)}}=\frac{2}{\sqrt{\pi}}\sqrt{(\kappa t)}.$$

(vi) *Interpretation of* $1/q^n$.

The method of (v) shows at once that

$$\frac{1}{q^{2m+1}}=\frac{(\kappa t)^{m+\frac{1}{2}}}{\Gamma(m+\frac{3}{2})},\qquad\frac{1}{q^{2m}}=\frac{(\kappa t)^m}{\Gamma(m+1)},$$

and we can sum up all the preceding formulae in the single statement

$$\frac{1}{q^n}=\frac{(\kappa t)^{\frac{n}{2}}}{\Gamma(1+\frac{1}{2}n)};$$

which may be stated even more simply in the shape

$$\frac{1}{p^n}=\frac{t^n}{\Gamma(1+n)},$$

where n may be taken as having any (positive or negative) value. The last result is obvious by successive integration when n is an integer: and it is natural to conjecture that the same formula will be valid generally.

(vii) *Interpretation of* qe^{-qx}, *where* x *is positive.*

This is readily obtained by expansion in powers of x; and the result is

$$q-q^2x+q^3\frac{x^2}{2!}-q^4\frac{x^3}{3!}+q^5\frac{x^4}{4!}-\ldots.$$

Using the results found above we obtain the series

$$\frac{1}{\sqrt{(\pi\kappa t)}}\left\{1-\frac{1}{2\kappa t}\frac{x^2}{2!}+\frac{1.3}{(2\kappa t)^2}\frac{x^4}{4!}-\frac{1.3.5}{(2\kappa t)^3}\frac{x^6}{6!}\cdots\right\}$$

$$=\frac{1}{\sqrt{(\pi\kappa t)}}\left\{1-\frac{x^2}{4\kappa t}+\frac{1}{2!}\left(\frac{x^2}{4\kappa t}\right)^2-\frac{1}{3!}\left(\frac{x^2}{4\kappa t}\right)^3+\ldots\right\}=\frac{1}{\sqrt{(\pi\kappa t)}}e^{-x^2/4\kappa t}.$$

If these operations appear to need further justification, it is easy to see that the direct expression of qe^{-qx} by means of a complex integral leads to the formula

$$\frac{2}{\pi}\int_0^\infty e^{-\kappa t\theta^2}\cos(\theta x)\,d\theta=\frac{1}{\sqrt{(\pi\kappa t)}}e^{-x^2/4\kappa t}.$$

(viii) *Interpretation of* e^{-qx} *and* $\frac{1}{q}e^{-qx}$, x *being positive.*

We have proved that

$$qe^{-qx} = \frac{1}{\sqrt{(\pi\kappa t)}}\, e^{-x^2/4\kappa t}.$$

Thus, on integrating with respect to x, we see that

$$e^{-qx} = -\int \frac{dx}{\sqrt{(\pi\kappa t)}}\, e^{-x^2/4\kappa t} + \text{const.}$$

$$= -\frac{2}{\sqrt{\pi}}\int^z e^{-v^2}dv + \text{const.}$$

where $$z = \tfrac{1}{2}x/\sqrt{(\kappa t)}.$$

Now when $x \to 0$, $e^{-qx} \to 1$, and so

$$e^{-qx} = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-v^2}dv = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-v^2}dv.$$

We have now a verification of the work; for as x tends to $+\infty$, e^{-qx} tends to 0 (since the real part of q is positive); and this property is seen to hold for the integral just found.

The value of $(1/q)\, e^{-qx}$ is easily found by observing that

$$\left(\frac{1}{q} + x\right)e^{-qx} = \frac{1}{q} - \frac{qx^2}{2!} - \frac{3q^3x^4}{4!} - \frac{5q^5x^6}{6!} - \ldots$$

$$= 2\sqrt{\left(\frac{\kappa t}{\pi}\right)}\left\{1 - \frac{x^2}{4\kappa t} + \frac{1}{2!}\left(\frac{x^2}{4\kappa t}\right)^2 - \ldots\right\} = 2\sqrt{\left(\frac{\kappa t}{\pi}\right)}\, e^{-x^2/4\kappa t}.$$

Thus $$\frac{1}{q}e^{-qx} = 2\sqrt{\left(\frac{\kappa t}{\pi}\right)}\, e^{-x^2/4\kappa t} - \frac{2x}{\sqrt{\pi}}\int_z^\infty e^{-v^2}dv,$$

where as before $$z = \tfrac{1}{2}x/\sqrt{(\kappa t)}.$$

This is a form suited for numerical work; but a more useful result is the simpler formula

$$\frac{1}{q}\, e^{-qx} = \frac{x}{\sqrt{\pi}}\int_z^\infty e^{-v^2}\frac{dv}{v^2}.$$

For large values of z this integral can be converted into an asymptotic series of which the first term is $\dfrac{x}{2\sqrt{\pi}}\cdot\dfrac{e^{-z^2}}{z^3}$.

(ix) The only point remaining is to establish the asymptotic property for the series in powers of q, used in §§ 2, 3 above; this series is derived from the function

$$f(q) = 1 - \frac{1}{1 - s + sqa},$$

and suppose that as a matter of algebraic expansion we obtain

$$f(q) = A_0 + A_1q + A_2q^2 + \ldots.$$

When $f(q)$ is converted into a complex integral along the path indicated above, there is a contribution from the small circle round the origin, which tends to the value A_0, when the radius of the circle tends to zero; and the straight paths contribute the integral

$$\frac{1}{\pi\iota}\int_0^\infty \{f(\iota\theta) - f(-\iota\theta)\}\, e^{-\kappa\theta^2 t}\,\frac{d\theta}{\theta} = \frac{2}{\pi}\int_0^\infty \phi(\theta)\, e^{-\kappa\theta^2 t}\, d\theta,$$

where $$\phi(\theta) = \frac{f(\iota\theta) - f(-\iota\theta)}{2\iota\theta} = \frac{sa}{(1-s)^2 + s^2a^2\theta^2}.$$

Thus $$f(q) = A_0 + \frac{2}{\pi}\int_0^\infty \phi(\theta)\, e^{-\kappa\theta^2 t}\, d\theta$$

gives the interpretation of the function $f(q)$.

Now $\phi(\theta)$ can be expanded in powers of θ, when θ is sufficiently small; and, having regard to the connexion between $\phi(\theta)$ and $f(q)$, it is clear that the expansion will be of the form

$$\phi(\theta) = A_1 - A_3\theta^2 + A_5\theta^4 - \ldots.$$

Denote the sum of the first n terms of this expansion by $S_n(\theta)$; then it is easy to see that we have the algebraic identity*

$$\phi(\theta) - S_n(\theta) = (-1)^n \frac{(1-s)^2 A_{2n+1}\,\theta^{2n}}{(1-s)^2 + s^2a^2\theta^2}.$$

Thus for all real values of θ, the difference between $\phi(\theta)$ and $S_n(\theta)$ is numerically less than $(-1)^n A_{2n+1}\,\theta^{2n}$; and accordingly the interpretation of $f(q)$ may be taken as

$$A_0 + \frac{2}{\pi}\int_0^\infty S_n(\theta)\, e^{-\kappa\theta^2 t} d\theta,$$

with an error which is numerically less than

$$\frac{2}{\pi}\int_0^\infty (-1)^n A_{2n+1}\theta^{2n} e^{-\kappa\theta^2 t} d\theta.$$

Thus we may say that the error in replacing $f(q)$ by

$$A_0 + A_1 q + A_3 q^3 + \ldots + A_{2n-1}\, q^{2n-1}$$

(or by $A_0 + A_1 q + A_2 q^2 + \ldots + A_{2n} q^{2n}$) is less than the numerical value of the following term $A_{2n+1}\, q^{2n+1}$ (when interpreted according to the rules already given). This establishes the asymptotic property assumed in regard to the series for $f(q)$.

* On multiplying up, we see that

$$\{\phi(\theta) - S_n(\theta)\}\{(1-s)^2 + s^2a^2\theta^2\}$$

is a polynomial of degree $2n$ in θ; further when θ is small it can be expanded in powers of θ, the first term being $(-1)^n (1-s)^2 A_{2n+1}\,\theta^{2n}$. Accordingly the product must reduce to this single term.

On the effect of a magnetic field on the intensity of spectrum lines. By H. P. Waran, M.A., Government of India Scholar of the University of Madras. (Communicated by Professor Sir Ernest Rutherford, F.R.S.)

(Plate III.)

[*Read* 16 May 1921.]

Since my last communication to the society* on the phenomenon of the changes in the general spectrum brought about by the influence of the magnetic field, I have come to know that the phenomenon has been noticed as early as 1858† and studied to a certain extent by various observers. Yet the complex changes taking place in the source when the radiation is emitted from

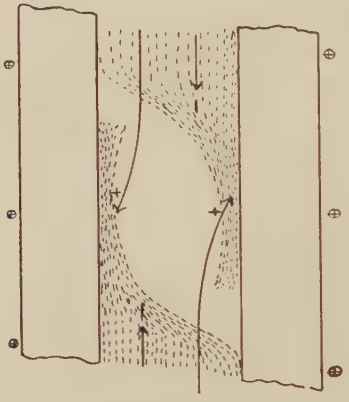


Fig. 1.

inside a magnetic field, does not seem to have had the attention it deserves. But in 1913‡ Messrs Kent and Frye made a study of the phenomenon in some particular cases with a view to its complete elucidation. They do not seem to have come to any definite conclusions and from their experiments they seem to be inclined to attribute the effects observed mainly to the chemical disintegration of the walls of the discharge tube brought about by the disruptive action of the blast of ions deflected to the sides by the magnetic field.

Considering a section of the discharge transverse to the magnetic field, as in Fig. 1, with the lines of force entering the plane of

* *Proc. Camb. Phil. Soc.* vol. 20, Part I, p. 45.
† A. de la Rive, *Annales de chimie et de physique* (3), 54, p. 238, 1858.
‡ *Astrophysical Journal*, 37, pp. 183–189, 1913.

the paper normally, it is seen that the discharge current through the tube which consists of the unrectified current from the secondary of an Induction Coil is split into two streams and deflected to the opposite sides of the walls of the tube. The brighter stream corresponding to the break current is arranged to be on the side facing the spectroscope. Since they are travelling in opposite directions both the ions forming the discharge are deflected in the same direction on to the glass and with a strong discharge in an intense field the magnitude of the corrosion of the sides of the glass under this sand blast action of the ions is considerable. Where this is predominant the real effect gets mixed up with the effects of the chemical disintegration and the phenomenon may get attributed entirely to the latter cause. This conclusion is inevitable from the intense fields and strong currents Messrs Kent and Frye used without taking adequate precautions to exclude the complications of the attendant chemical disintegration of the tube.

To examine the extent of such disintegration heavy currents and strong fields were employed for a few minutes and Fig. 2 shows an enlarged view of a side of the capillary of the glass discharge tube in the region opposite the pole pieces. The deep channels found to have been cut in the glass show that the disintegration of the glass is considerable. The consequent complications are sure to lead one to erroneous conclusions in case proper precautions are not taken to exclude them and make allowances for such effects. To study the phenomenon free of such complications, this disruptive action of the ions should be reduced to a minimum by the employment of a moderate field and current and even then the tube must be of a material like quartz not liable to such easy decomposition, arising from any local heating.

In my experiments I used a quartz discharge tube with aluminium electrodes the diameter of whose capillary was of the order of about 2 mm. The current generally used was about 2 milliamperes and the field was of the order of about 5000 C.G.S. units. Further the tube was in permanent communication with a mercury pump fitted with McLeod's gauge and drying tube of phosphorous pentoxide. Thus owing to the large volume in connection with the discharge tube the pressure of gas in the tube remained constant throughout the experiment and the disturbing influence of the gradual absorption or emission of gas by the electrodes or walls of the tube during the course of an experiment was considerably compensated. Under such circumstances the results obtained with the quartz tube are identical with those obtained with the glass tube and hence the phenomenon cannot be attributed to the chemical disintegration of the walls of the tube.

Another possible view of the phenomenon put forward by Messrs Kent and Frye and others is that the change is due to the

decrease in cross-section of the discharge brought about by the deflecting action of the field. From Fig. 1 it was seen that the discharge is pushed on either side into two thin filaments leaving the centre of the tube more or less free, thus reducing the effective cross-section of the discharge. They regard it as tantamount to a change to a narrower capillary. But that the effect is not entirely due to this can be shown by the following considerations.

Though a narrower capillary does generally tend to enhance the weaker lines and bring about a trace of continuous spectrum, its effect is not of such a selective nature tending to the enhancement of some lines previously almost invisible, at the same time decreasing the intensity of some others as observed in the case of mercury and many other gaseous spectra. Further the effect of the field is to weaken the continuous spectra in many cases though under the same circumstances a narrower capillary tends to intensify the continuous spectrum.

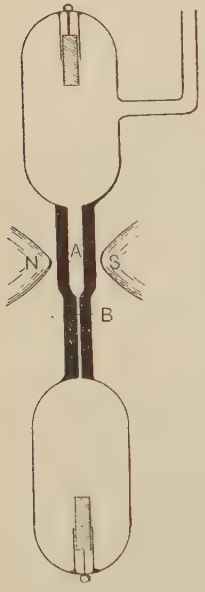


Fig. 3.

To test this point further a tube of the form shown in Fig. 3, in which the capillary portion was made of two sections in series, with sectional areas in the ratio of about 10 : 1, was made and placed transverse to the field as indicated. An image of the portion of the tube between *A* and *B* was thrown on the slit of the spectroscope and the spectra given by these two capillaries were compared. There was very little change except that the portion of the spectrum corresponding to the narrower capillary was distinctly brighter throughout the whole spectrum. A magnetic field was now applied transversely at *A* to see how far the narrowed section of the discharge through *A* gave a spectrum resembling that given by *B*. The change introduced by the field was quite different. With hydrogen at low pressure, while the effect of the narrowed capillary *B* was to increase the intensity of the hydrogen lines the effect of the field on *A* was to decrease the intensity of the hydrogen to a slight extent and increase the intensity of the mercury lines as shown in Fig. 4 (1). Such results clearly show that though the decrease in the cross-section may be the cause of some of the changes observed, yet there are others that cannot go under this simple explanation and they point to the existence of some other influence exerted by the magnetic field.

Further, experimenting with a tube of the form illustrated below, Fig. 5, in which the discharge is in the direction of the

field effects are observed similar to those obtained with the discharge going transverse to the field. With a leyden jar in parallel with the coil and a spark gap in series with the tube shunted across the condenser, at low pressures the stream-like character of the discharge is lost and the discharge passes through the whole section of the tube without seeming to suffer any visible change in the cross-section. But the spectroscopic changes observed are similar. The fact that in many cases the change over the spectrum is brought about practically instantaneously on the application of the field is also against the possibility of any progressive changes

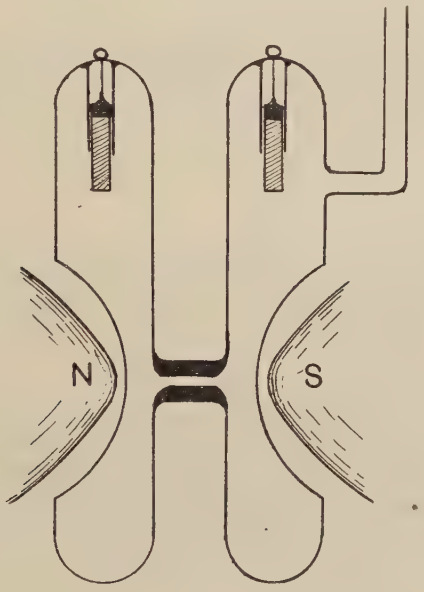


Fig. 5.

resulting from the absorption or liberation of some of the component gases.

There is at times a great similarity between the effect produced by the condensed discharge through the tube and that brought about by the magnetic field, and from that, though we may conclude that the magnetic field brings about a similar violent excitation of the spectrum in some way, we cannot as yet go so far as to say that their effects are identical. For there are important differences between the effects produced by the two. Experimenting with hydrogen the effect of a condensed discharge is found to be mainly to broaden the hydrogen Balmer series lines, especially those towards the violet, where as the magnetic field under the same conditions

leaving the Balmer series lines unaffected brings about the secondary spectrum very prominently as shown in Fig. 4 (2).

It is possible that the change in the spectrum is purely mechanical in origin. In Fig. 1 we saw that under the influence of the magnetic field the discharge was deflected to either side. In each of these streams the electrons being much more easily deflected out of their path than the positive ions, they might be going in a layer much closer to the walls than the positive ions as shown in Fig. 1. In such a case we may regard the massive positive particles as bombarding the layer of electrons adjacent to the walls, thus giving rise to mechanical and electrical reactions that cause a peculiar excitation of their spectra. Orbits and frequencies previously not natural might then become possible and this would account for the change observed.

The spectrum of hydrogen with its simple structure is only very slightly susceptible to the influence of the field, excepting of course the secondary spectrum which it brings about. When the gas is pure and at a low pressure the Balmer series lines undergo very little change in their intensity. But when it is present mixed with other gases or at higher pressures it is brought out intensely by the field though not prominent without it. From our observation of the enhancement of the lines of the monatomic gases, now confirmed independently by the study of Messrs Kent and Frye on Argon, if we are to attribute it to the enhancing effect of the magnetic field on the atoms, this would mean that at higher pressures atoms predominate giving the Balmer series, while at lower pressures the molecules predominate giving the secondary spectrum, a conclusion in accordance with that arrived at by G. P. Thomson in his study of the spectrum of the hydrogen positive rays. The effect of the field in enhancing the secondary spectrum shows however that this division of share in the radiation is by no means clear cut between the atom and the molecule.

From the enhancement of the Balmer series lines observable at higher pressures it is possible that the atoms are mainly responsible for them and by the employment of a magnetic field in conjunction with Prof. Wood's† long discharge tube the lines of the Balmer series obtainable in the laboratory could probably be still further increased.

Experimenting with nitrogen having a constitution and spectrum considerably more complex than hydrogen the effects observed are very complicated. When a condensed spark is not employed to excite the tube, the effect of the field is to enhance the band spectrum and to bring in some of the lines belonging to the line spectrum of nitrogen with a different intensity distribution

* G. P. Thomson, *Phil. Mag.* vol. 40 (1920), p. 240.
† R. W. Wood, *Proc. Roy. Soc.* A, vol. 97, p. 455.

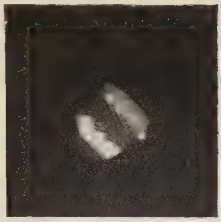


Fig. 2.

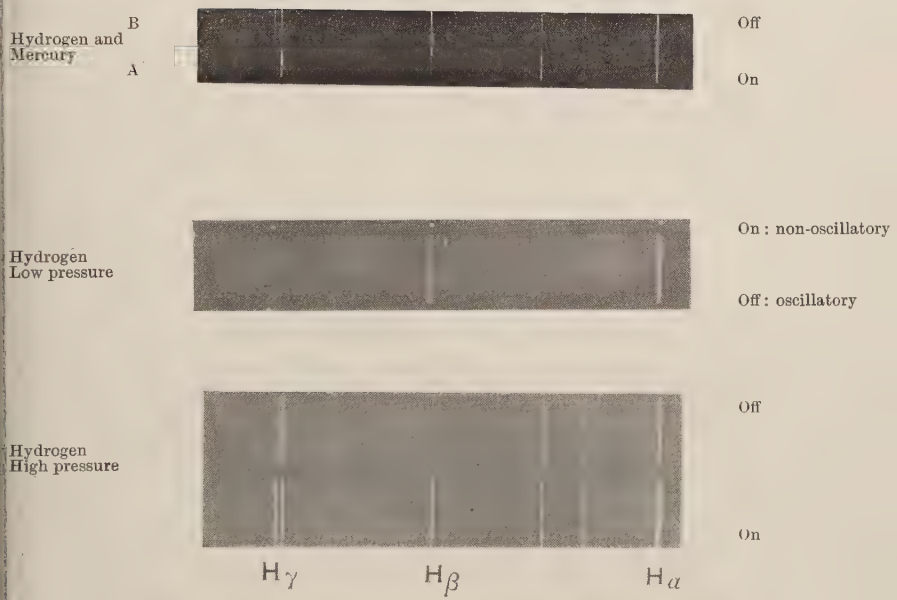

Hydrogen and Mercury
B
A
Off
On
Hydrogen Low pressure
On : non-oscillatory
Off : oscillatory
Hydrogen High pressure
Off
On
Hγ
Hβ
Hα
Nitrogen
On
Non-oscillatory discharge
Off
Nitrogen
On
Oscillatory discharge
Off

Fig. 4.

from the line spectra obtained with a condensed spark. The dense line at 4277·7 A.U. brought out very strongly by the field is generally absent without it, especially when the gas is not quite pure. With a condensed discharge it is rather faint. It has been catalogued as a negative pole band by Exner and Haschek*.

The phenomena attending the employment of the magnetic field are many and varied and complications arising from local variations of potential current, temperature and pressure known to occur there are difficult to eliminate entirely. But by independent variations of some of these factors it has not been possible to reproduce the phenomenon observed and hence the phenomena cannot be attributed entirely to these disturbing causes. It is possible that it might be due to some other influence brought about by the field. Further study of the phenomenon is in progress.

In conclusion I beg to express my indebtedness to the kind and sympathetic help of Professor Sir Ernest Rutherford throughout this work.

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CAMBRIDGE.

* Kayser, *Handbuch der Spectroscopie*, vol. 5, p. 826.

On a property of focal conics and of bicircular quartics. By C. V. Hanumanta Rao, University Professor, Lahore. (Communicated by Prof. H. F. Baker.)

[*Received* 20 April. *Read* 2 May 1921.]

The present note arises directly out of a note with the same title by Prof. H. F. Baker in the *Proceedings* (vol. xx. pp. 122–130). The property referred to in the title is that a varying circle, of one mode of generation, makes with two fixed circles, of a second mode, angles with a constant sum; and this result is here deduced from a particular case of it, viz. that where the curve consists of two circles. A preliminary series of results is inserted of some interest in themselves. By distances and angles will be meant throughout the Cayley separations, and the quadric or conic with respect to which the homographies are considered is indicated in each case.

1. Given two conics α, β in a plane, they have four common tangents meeting in three pairs of points. Indicate by V_1, V_2 one such pair of points. Then the sum of the distances, with respect to α, of $V_1 P$ and $V_2 P$, is constant as P moves on β; and this constant remains the same if α, β are interchanged.

The proof of this result depends on the existence of two fixed points on the line $V_1 V_2$, and this fact in turn is an easy consequence of the space figure of two conics with two common points.

Conversely, given a conic α and two fixed points V_1, V_2 in its plane, the locus of a point P, which moves so that the sum of the distances with respect to α of $V_1 P$ and $V_2 P$ is constant, is a conic β touching the four tangents from V_1, V_2 to α.

In particular, when one of the three pairs of points such as V_1, V_2 is projected into the circular points at infinity, the other two pairs are the foci and the conics are confocal. This leads to a slightly more general definition of confocal conics than the usual one, namely taking a fundamental conic Σ the system of conics touching any four arbitrary fixed tangents of Σ may be called a confocal system. Or again when the conic Σ is made to degenerate tangentially into the two circular points at infinity, we have the usual definition of confocal conics; it was this idea in fact which suggested the theorems of this note.

2. Precisely similar results hold in space of three dimensions. Given a quadric α and two points V_1, V_2, the locus of a point P such that the sum of the distances with respect to α of $V_1 P$ and $V_2 P$ is constant, is a quadric β which is enveloped by the enveloping

cones of α from V_1, V_2. In fact taking α as $\Sigma x^2 = 0$, the equation to β is of the form

$$\frac{(\Sigma x x_1)^2}{\Sigma x_1^2} - 2\cos C\,\frac{(\Sigma x x_1)(\Sigma x x_2)}{\sqrt{(\Sigma x_1^2)(\Sigma x_2^2)}} + \frac{(\Sigma x x_2)^2}{\Sigma x_2^2} = (\Sigma x^2)\sin^2 C,$$

where C is a constant, and this is clearly a quadric enveloped by the two cones like $(\Sigma x^2)(\Sigma x_1^2) = (\Sigma x x_1)^2$.

Conversely, given two quadrics α, β having two common enveloping cones from V_1, V_2, either quadric may be thought of as the locus of a point P such that the sum of the distances with respect to the other quadric, of $V_1 P$ and $V_2 P$, is constant; and the constant is the same whichever quadric is considered as the locus.

In particular, for some definite value of the constant the latter quadric will degenerate into a pair of planes, viz. the two planes of intersection of the two enveloping cones. Thus, given two conics β_1, β_2 intersecting in two points, V_1, V_2 indicating the vertices of the two cones through both of them, the locus of a point P such that the sum of the distances with respect to β_1, β_2, of $V_1 P$ and $V_2 P$, is constant, is a quadric enveloped by the two conès.

Taking the two points V_1, V_2 in the first result of this article to coincide, we find that if two quadrics have ring contact with V for the pole of the plane of contact, and if P be an arbitrary point on either quadric, then the distance VP with respect to the other quadric is a constant.

3. Reciprocally, take two quadrics having two common conics. Then an arbitrary tangent plane to either quadric makes with the planes of the two conics angles (measured with respect to the other quadric) whose sum is constant. Indicating the quadrics by

$$\Sigma x^2 + 2\lambda zt = 0, \quad \Sigma x^2 + 2\mu zt = 0,$$

the constant referred to is found to be $\arctan \dfrac{\sqrt{(1-\lambda^2)(\mu^2-1)}}{1-\lambda\mu}$.

In particular taking the two quadrics as a cone and a sphere, we have the well-known theorem that a varying tangent plane to a cone makes, with two circular sections of opposite systems, angles with a constant sum.

Conversely, given a quadric α and two arbitrary planes z, t, the envelope of a plane which makes with them two planes angles with a constant sum, is a quadric β intersecting the given quadric along the two given planes z, t.

We observe that in the above results the second quadric consists of two planes, and two conics having two common points represent the elliptic quartic curve which by projection is to yield the bicircular quartic. From this particular result we shall deduce the theorem in the general case.

For this purpose consider a sub-group of the possible positions of the moving plane, namely the planes all of which pass through a fixed point. They are then tangent planes to an enveloping cone of β, and we have the result that through the curve of intersection of this enveloping cone and the quadric α there passes another cone touched by the planes z, t. This is so in virtue of the following simple result:

Given two quadrics λ, μ with two common conics z, t, through the common curve of λ and any enveloping cone of μ there passes another cone touched by the planes of z, t. For if $S = 0$ and $S = zt$ be the two quadrics, any enveloping cone of the first is $SS' = P^2$ and meets the second in a curve lying on the cone $P^2 = S'zt$, which clearly has z, t for tangent planes.

We have thus established the well-known result for the common curve of two arbitrary quadrics, viz. that a varying tangent plane to one of the cones through this curve makes with two fixed tangent planes to a second such cone, angles with a constant sum.

4. Quadrics having ring contact are just concentric spheres, and two quadrics with two common conics are equivalent to two spheres. But no such complete projective reduction can be effected in the case of two arbitrary quadrics, and what we have done in Art. 3 is to establish a general result for the common curve of two arbitrary quadrics by deducing it from the particular case where one quadric degenerates into two planes.

Convex Solids in Higher Space. By Dr W. Burnside, Honorary Fellow of Pembroke College.

[*Received* 14 July 1921.]

Definitions. A set of linear $(n-1)$-spreads in n-dimensional space is said to be "general" when no $n+1$ of them meet in a point, no n in a line, no $n-1$ in a 2-spread, ..., and no three in an $(n-2)$-spread.

A set of points in n-dimensional space are said to be the internal points of a convex polyhedron, when each pair A and B of them satisfy the following conditions: (i) no point of the finite line AB lies on any one of a certain set of m $(n-1)$-spreads; (ii) the line AB produced from B meets at least one of the m $(n-1)$-spreads at a finite distance; (iii) the line BA produced from A meets at least one of the m $(n-1)$-spreads at a finite distance. If, in addition, it is always possible to choose B so that AB produced from B meets any assigned one of the m $(n-1)$-spreads before it meets any of the others, then each of the m $(n-1)$-spreads is said to form part of the boundary of the convex polyhedron.

Consider five 3-spreads A, B, C, D, E in 4-dimensional space of "general" position and such that no one of their five points of intersection is at infinity. Denote by a, b, c, d, e the points of intersection of B, C, D and E; ....; A, B, C and D. If e and q are on opposite sides of E, then eq produced from q does not meet A, B, C, D or E; and therefore q cannot be an internal point of a convex polyhedron bounded by the five 3-spreads. If e and q are on the same side of E, let eq meet E in p. In E the four points a, b, c, d are the vertices of a tetrahedron. If p is outside this tetrahedron, it must be separated from one of the vertices, say a, by the plane through the other three. Hence p and a, and therefore also q and a, are on opposite sides of A: and q cannot be an internal point of a convex polyhedron bounded by the five 3-spreads. It follows that the only points that can be internal points of a convex polyhedron bounded by the five 3-spreads are the points of the finite lines joining e to every internal point of the tetrahedron $abcd$: and these points clearly satisfy all the conditions.

Hence five 3-spreads of general position in 4-dimensional space, whose intersections are all finite points, form the boundary of just one convex polyhedron. An obvious extension of this reasoning shows that $n+1$ $(n-1)$-spreads in n-dimensional space, of general position, no one of whose intersections is at infinity, form the boundary of just one convex polyhedron.

There are just $\frac{1}{2}(n+1)(n+2)$ points of intersection and $\frac{1}{6}n(n+1)(n+2)$ lines of intersection of a general set of $(n-1)$-spreads, $n+2$ in number, in n-dimensional space. The lines pass n by n through the points and the points lie 3 by 3 on the lines. If each $(n-1)$-spread is denoted by a single symbol i, the point in which all the $(n-1)$-spreads except i and j meet may be denoted by ij and the line of intersection of all the $(n-1)$-spreads except i, j and k by ijk. The three points ij, ik, jk lie on the line ijk. If this configuration is projected from an arbitrary point of the n-dimensional space upon an arbitrary $(n-1)$-spread in it, the configuration becomes a like one in $(n-1)$-dimensional space. If the points of the original configuration are all finite points, the projection may clearly be carried out so that if ij is between ik and jk in the original configuration, the same is true after projection. Taking again an arbitrary point and an arbitrary $(n-2)$-spread in the $(n-1)$-dimensional space, the configuration may be projected into a like one in $(n-2)$-dimensional space; and the process may be continued. Moreover if all the points of the original configuration are finite points (so that for each set of three such as ij, ik, jk one is actually between the other two) and if each projection is carried out as suggested above, then in the final two-dimensional figure ij will be between ik and jk if it was so in the original configuration.

It will be said that ij and ik are opposite or adjacent according as jk is or is not between them. If ij, ik are adjacent and also ij and il, then ik and il are adjacent. Hence, with m single symbols, the $m-1$ points $i1, i2, \ldots . im$ may be divided into two sets such that all those of either set are adjacent, while any two taken one from each set are opposite. A suitable symbol to indicate the separation is 12 13 | 14 15 16, all those on either side of the bar being adjacent.

It will now be shown that, apart from permutation of the single symbols, there is just one scheme for the separations of the set of $\frac{1}{2}m(m-1)$ points arising from m symbols.

When n is odd, 7 for example, the typical separation is

| 12	13	14	15	16	17
| 21	23	24	25	26	27
13 |	23	34	35	36	37
24 |	14	34	45	46	47
15	35 |	25	45	56	57
26	46 |	16	36	56	67
17	37	57 |	27	47	67

.....(i),

which is associated in an obvious way with the symbol

$$\{12\}\{34\}\{56\}\{7\}.$$

When n is even, say 8, the typical separation is

$$\begin{array}{rrrrrrr}
| & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
| & 21 & 23 & 24 & 25 & 26 & 27 & 28 \\
13 & | & 23 & 34 & 35 & 36 & 37 & 38 \\
24 & | & 14 & 34 & 45 & 46 & 47 & 48 \\
15 & 35 & | & 25 & 45 & 56 & 57 & 58 \\
26 & 46 & | & 16 & 36 & 56 & 67 & 68 \\
17 & 37 & 57 & | & 27 & 47 & 67 & 78 \\
28 & 48 & 68 & | & 18 & 38 & 58 & 78
\end{array} \qquad \ldots\ldots(\text{ii}),$$

which is associated in a similar way with the symbol

$$\{12\}\{34\}\{56\}\{78\}.$$

There is no difficulty in verifying that for three symbols and for four symbols, all possible separations arise from these by permutations of the single symbols. It will be shown here that if the scheme is general for 7 it is general for 8. If the proof is examined it will be quite clear that the same method may be used for any two consecutive numbers. Assuming that the separation of 21 points is given by the scheme (i), the question to be settled is how the seven points 18, 28, ... , 78 fit into it and how they are separated among themselves.

Suppose for instance that 48 is the first of the new symbols that occurs to the left of the bar. Then 48 is opposite to 45, 46 and 47; so that 45 and 58, 46 and 68, and 47 and 78 are adjacent. It follows that 68 occurs to the left of the bar and 58, 78 to the right. Since 18, 28 are both adjacent to 12, 28 is opposite to 18. So 23, 28 and 23, 38 being adjacent, 28 and 38 are opposite: 24, 28 and 24, 48 being adjacent, 28 and 48 are opposite. In a similar way 28 is shown to be opposite to 58, 68, 78. The scheme that thus arises for the 28 points is

$$\begin{array}{rrrrrrr}
| & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
| & 21 & 23 & 24 & 25 & 26 & 27 & 28 \\
13 & | & 23 & 34 & 35 & 36 & 37 & 38 \\
24 & 48 & | & 14 & 34 & 45 & 46 & 47 \\
15 & 35 & | & 25 & 45 & 56 & 57 & 58 \\
26 & 46 & 68 & | & 16 & 36 & 56 & 67 \\
17 & 37 & 57 & | & 27 & 47 & 67 & 78 \\
28 & | & 18 & 38 & 48 & 58 & 68 & 78
\end{array}$$

Comparing this with scheme (ii) it is changed into the latter by the permutation (846), the other single letters remaining unchanged. Similarly if 38 is the first to occur to the left of the bar the scheme that arises is changed in (ii) by the permutation (8357).

The general statement is that if, in forming a scheme for $2n$ from the standard scheme for $2n-1$, the first of the new symbols that enters to the left of the bar is $2r+1$, $2n$, then the permutation

$$(2n, 2r+1, 2r+3, \ldots, 2n-1)$$

will change the resulting scheme for $2n$ into the standard scheme for $2n$: and that if $2r$, $2n$ is the first that enters to the left of the bar, the permutation

$$(2n, 2r, 2r+2, \ldots, 2n-2)$$

has a similar effect. There is no difficulty in establishing similar results for building up the scheme for $2n+1$ from that for $2n$.

Returning now to the configuration of $n+2$ $(n-1)$-spreads in space of n dimensions, of general position, whose $\frac{1}{2}(n+1)(n+2)$ points of intersection are all finite points, the separation

$$ia, ib, ic, \ldots . | id, ie, \ldots .$$

implies that of the $n+1$ points which do not lie on the ith $(n-1)$-spread, the set ia, ib, ic,... are separated by the ith $(n-1)$-spread from the set id, ie,....

In particular for five planes in ordinary space the scheme is

| 12	13	14	15
| 21	23	24	25
13 |	23	34	35
24 |	14	34	45
15	35 |	25	45

The first line implies that plane 1 does not divide the tetrahedron 2345 into two parts; and the second line implies that plane 2 does not divide the tetrahedron 1345 into two parts.

The third line implies that plane 3 does divide the tetrahedron 1245 into two parts, one of which is the tetrahedron 2345, while the other is a polyhedron with vertices 12, 14, 15, 23, 34, 45 bounded by each of the five planes. Similarly plane 4 divides the tetrahedron 1235 into the tetrahedron 1345 and a polyhedron with vertices 12, 13, 25, 14, 35, 45 bounded by each of the five planes. Plane 5 divides the tetrahedron 1234 into two polyhedra, each of which is bounded by all the five planes, 12, 14, 23, 34, 15, 35 being the vertices of one and 12, 14, 23, 34, 25, 45 those of the other.

Hence any five planes of general position in space, whose points of intersection are all finite, form the five faces of just two distinct convex polyhedra each of which has six vertices. The scheme for six 3-spreads in 4-dimensional space may be similarly dealt with. Any five of the 3-spreads bound a convex polyhedron, and the scheme shows how this polyhedron is divided by the remaining 3-spread. The result shows that there are just three distinct convex

polyhedra each of which is bounded by all the six 3-spreads. Two of these have eight vertices, viz.

12 23 25 26 14 34 45 46
12 14 15 16 23 34 35 36

It is obvious that two of the 3-dimensional faces of these are tetrahedra and the other four polyhedra with five faces and six vertices.

The remaining one has nine vertices, viz.

12 14 16 23 25 34 36 45 56

Its 3-dimensional faces are all polyhedra with six vertices and five faces.

A similar examination of the next scheme shows that seven 4-spreads in 5-dimensional space bound just four distinct convex polyhedra. The vertices are

23 24 37 47 12 17 25 26 57 67.
23 24 37 47 13 14 35 45 36 46
13 14 16 23 24 26 37 47 67 15 25 57
13 14 16 23 24 26 37 47 67 35 45 56

Note on the Velocity of X-ray Electrons. By R. Whiddington, M.A.

[*Received* 5 August 1921.]

It has been known for many years that X-rays have the power of ejecting high speed electrons from the surface of materials on which they are incident.

The earliest attempts to determine the velocities of these electrons were made by Dorn in 1900 and Innes in 1907. Their general results being that the velocity, as measured by photographic records of the curvature in a known magnetic field, lay between 6 and 8×10^9 cm./sec. and was independent of the intensity of the exciting radiation.

In 1912 the problem was attacked from a different angle, by interpreting the absorption experiments of Beatty. It had previously been shown that a fourth power law of velocity absorption was quite fairly accurately true for fast moving electrons such as those ejected by X-rays*.

Beatty having measured, in essence, the range in air of the electrons ejected by various qualities of X-ray it was therefore possible to deduce their velocity*.

In the paper cited it was shown that the velocity of the *fastest* electrons ejected by X-rays was very nearly equal to $10^8 \times A$, where A is the atomic weight of the radiator supplying the X-rays. It was further predicted that future work should show the existence of two sets of electrons of definite speeds.

In the early months of this year de Broglie† published a short account of some highly important results in which by using a Coolidge X-ray tube he was able to take magnetic photographs of X-ray electrons with only an hour's exposure.

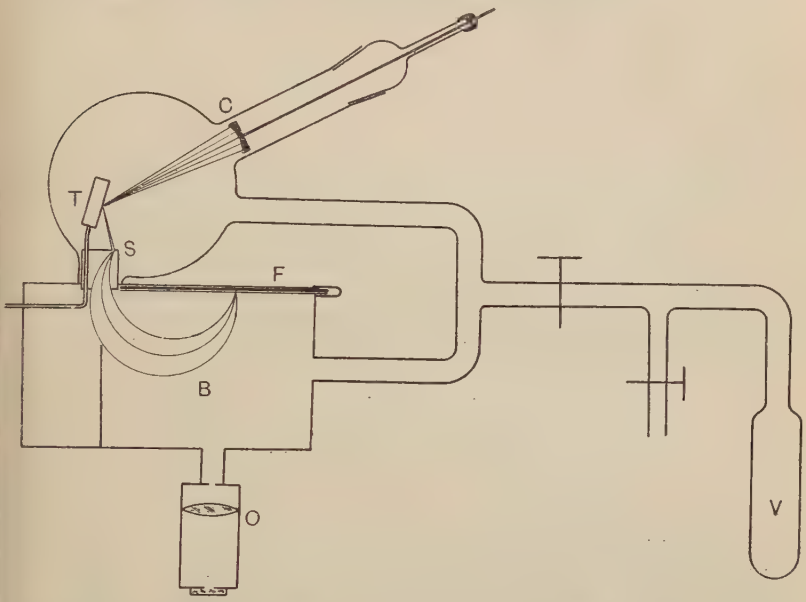
The present experiments were then in progress and it seems worth while now to publish a preliminary account of the apparatus finally adopted and initial results obtained.

The problem resolves itself into obtaining as strong a source of X-rays as possible, causing them to pass through a thin sheet of solid matter and observing the magnetic spectrum on the emergent side. De Broglie used a Coolidge tube to provide a powerful source of X-rays, a method which commends itself on the grounds of simplicity and ease of working.

The first attempts I made were with an apparatus consisting of a fine slit covered with thin metal foil on which Cathode-rays were focussed, the electrons ejected from the emergent side being subject to magnetic deviation on a photographic film. Photographs

* Whiddington, *Proc. Roy. Soc.* 1912.
† De Broglie, *Comptes Rendus*, 1921.

were obtained after considerable difficulty, but the method was abandoned since it was found that in spite of all precautions the thin foil rapidly disintegrated, wearing thin and eventually breaking. The apparatus finally adopted is shown in the figure.



Cathode-rays from the concave cathode C are focussed on the water-cooled target T. Immediately below T is the fine slit S (·5 mm. wide and 5 mm. long). About 2 cm. below this slit is another wider slit leading the rays into the evacuated box B on the under surface of which a photographic film F can be placed. The collimator O projects a little spot of light on the film for reference purposes. The whole system of X-ray bulb and camera is evacuated by a liquid air charcoal bulb V.

At right angles to the plane of the figure a nearly uniform magnetic field is applied so that the X-ray electrons streaming into the box are focussed on the film. This method was originally used by Rutherford for the determination of the speed of β-rays, was later used by Rawlinson and Robinson* in an experiment on X-ray electrons and was also used by de Broglie† in the experiments just cited.

Under good conditions as much as five milliamperes can be passed through the bulb and it is interesting to compare the efficiency of the arrangement above illustrated with that of de Broglie using a Coolidge tube.

* *Phil. Mag.* 1913. † *Loc. cit.*

If the distance between T and S is 1 cm., and we assume that the distance between the target of a Coolidge tube and the slit is (say) 8 cm., it follows that with the arrangement above, there is an available intensity 64 times as great.

Even if only 1 milliampere is used the available intensity will be more than ten times as great.

One disadvantage of the arrangement is that the coils producing a field in B, produce a small but appreciable magnetic field in the region STC.

The result is that the cathode-rays from C are deflected. This deflexion must be balanced out by an additional compensating coil in series with the main coils. It is fortunate in this connexion that it is the fastest rays arriving at T which are the most effective X-ray parents and which are least affected by the stray field*.

The photographic film used was 10 cm. long and the current in the field coils was adjusted to give a range of velocities from 3×10^9 to 10^{10} cm./sec. approximately, yielding a dispersion of $0{\cdot}071 \times 10^9$ cm./sec. per millimetre.

It is possible to take a photograph with the apparatus in half an hour although longer exposures are desirable. Using a platinum or rhodium target, for example, and a copper foil over the slit, the strongest lines on the film correspond to velocities $6{\cdot}02 \times 10^9$ and $5{\cdot}74 \times 10^9$ cm./sec.

If we apply the quantum relation to the X-ray K doublet of copper we get the corresponding velocities of electrons carrying the same energy to be $5{\cdot}61 \times 10^9$ and $5{\cdot}33 \times 10^9$ cm./sec.

It is interesting to note that while the differences are precisely the same in both experimental and calculated cases, the actual values differ by about 10 per cent. Whether this difference is real or due to a defect somewhere in the apparatus must be determined by further investigation.

It is worth remembering, however, that both the present writer and Hull† found that velocities of parent electrons distinctly in excess of the value demanded by the quantum relation were required for the production of a fluorescent radiation of any definite wave length.

It seems possible that the difference if it be real may be accounted for by taking into account the sums of the individual energies required for the simultaneous excitation of the radiations of both K and L series.

It is hoped to publish a full and extended account of this work very shortly.

* See also Whiddington, *Proc. Roy. Soc.* 1911.
† *Physical Review*, 1916.

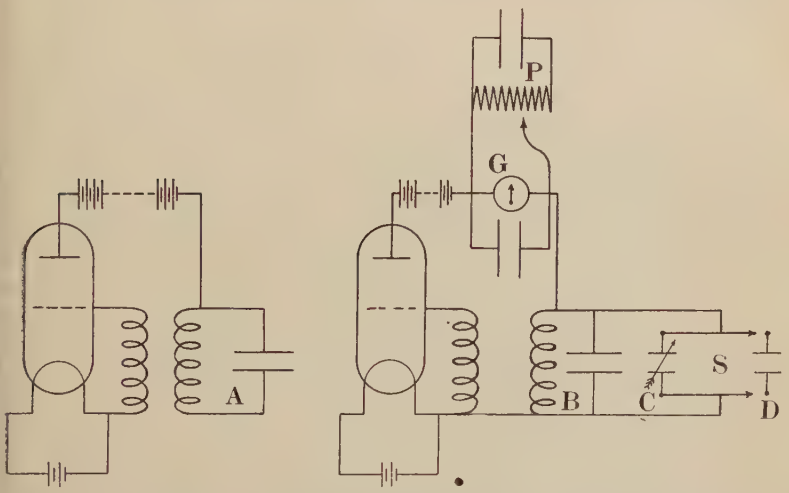
A Laboratory Valve method for determining the Specific Inductive Capacities of Liquids. By R. Whiddington, M.A.

[*Received* 30 July 1921.]

The method outlined below has been used successfully by students as a laboratory exercise in the Physics Laboratories of Leeds University.

The method employs alternating electro-motive forces of low frequency generated by a thermionic valve linked with the usual reacting circuits.

The apparatus is shown diagrammatically below. On the left is shown a standard circuit generating oscillations in the closed circuit *A* consisting of a fixed condenser of about ·5 m.f.d. and a large air core coil. The natural frequency of this circuit is about 1000.



Very loosely linked to this circuit is a similar one *B* also comprising an air core coil and condenser box, but in parallel with the latter is arranged a small variable condenser *C* and a mercury cup switch *S* so that an additional small condenser *D* can be switched in at will. *C* is a moving vane air condenser fitted with pointer and scale, *D* is a small parallel plate condenser the dielectric medium of which can be easily changed without altering the distance between the plates. Included in the anode circuit of this arrangement is a condenser shunted, and an aperiodic needle galvanometer *G* connected as shown to a potentiometer *P* to balance the steady current flowing.

This galvanometer indicates in a very convenient manner the slow beats set up when A and B are suitably adjusted.

To determine the capacity of D it is merely necessary to adjust C until a definite beat rate between A and B is established and then switch in D. A new beat rate will be set up which can be restored to the original rate by decreasing C, until the decrease in C is equal to the additional capacity of D. The experiment is then repeated with a liquid dielectric in D, the ratio of the two capacities being the Specific Inductive Capacity required.

It is to be noticed that the actual capacity change indicated by C is not required so long as the scale is known to be uniform by previous calibration.

It was found in the case of the particular condenser used in the position C that the scale was for all practical purposes uniform over the central portion of its scale, each division representing ·0000065 m.f.d.

In an experiment carried out by Mr L. G. Stanton the following results were obtained using olive oil as the dielectric.

Dielectric Air.

D in	D out	Difference	
116	160	44	Mean difference 44
118	162	44	

Dielectric Olive Oil.

D in	D out	Difference	
36	166	130	Mean difference 131
32	164	132	

The Specific Inductive Capacity of the oil is therefore 2·98 which is very close to that usually given in Physical Tables.

The Theoretical Value of Sutherland's Constant in the Kinetic Theory of Gases. By C. G. F. James, Trinity College, Cambridge. (Communicated by Mr R. H. Fowler.)

[*Received* 28 April: *read* 2 May 1921.]

§ 1. In any attempt to deduce, from observations on viscosity and diffusion for gases, facts as to the nature of the intermolecular forces, it is necessary to find the theoretical relation that holds between the so-called Sutherland's constants, and any assumed intermolecular force. It is assumed that the molecule behaves as a perfectly elastic sphere surrounded by a field of attractive force. This is in fact the only known model, capable of satisfactorily predicting the observed laws of variation with temperature of these quantities, at least at ordinary temperatures. The relations in question have been worked out by Professor Chapman in various papers*. His formulae for Sutherland's constants are given on p. 459 of his first paper on the subject.

It appears however that the formulae in question are affected by a certain error explained below. This mistake affects the relations between Sutherland's constant S for a single gas, and analogous quantities, and the potential of the intermolecular field. It must be definitely understood however that this in no way affects the rest of his theory, or the numerical values of σ, the molecular diameter, obtained. These are in fact deduced directly from observed values of S. It is only when theoretical values of S become important, that this mistake is of any significance.

It was suggested to me by Mr Fowler that the correct theoretical determination of this constant is of considerable importance. This, then, forms the subject of this paper.

It is known, that with fair accuracy the relation between the viscosity μ of a single gas and the temperature T is, when T is sufficiently large,

$$\mu \propto T^{\frac{1}{2}}/(1 + S/T),$$

where S is Sutherland's constant; and can be calculated in terms of the forces in action. Thus if $\phi(r)$ is the potential of the force between two molecules whose distance apart is r, Prof. Chapman's result was

$$S = \phi(\sigma)/3R,$$

R being the usual gas constant for one molecule ($1{\cdot}372 \times 10^{-6}$).

* *Phil. Trans.* (A), vol. 211 (1912), p. 432; vol. 216 (1915), p. 276; vol. 217 (1917), p. 115.

The correct result is not so simple as this, and cannot be expressed so generally.

In the same way, the coefficient of diffusion for a mixture of two gases satisfies a relation

$$D_{12} \propto T^{\frac{3}{2}}/(1 + S_{12}/T),$$

where if σ_{12} denotes the mean of the two molecular diameters,

$$S_{12} = \lambda\phi\,(\sigma_{12})/R,$$

where λ is a number depending on the law of force assumed. Prof. Chapman gives $\lambda = \frac{1}{2}$ for all laws of force.

Prof. Chapman informs me that Enskog* has already pointed out the necessity for the corrections here referred to, and that his results agree with those found in § 6. The results of the other sections have not, I understand, been given by him.

§ 2. *A note on the approximations employed.* It is necessary to regard the absolute temperature T as large, since the theory gives for the denominators of μ and D_{12} series in descending powers of T. We shall also neglect squares and higher powers of $\phi\,(\sigma)$. Actually it will be seen that all terms involving $\phi\,(\sigma)$ to any power are multiplied by $1/T$ to the same power. Thus the assumption is really that $\phi\,(\sigma)/T$ is negligible in higher powers than the first.

As regards the terms involving $\{\phi(\sigma)/T\}^2$ it will be found that Prof. Chapman's statement, that these are positive in each case, is not affected.

§ 3. *Statement of the Problem.* The first step is the determination of the deflection of a typical molecule, relative to a selected molecule which is conveniently supposed reduced to rest. In the diffusion problem these molecules will be of opposite kinds. In the viscosity case we will for simplicity consider a single gas only.

To the order of approximation explained above it is only necessary to consider molecules that actually suffer impact with the selected molecule. Let A and B be the centres of the respective molecules at impact, so that in the general case:

$$AB = \sigma_{12} = \tfrac{1}{2}\,(\sigma_1 + \sigma_2).$$

Let TN, NT' be the asymptotes of the initial and final relative paths; QB, BQ' the directions of motion immediately before and immediately after impact. Let V be the relative velocity at a great distance, p the perpendicular onto the asymptotes. The molecule B is typified by the direction and magnitude of V, by p, and by an azimuthal angle ϵ, determining the direction of p, in a plane at right angles to V. The true deflection required is

$$2\chi = T_1\hat{N}T',$$

* Inaugural Dissertation, Uppsala, 1917.

while the angle taken by Prof. Chapman is

$$2\chi' = Q_1\hat{B}Q'.$$

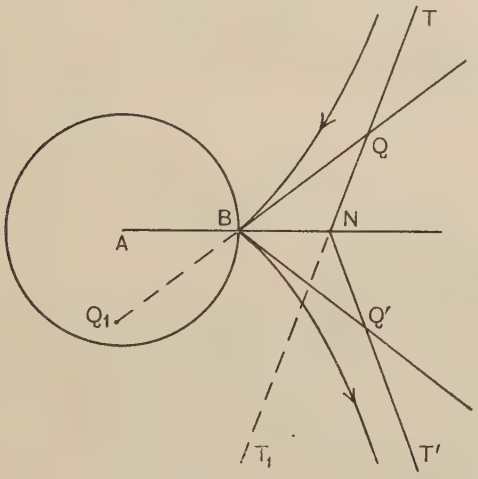
We propose to calculate the effect of this difference on the viscosity and diffusion problems. Two laws of force will be considered:

(1) The inverse power law.

(2) A shell of constant force from $r = \sigma_{12}$ to $r = d = \alpha\sigma_{12}$, where α is a moderately small number.

§ 4. *Calculation of the Correction in the angle* χ. This correction is seen from the figure to be negative and equal in magnitude to

$$\psi = B\hat{Q}N.$$



Let $\{-f(r)\}$ be the acceleration of B in the relative orbit. Thus $\phi(r)$ being the potential of the intermolecular field,

$$f(r) = -\frac{d\Phi(r)}{dr} = -\frac{m_1 + m_2}{m_1 m_2}\frac{d\phi(r)}{dr},$$

$\Phi(r)$ being introduced for convenience merely. In our case, the forces being attractive, $f(r)$ and $\Phi(r)$ are positive.

Let ϖ be the perpendicular onto the tangent at any point. Then we have

$$\frac{d}{dr}\left(\frac{1}{\varpi^2}\right) = -\frac{2}{c^2}f(r),$$

where c is the angular momentum pV. Integrating:

$$\frac{1}{\varpi^2} = \frac{1}{p^2}\left(1 + \frac{2\Phi(r)}{V^2}\right).$$

Now, if ρ is the radius of curvature at any point of the path, and μ the inclination of the radius vector to the tangent,

$$-f(r) = \frac{v^2}{\rho \sin \mu} = \frac{V^2 p^2 r}{\varpi^3 \rho} = \frac{V^2 r}{\rho p}\left\{1 + \frac{2}{V^2}\Phi(r)\right\}^{\frac{3}{2}},$$

or to the first order

$$\frac{1}{\rho} = -\frac{p f(r)}{V^2 r}.$$

If s is the arc of the path this is

$$\frac{d\psi}{ds} = -\frac{p f(r)}{V^2 r},$$

leading to

$$\psi = -\frac{p}{V^2}\int_\infty \frac{f(r)}{r}\frac{ds}{dr}\,dr.$$

Also, from the equation

$$\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2 = \frac{2r^2}{p^2 V^2}\{\tfrac{1}{2}V^2 + \Phi(r)\} - 1,$$

we obtain

$$\frac{ds}{dr} = \left\{\frac{\{V^2 + 2\Phi(r)\}\, r^2}{r^2\{V^2 + 2\Phi(r)\} - p^2 V^2}\right\}^{\frac{1}{2}},$$

which to the first order is

$$\frac{ds}{dr} = \frac{r}{(r^2 - p^2)^{\frac{1}{2}}},$$

whence

$$\psi(\sigma) = \frac{p}{V^2}\int_\sigma^\infty \frac{f(r)\,dr}{(r^2 - p^2)^{\frac{1}{2}}}.$$

It will be observed that $f(r)$ may be of any order of magnitude, provided only $\Phi(r)$ remains small.

§ 5. *The Viscosity of a Single Gas. General Analysis.* On reference to the papers cited* it will be seen that we have now to evaluate

$$\Omega''(V) = \pi V \int_0^{p_\nu} \sin^2 2\chi\, p\,dp,$$

where p_ν is the greatest value of p for orbits involving collision, so that $p_\nu = \sigma\{1 + 2\Phi(\sigma)/V^2\}^{\frac{1}{2}}$. To our order we may put $p_\nu = \sigma$. Putting $\chi = \chi' - \psi$, and expanding in ψ to the first power only,

$$\Delta\Omega''(V) = -\pi V \int_0^\sigma 4\psi \sin 2\chi' \cos 2\chi' p\,dp.$$

* First Paper, p. 453: $\Omega''(V)$ is denoted there by $\Omega''_{11}(V_0)$. See also pp. 457, 458.

Now $\cos\chi' = p/p_\nu = p/\sigma = t$ (say). This substitution gives

$$\Delta\Omega''(V) = -8\sigma^2\pi V\int_0^1 \psi t^2(1-t^2)^{\frac{1}{2}}(2t^2-1)\,dt.$$

It is now necessary to specialize the law of force.

§ 6. *The Inverse Power Law.* We take

$$f(r) = k/r^n,\quad \Phi(r) = k/(n-1)r^{n-1},$$

giving

$$\psi = \frac{kp}{V^2}\int_\sigma^\infty \left\{\sum_0^\infty s_r\left(\frac{p}{r}\right)^{2r}\right\}\frac{dr}{r^{n+1}},$$

where s_r is the coefficient of ξ^r in the expansion of $(1-\xi)^{-\frac{1}{2}}$. Thus

$$\psi = \frac{kp}{V^2}\sum_0^\infty \frac{s_r}{n+2r}\frac{p^{2r}}{\sigma^{n+2r}} = \frac{k}{V^2\sigma^{n-1}}\sum_0^\infty \frac{s_r}{n+2r}t^{2r+1},$$

$$\Delta\Omega''(V) = -\frac{8\pi kV}{V^2\sigma^{n-3}}\int_0^1\sum_0^\infty t^{2r+3}\frac{s_r}{n+2r}(1-t^2)^{\frac{1}{2}}(2t^2-1)\,dt,$$

which readily evaluates to

$$\Delta\Omega''(V) = -\frac{16\pi k}{V\sigma^{n-3}}h_n,$$

where h_n is the sum of the series $\sum_0^\infty \dfrac{r+1}{(2r+7)(2r+5)(2r+3)(2r+n)}$.

The next step is the evaluation of

$$\begin{aligned}\Delta R'' &= \int_0^\infty \Delta\Omega''(V)\,V^6\,e^{-\frac{1}{2}hmV^2}\,dV\\ &= -\frac{16k\pi h_n}{\sigma^{n-3}}\int_0^\infty V^5\,e^{-\frac{1}{2}hmV^2}\,dV\\ &= -\frac{16k\pi h_n}{\sigma^{n-3}}\left(\frac{2}{hm}\right)^3.\end{aligned}$$

Now $\Phi(\sigma) = k/(n-1)\sigma^{n-1}$. Substituting,

$$\Delta R'' = -\frac{16\pi\sigma^2}{(hm)^4}\left\{\frac{hm\,\Phi(\sigma)}{3}h_n'\right\},$$

where $h_n' = 24(n-1)\sum_0^\infty \dfrac{r+1}{(2r+7)(2r+5)(2r+3)(2r+n)}$.

This will be found to lead to

$$\mu = \frac{5m}{16\sigma^2\pi^2}\left(\frac{RT}{m}\right)^{\frac{1}{2}}\Big/\left\{1+\frac{hm\,\Phi(\sigma)}{3}(1-h_n')\right\}.$$

Now $h = 1/2RT$, $\Phi(\sigma) = 2\phi(\sigma)/m$. Thus

$$S = \frac{\phi(\sigma)}{R}\frac{1-h_n'}{3}.$$

The numerical values of $RS/\phi(\sigma)$ are as follows:

n	4	5	6	7	8	9
$RS/\phi(\sigma)$	0·213	0·196	0·183	0·172	0·163	0·156

Prof. Chapman's value is 0·333 in all cases. The calculation is very simple for n odd, from $n = 9$ (inclusive). The other values are more troublesome. The value for $n = 8$ was interpolated merely.

§ 7. We proceed now to the same calculation for the law of force.

$$f(r) = A, \quad \sigma \leqslant r \leqslant \alpha\sigma,$$
$$f(r) = 0, \quad r > \alpha\sigma.$$

Thus
$$\Phi(\sigma) = A\sigma(\alpha - 1).$$

Hence
$$\psi = \frac{Ap}{V^2}\int_\sigma^{\alpha\sigma} \frac{dr}{(r^2 - p^2)^{\frac{1}{2}}} = \frac{Ap}{V^2}\int_\sigma^{\alpha\sigma} \sum_0^\infty s_r \left(\frac{p}{r}\right)^{2r} \frac{dr}{r},$$

$$\psi = \frac{Ap}{V^2}\left\{\log\alpha - \left[\sum_1^\infty \frac{s_r}{2r}\frac{p^{2r}}{r^{2r}}\right]_\sigma^{\alpha\sigma}\right\}$$

$$= \frac{A\sigma}{V^2}\left\{t\log\alpha + \sum_1^\infty \frac{s_r}{2r} t^{2r+1}\left[1 - \frac{1}{\alpha^{2r}}\right]\right\}.$$

This leads to
$$\Delta\Omega''(V) = 16A\sigma^3\pi h_\alpha/V,$$

where
$$h_\alpha = \sum_1^\infty \frac{(r+1)}{2r(2r+3)(2r+5)(2r+7)}\left\{1 - \frac{1}{\alpha^{2r}}\right\} + \frac{1}{105}\log\alpha.$$

By comparison with § 6 we can at once write down the number corresponding to h_α', namely
$$h_\alpha' = 24h_\alpha/(\alpha - 1).$$

In this way we draw up the table

α	1·25	1·50	1·75	2·00	2·50	3	4
$RS/\phi(\sigma)$	0·166	0·206	0·227	0·241	0·259	0·270	0·283

α	5	6	7	8	9	10
$RS/\phi(\sigma)$	0·291	0·297	0·301	0·304	0·307	0·309

§ 8. For a given value of $\phi(\sigma)$, Sutherland's constant S decreases, so that finally for a thin shell of intense force it vanishes. The result is thus, to this order at any rate, the same as for elastic spheres exerting no force.

Thus suppose $f(r) = 0$ for $r > \sigma + \epsilon$, while $\Phi(\sigma) = \int_{\sigma+\epsilon}^{\sigma} f(r)\,dr$ remains finite. This gives without approximation

$$\psi = \int_{\sigma}^{\sigma+\epsilon} \frac{p}{V^2} \frac{f(r)}{r} \left\{1 + \frac{2\Phi(\sigma)}{V^2}\right\}^{-\frac{3}{2}} \frac{ds}{dr}\, dr$$

$$= \frac{p\sigma^2}{V^2 p_\nu{}^3} \operatorname{cosec} \chi',$$

in the limit $\epsilon \to 0$. Approximately

$$\Delta\Omega''(V) = -\, 8\pi\sigma^2 \Phi(\sigma)/V \int_0^1 (2t^2 - 1)\, t^3\, dt$$

$$= -\, 2\sigma^2 \Phi(\sigma)/3V,$$

whence

$$\Delta R'' = -\, \frac{16\pi\sigma^2}{(hm)^4} \left\{\frac{hm\, \Phi(\sigma)}{3}\right\},$$

and it will be seen that this leads to $S = 0$. The approximation consists only in the neglect of higher terms in the expansion of $\sin^2 2(\chi' - \psi)$. These terms will not disappear.

§ 9. *The Second Order Terms.* It will be sufficiently typical to consider only the inverse power law. These terms arise in five ways.

I. From the second term in the expansion of $\psi(\sigma)$.

II. From the first term in the expansion of $\psi(\sigma)$; and the second term in the expansion of $\left(\frac{ds}{dr}\right)_{r=\sigma}$.

III. From the first term in $\psi(\sigma)$ arising from the approximate value of the upper limit p_ν. This is conveniently dealt with simultaneously with the last.

IV. From the ψ^2 term in the expansion of $\sin^2 2(\chi' - \psi)$.

V. From the deflection of molecules, which do not actually suffer collision.

These and these only lead to terms of type $1/T^2$ in the denominator of the expression for μ. I have verified by actual calculation that taken together they give a term of the type $K\{\phi(\sigma)/RT\}^2$, where K is a positive number.

§ 10. *Diffusion of a Mixture of Two Gases.* We have to calculate*

$$\Omega'(V) = 4\pi V \int_0^{p_\nu} \sin^2 \chi\, p\, dp,$$

and $\sin^2 \chi = \sin^2 \chi' - 2\psi \sin \chi' \cos \chi'$, so that

$$\Delta\Omega'(V) = 8\pi V \int_0^{p_\nu} \psi \sin \chi' \cos \chi'\, p\, dp.$$

For the inverse power law we find

$$\Delta\Omega'(V) = -\, 8\pi k H_n / V_\sigma{}^{m-3},$$

* $\Omega'(V)$ is Prof. Chapman's $\Omega'_{12}(V_0)$.

where $$H_n = \sum_0^\infty \frac{2(r+1)}{(2r+1)(2r+3)(2r+5)(2r+n)}.$$

The next step in the integration is to calculate

$$\Delta P'_{12} = -\int_0^\infty V^4 \Delta\Omega'(V) e^{-\xi V^2} dV, \quad \text{where } \xi = -\frac{hm_1m_2}{m_1+m_2},$$

or $$\Delta P'_{12} = -\frac{4\pi k H_n}{\sigma^{n-3}} \left\{\frac{hm_1m_2}{m_1+m_2}\right\}^{-2}.$$

The coefficient of diffusion D_{12} contains a factor $\left(1+\frac{S_{12}}{T}\right)^{-1}$, namely

$$D_{12} = \frac{3}{16}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sigma^2_{12}} \left(\frac{RT}{h}\right)^{\frac{3}{2}} \left(\frac{1}{m_1}+\frac{1}{m_2}\right)(1+S_{12}/T)^{-1},$$

and on reference to the papers cited, together with the above analysis, it will be seen that

$$S_{12} = \phi(\sigma)\{1-4(n-1)H_n\}/2R.$$

By comparison with the above, and with the viscosity calculations, we can at once write down, for the shell of force case

$$S_{12} = \phi(\sigma)\{1-4H_\alpha/(\alpha-1)\}/2R,$$

where

$$H_\alpha = \left[\sum_1^\infty \frac{2(r+1)}{(2r+1)(2r+3)(2r+5)}\left\{1-\frac{1}{\alpha^{2r}}\right\} + \frac{2}{105}\log\alpha\right].$$

The values of $RS_{12}/\phi(\sigma)$ are given in the following tables:

Inverse Power Law.

n	4	5	6	7	8	9
$RS_{12}/\phi(\sigma)$	0·227	0·201	0·182	0·167	0·154	0·144

The numerical calculation becomes simple for n odd, from $n = 7$ inclusive. $n = 8$ was interpolated merely.

The Shell of Constant Force.

α	1·25	1·50	1·75	2	2·5	3	4
$RS_{12}/\phi(\sigma)$	0·153	0·208	0·243	0·268	0·303	0·327	0·358

α	5	6	7	8	9	10
$RS_{12}/\phi(\sigma)$	0·378	0·393	0·404	0·413	0·420	0·425

Prof. Chapman's value was 0·5 in all cases. That this sequence tends to zero as α tends to unity can be verified as before.

It can also be verified, in the same way, that the next term in the series in inverse powers of T is still positive.

On the Stability of the Steady Motion of viscous liquid contained between two rotating coaxal circular cylinders. By W. J. Harrison, M.A., Fellow of Clare College, Cambridge.

[*Received* 29 August 1921.]

In the previous paper* on this subject an error has been made which invalidates the results given in Part I. The solution given in equation (26) satisfies equation (25), but is not sufficiently general to provide a solution of the problem, as it makes two values of m identical. This error has been pointed out by Prof. W. McF. Orr. Between equations (4) and (12) there are also various errors and misprints which are misleading, although they do not affect the subsequent work. A brief statement of these and a slight modification of the method of obtaining equation (17) will be given first.

The equation following (4) is correct. Integrating the last three integrals by parts we obtain the equation preceding (5), except that the final expression in it should be written

$$-\int\left[p_{xx}\frac{\partial u}{\partial x}+p_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+p_{yy}\frac{\partial v}{\partial y}\right]d\tau.$$

Equation (5) is correct after making the same alteration. Equations (6) should be written

$$p_{xx}=-p-\tfrac{2}{3}\mu\,\mathrm{div}\,q+2\mu\frac{\partial u}{\partial x},\ \text{etc.},$$

$$p_{xy}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).$$

In equation (7), $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2$ should be replaced by $\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2$, and (8) should be written $p'=p+\frac{2}{3}\mu\,\mathrm{div}\,q$.

The equation giving the critical value of μ for a given disturbance is obtained by equating the left-hand side of equation (7) to zero. Varying u, v, μ in this equation, putting $\delta\mu=0$, and integrating by parts where necessary, we obtain equations determining a state of disturbed motion such that the relative kinetic energy is stationary, μ being at the same time stationary and a maximum. After performing these operations the fluid is treated as incompressible.

Thus we arrive at equations determining the mode of disturbance which is most likely to cause the steady motion to change to turbulent motion. Hence for all greater values of μ than that which

* Tamaki and Harrison, *Trans. Camb. Phil. Soc.*, vol. xxii, No. 22.

gives stationary relative kinetic energy for this mode of disturbance, the steady motion is bound to be stable whatever the nature of the disturbance.

These equations are (compare with (9) in the previous paper)

$$2\mu\nabla^2 u - \rho\left(2u\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} + v\frac{\partial V}{\partial x}\right) = \frac{\partial p}{\partial x},$$

$$2\mu\nabla^2 v - \rho\left(u\frac{\partial V}{\partial x} + u\frac{\partial U}{\partial y} + 2v\frac{\partial V}{\partial y}\right) = \frac{\partial p}{\partial y}.$$

Writing

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x},$$

$$U = -\frac{\partial\Psi}{\partial y}, \quad V = \frac{\partial\Psi}{\partial x},$$

eliminating p, and transforming to polar coordinates (r, θ), noticing that Ψ is a function of r only, we find

$$2\mu\nabla^4\psi + 2\rho\left(\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r}\right)\frac{1}{r}\left(\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \rho\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\right) = 0.$$

Now

$$\Psi = A\log r + \tfrac{1}{2}Br^2.$$

Hence

$$\nabla^4\psi - \frac{2A\rho}{\mu r^3}\left(\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) = 0.$$

Assume that ψ varies as $e^{\iota\lambda\theta}$, where λ can take integral values, and we have equation (17).

Let the notation be changed by writing

$$k = 2A\lambda\rho/\mu,$$

$$\psi = r^{\iota m+1}\, e^{\iota\lambda\theta},$$

so that (21) becomes

$$m^4 + 2(1+\lambda^2)m^2 + km + (1-\lambda^2)^2 = 0.$$

Write $\log_e(b/a) = n$, then equation (25) becomes

$$2\sigma s\cos\sigma n\cos sn - (4p^2 - \sigma^2 - s^2)\sin\sigma n\sin sn - 2\sigma s\cos 2pn = 0 \quad \ldots\ldots(1).$$

This equation is identical with equation (15) on page 125 of Orr's paper in the *Proceedings of the Royal Irish Society*, vol. XXVII, Section A. He shows that either s or σ must be imaginary.

Write $\sigma = \iota\sigma'$, and (1) becomes

$$2\sigma' s\cosh\sigma' n\cos sn - (4p^2 + \sigma'^2 - s^2)\sinh\sigma' n\sin sn - 2\sigma' s\cos 2pn = 0 \quad \ldots\ldots(2).$$

Thus equations (23) are replaced by

$$\left.\begin{aligned} \sigma'^2 - s^2 - 2p^2 &= 2\,(1+\lambda^2), \\ (p^2+\sigma'^2)\,(p^2-s^2) &= (1-\lambda^2)^2, \\ 2p\,(\sigma'^2+s^2) &= -k. \end{aligned}\right\} \quad \text{.....(3).}$$

Expressing σ' and s in terms of p, we have

$$s^2 = 2\sqrt{\{\lambda^2 + (1+\lambda^2)\,p^2 + p^4\}} - (1+\lambda^2+p^2) \quad \text{...(4),}$$

$$\sigma'^2 = 2\sqrt{\{\lambda^2 + (1+\lambda^2)\,p^2 + p^4\}} + (1+\lambda^2+p^2) \quad \text{...(5).}$$

Thus $$\frac{\lambda\rho A}{\mu} = -4p\sqrt{\{\lambda^2 + (1+\lambda^2)\,p^2 + p^4\}}.$$

Now p may be taken either positive or negative, and A is either positive or negative according to the nature of the motion of the cylinders. Putting

$$A = \frac{a^2b^2\omega}{b^2-a^2}, \quad (b > a)$$

and assuming p and ω to be treated as positive, we have

$$\frac{\rho\omega a^2}{\mu} = \left(1 - \frac{a^2}{b^2}\right)\frac{p}{\lambda}\sqrt{\{\lambda^2 + (1+\lambda^2)\,p^2 + p^4\}} \quad \text{.....(7).}$$

The discussion of equation (2) proceeds exactly as in Orr's paper, using the values for s and σ' obtained above, and it is easily shown that there is no solution for which sn is less than π except one for which sn is zero.

Putting $s^2 = p^2 + \epsilon$ in (4), we see that

$$(1-\lambda^2)^2 + 2\epsilon\,(2p^2 + 1 + \lambda^2) = 0.$$

Hence ϵ is negative, and p is always numerically greater than s. Thus from (5) $\sigma' > \sqrt{3}p > \sqrt{3}s$, and, therefore, $\sinh \sigma' n$ and $\cosh \sigma' n$ each exceed 100 and are approximately equal. Thus (2) becomes

$$\tan sn = \frac{2\sigma' s}{4p^2 + \sigma'^2 - s^2}$$

$$= \frac{\sqrt{\{3p^4 + 2\,(1+\lambda^2)\,p^2 - (1-\lambda^2)^2\}}}{3p^2 + 1 + \lambda^2} \quad \text{.....(8).}$$

We have to solve the equations (4) and (8) for p and s, using integral values of λ. The result of the substitution in (7) gives the corresponding critical values of μ. Now for a given λ, $\tan sn$ given by (8) has no real stationary value, and thus $\tan sn$ must be less than $\sqrt{\frac{1}{3}}$. Hence the smallest value of sn, except the zero value, satisfies the inequality $\pi \leqslant sn \leqslant 7\pi/6$. The corresponding value of p is also the smallest of the series of roots in p, and therefore gives the left-hand side of (7) its minimum value for given values of b/a and λ.

The calculated values of $\rho\omega a^2/\mu$ for series of values of b/a and λ are shown below.

$\rho\omega a^2/\mu$.

λ	b/a		
	1·1	2	10
1	39480	459	22
2	19760	242	17*
3	13200	176	22
4	—	149	—
5	—	140	—
6	—	137*	—
7	—	146	—
40	424	—	—
41	423*	—	—
42	424	—	—
43	425	—	—
45	428	—	—
50	443	—	—

The values marked * give the critical values of μ for the corresponding values of b/a.

Write $b - a = D$, then we have

$b/a =$	1·1	2	10
$D/a =$	·1	1	9
Critical value of $\rho\omega a^2/\mu =$	423	137	17
Critical value of $\rho\omega D^2/\mu =$	4·23	137	1380

The critical values of $\rho\omega D^2/\mu$ may be compared with Orr's result $B\rho D^2/\mu = 177$ for the shearing motion between parallel planes, showing that the shearing motion between circular cylinders is relatively more stable than the shearing motion between parallel planes if b/a is small, and relatively less stable if b/a is large.

There are some approximations which are of interest.

(*a*) λ *large.*

Since s^2 is positive, it follows that p^2 is greater than the positive root of

$$3p^4 + 2(1 + \lambda^2)p^2 - (1 - \lambda^2)^2 = 0,$$

considered as a quadratic equation in p^2. The same result follows by considering the reality of tan sn. Hence

$$p^2 > \tfrac{1}{3}\{2(\lambda^4 - \lambda^2 + 1)^{\frac{1}{2}} - (1 + \lambda^2)\}.$$

Therefore p is of the same order as λ, if λ is large. Accordingly

$$\frac{\rho\omega a^2}{\mu} = \left(1 - \frac{a^2}{b^2}\right)\frac{4p^2\sqrt{(\lambda^2 + p^2)}}{\lambda} \quad \text{.....(9),}$$

$$\tan sn = \sqrt{(3p^4 + 2\lambda^2p^2 - \lambda^4)}/(3p^2 + \lambda) \quad \text{.....(10),}$$

$$s^2 = 2p\sqrt{(\lambda^2 + p^2)} - \lambda^2 - p^2 \quad \text{.....(11).}$$

(10) and (11) are the same as Orr's (30) and (21).

(*b*) *b/a nearly unity; λ not large.*

We find $s = p, \quad sn = 7\pi/6.$

(*c*) *a and b large; b − a finite; λ not large.*

We have $s = p, \quad sn = pn = 7\pi/6.$

Write $b - a = d.$

Then $$n = \log_e\left(\frac{b}{a}\right) = \log_e\left(1 + \frac{d}{a}\right) = \frac{d}{a},$$

and $$p = \frac{7\pi a}{6d}.$$

Therefore $$\frac{\rho\omega d^2}{\mu} = \frac{8}{\lambda}\left(\frac{7\pi}{6}\right)^3$$
$$= 394/\lambda.$$

This approximation may be compared with the calculations given above for $b/a = \cdot 1$. If, however, we make a and b infinite, the restriction of λ to finite integral values is meaningless, and therefore we cannot expect to deduce the results for shearing motion between two parallel planes.

The soaring flight of dragon-flies. By E. H. Hankin, M.A., Sc.D., Agra, India.

[*Read* 16 May 1921.]

I. *Introduction.*

Dragon-flies of the larger species to be found in Agra, such as *Hemianax ephippiger*, frequently spend the day gliding at a height of several metres above tree-top level. At sunset they come down to low levels in search of prey, and hence have been described as of crepuscular habits. When flying at a height during the day they may be watched through a binocular till one's arms are tired without a single flap of the wings being observed. Proofs that such flight is not due to undiscovered wing movements will be found in a later paragraph, thus leading to the conclusion that it is an instance of soaring flight. Smaller species, such as *Pantala flavescens* and *Tramea burmeisteri*, that habitually fly nearer to ground level are more suitable for detailed observation and on these most of my work has been done.

II. *Soaring flight at low speed.*

The mode of flight of *Pantala flavescens* during the months of July and August and in fine weather is as follows.

If, in the early morning, one watches a group of these dragon-flies, it will be seen that their flight consists of alternate periods of flapping and gliding. The flaps at this time are of the hind wings only. As the sun gets stronger the amount of flapping is seen to decrease and to be more and more replaced by gliding. By about nine o'clock the dragon-flies are showing two methods of flight; either they glide with the abdomen horizontal, aided by two or three flaps of the hind wings every two or three yards of their course (Fig. 1), or, on the other hand, they glide with the abdomen elevated, travelling horizontally for the most part, and apparently may proceed in this way for indefinite distances (Fig. 2). Glides of 10 or 15 seconds without a flap may be observed. Flaps are apt to occur when two dragon-flies happen to approach each other. In the case of another species (*Rhyothemis variegata*), observed in Calcutta, proofs were obtained that such occasional flapping of their brilliantly coloured hind wings is used as a signal to other individuals (*Animal Flight*, p. 388).

The mode of flight continues to vary with the time of day. If there is strong sunshine and but little wind, then, at about eleven o'clock, the dragon-flies may be seen gliding with the abdomen and hind legs hanging down as shown in Fig. 3. This adjustment is maintained till about three o'clock when the abdomen is seen

to be again elevated above the level of the thorax. Then, later in the afternoon as the sun gets weaker, the gliding is interrupted by periods of flapping of the hind wings. The amount of flapping gradually increases till, towards twilight, the dragon-flies descend to within a few inches of the ground in search of prey. They are then in fast jolting flight caused by flapping of all four wings.

Fig. 1. Dragon-fly in flap-gliding flight seen from the side and below. The hind legs are parallel with the abdomen. The latter is horizontal.

Fig. 2. Dragon-fly in slow speed soaring flight seen from the side. The abdomen is elevated above the level of the thorax.

Fig. 3. Dragon-fly in soaring flight with hind legs and abdomen hanging down.

III. *Object of flight with abdomen hanging down.*

A clue to the meaning of the adjustment was obtained by observing instances in which it occurred in single individuals while others near by were not using it.

On several occasions it has been noticed that dragon-flies with abdomens down show less speed than others whose abdomens are up. With various species of dragon-fly gliding over water it has been repeatedly noticed that the abdomen is lowered when they need to check speed, either to avoid another dragon-fly or on turning as they reach some obstruction. In a light wind dragon-flies gliding to and fro over a restricted course, as is usual with *Pantala*, have been seen to have abdomens down when going with the wind and to raise their abdomens each time they go in the opposite direction.

A dragon-fly has been observed to lower the abdomen for a moment, apparently to check speed, while catching an insect.

We are thus led to the conclusion that lowering the abdomen and hind legs is an adjustment intended to check speed. It is obvious that in the lowered position the abdomen and legs must present more resistance to speed ahead than occurs when these organs are in the up position.

IV. *Conditions under which continued flight occurs with abdomen down.*

From the diary of my observations it soon appeared that this adjustment did not occur on days when the presence of thin cirrus cloud had been noted.

On one occasion it was noticed that the dragon-flies kept changing the position of their abdomens from up to down every few minutes. Suspecting the cause of this I obtained several pieces of coloured glass through which the sun could be observed without inconvenience. It was found that small thin clouds were passing over it which were too small to cast a shadow or even to cause any appreciable decrease in the intensity of the sunshine. Whenever one of these clouds was over the sun the dragon-flies held their abdomens up, downwards whenever the sun was clear.

It was further found that if small cumulus clouds are rapidly crossing the sky, and if there is not much wind at ground level, dragon-flies glide with abdomens up when there is shadow and with abdomens down when there is sunshine. After the sun comes out there is a short interval before the abdomens are lowered. If the edges of the cloud shadow are sharp cut this interval may be measured and this was done on three occasions the times being 22, 23 and 27 seconds.

The depressed position of the abdomen in continued flight is seen only in the hotter months of the year and then only when the sun is shining in full strength.

V. *Significance of the use of a brake in continued flight.*

The interest of the above observations lies in their bearing on the question whether the soaring flight of dragon-flies is due to undiscovered wing movements. If this were the case, it may be asked, why could not their flight be checked by cessation of such movements and why should the use of a brake in continued flight be necessary?

These observations also furnish a proof of the dependence of soaring flight in light winds on sunshine. It is remarkable that the presence of an amount of thin cloud far too small to have any known effect on the flight of birds should be able to influence the flight of dragon-flies.

VI. *High speed soaring flight of dragon-flies.*

The first observation of high speed flight of dragon-flies was made by Leeuwenhoek about 200 years ago. In discussing the compound eyes of insects and as a proof of the quickness of sight compatible with eyes of this description, he relates how he watched a swallow chasing a dragon-fly over the surface of a large pond and how the swallow was baffled by the speed and the unexpected turns of the insect which kept it always several feet in front of the enemy*.

In my experience such high speed flight over water is due to a combination of soaring with occasional flapping. This form of rapid flight in which a dragon-fly can easily outdistance a swallow, so far as I have seen, occurs only in the presence of sunshine and wind.

On rare occasions it has been my fortune to observe dragon-flies in a strong wind and to be able to form a definite opinion as to the absence or presence of flapping in their flight. The following instance appears to be of sufficient interest to be worth transcribing from my diary in detail:

> August 10th, 1915, at 10.05 a.m. on the Bharatpur Road, 22 miles from Agra.
>
> A strong wind that tended to blow off my hat. A cluster of dragon-flies (*Pantala*) were gliding to leeward of a smal tree but generally remaining a little to one side of it so that they met the full force of the wind. They were in continued gliding flight. Long grass about 2 metres high was below

* Miall, *The Early Naturalists, Their Lives and Works* (Macmillan and Co., 1912, p. 206).

them. They kept at a distance of 1 to 3 metres above it. The grass was waving and showed, where the dragon-flies were gliding, no sign of its being sheltered by the tree. Sometimes, in a stronger gust, a few of the dragon-flies came up to leeward of one of the branches. Rarely a few went further to leeward than usual. The group generally extended from near the tree to about 35 metres to leeward of it. The dragon-flies showed lateral swaying, and one of them showed lateral instability for an instant just after a gust.

10.20. When quite near the tree the dragon-flies occasionally flapped. Away from the tree they appeared to glide only. Glides of at least 10 metres up wind without a flap were observed.

In view of the probable speed of the wind on this occasion, it is likely that the speed of these dragon-flies through the air was above rather than below 15 metres per second.

When in low speed flight dragon-flies appear to travel a little faster than a locust. The speed of locusts, whose flight depends on flapping, has been measured by me on different occasions and found to be 4 metres per second.

VII. *The relation of ascending currents to the soaring flight of dragon-flies.*

The facts of the case are simple. Whenever the air is soarable or, in other words, whenever there is sunshine, dragon-flies avoid such currents.

For instance, on one occasion, observations were being made at a time when the sky was partially clouded with short intervals of sunshine. Dragon-flies of two species were gliding, so long as there was cloud shadow, in the current reflected up from the windward side of a stable. Whenever there was slight sunshine the dragon-flies elevated their abdomens and glided away, generally only for a short distance from the ascending current. But twice during the period of observation the sun came out so strongly that the dragon-flies put on a brake, that is to say they were gliding with abdomens depressed. On each of these occasions every dragon-fly went a long distance, 50 yards or more, from the ascending current.

Dragon-flies appear to remember the position of an ascending current. At Futteypur-Sikri they have been seen collecting on the windward side of the hill towards sunset, coming from distances of half a mile or more to get there.

Thus observations on dragon-flies yield valuable evidence that soaring flight is not due to the use of ascending currents. As happens with soaring birds, such currents are avoided so long as the air is capable of supporting soaring flight.

VIII. *Comparison of low and high speed flight.*

As above stated, low speed flight is dependent on the presence of sunshine. In the early morning, before the sun has attained its full strength, it is clearly favoured by the absence of wind. On occasions when the air was uncomfortably hot and very dry and when, therefore, the slightest movement of the air could be recognised by its cooling effect on the skin and when the dragon-flies were gliding often within two or three feet of my head, it has been noticed that the coming of the slightest draught of wind caused gliding to be replaced by flapping.

Sometimes during the daytime, if the air is nearly calm, wind seems to be unwelcome to the dragon-flies. On such occasions they retire to the shelter of the leeward side of a tree on the coming of a puff so light that it causes only a gentle movement of the leaves.

On the other hand, if conditions are suitable for high speed flight the presence of wind appears to be helpful rather than harmful. The speed of flight appears to increase with the strength of the wind. In such "soarable" winds, dragon-flies travelling up wind may be seen to glide ahead with a distinct increase of speed whenever they are struck by a gust. A similar increase of speed on entering a gust occurs in the case of cheels and other soaring birds.

In low speed flight the wings of dragon-flies are in the "up" position, the wing tips being on a higher level than the body. The tips of the front pair of wings are more elevated than those of the hind wings. Hence the gliding dragon-fly has a distant resemblance to a staggered biplane. The abdomen is elevated above the level of the thorax (Fig. 2).

In high speed flight the abdomen appears to be generally if not always horizontal and the wings appear to be less elevated than when flight is at low speed.

IX. *Conclusions.*

Various species of dragon-flies have been found to possess the power of soaring flight.

Facts have been adduced that are incompatible with the idea that this soaring flight is due either to undiscovered wing movements or to the use of ascending currents.

Soaring flight is now known to occur in three different classes of animals, namely dragon-flies, flying-fishes and birds. It is remarkable that, despite their widely different structure, size, and weight, and the very different conditions under which they soar, there should be such similarity of the flight in these three classes of animals. In each class evidence is available that low speed flight depends on the presence of sunshine and high speed flight on the presence of wind. The speeds attained in the three classes are comparable if not identical.

The Gluteal Region of Tarsius spectrum. By A. B. Appleton.

[Plate IV.]

[*Read* 16 May 1921.]

Attention will be here called to certain anatomical features which have come to light in the course of a special examination of the thigh musculature of *Tarsius*, an animal interesting not only for its peculiar saltatory method of locomotion in the trees, but also for the varied Primate features exhibited in its anatomy. It combines features of the Insectivora, of Lemuroidea and of Anthropoidea, and it is in comparison with these animals that its musculature must be chiefly considered.

The dissections on which the following account is based were made upon two Tarsii kindly provided by Dr W. L. H. Duckworth*.

Burmeister's classical description of 1846 provides a careful and accurate description, but he had not the advantage of that masterly study by Leche† in 1883 of the pelvic region of Insectivora for help in a determination of the identity of the various muscles found in *Tarsius*.

Attention in this paper will be centred on two muscles which are absent in some Primates and in many other Mammalia; but which are not uncommonly found in the more primitive of living Mammalia. These muscles are the femorococcygeus and the caudofemoralis—employing Leche's nomenclature. They apparently have their counterparts among Monotremes (Leche) and the latter among Reptilia‡.

The superficial gluteal musculature of *Tarsius* consists of a thin sheet attached to the 3rd trochanter of the femur, and a thicker band passing from the caudal vertebrae to the back and outer margin of the femur.

The latter of these we here describe as the femorococcygeus muscle.

The thin muscle sheet attached to the 3rd trochanter, which, with Burmeister, we regard as the conjoined tensor fasciae latae and superficial gluteus, arises from the crest of the ilium, and, through the lumbodorsal aponeurosis, from sacral spines, extending as far back as the 1st caudal vertebra.

* References are made to dissections of specimens of *Tupaia*, *Lemur*, *Simia* and *Troglodytes*, also so provided.

† Leche, *K. Svensk. Vet. Akad. Handl.* 1883.

‡ Caudifemoralis of Gadow, *Morph. Jahrb.* 1882.

A thicker band of musculature forms its cephalad margin and this has a special attachment to the 3rd trochanter by a small tendon. Supplied by the superior (cephalad) gluteal nerve, it is thus distinguishable from the superficial gluteus muscle which forms the hinder part of the muscle sheet and is supplied by the inferior (caudad) gluteal nerve. The latter nerve reaches the hinder edge of the muscle from behind the gluteus medius and enters its deep surface.

The cephalad gluteal nerve enters the deep surface of the tensor fasciae latae after passing through the gluteus minimus.

A complete deficiency of muscular tissue occurs behind the tensor fasciae latae, a thick fascia only being present; and through this the fibres of the superficial gluteus gain attachment to the 3rd trochanter. In front of the tensor fasciae latae the fascia lata of the thigh is thinner than is the case in those numerous Mammalia where it receives the pull of the "tensor" muscle. This muscle in fact acts in *Tarsius* as a flexor and rotator of the thigh on the trunk, and not as an accessory extensor of the knee (which occurs in the cursorial groups, *e.g.* Artiodactyla and Carnivora).

Our description differs from that of Burmeister who states that the tensor fasciae latae of *Tarsius* loses itself below in the fascia lata. He mentions no tendinous attachment to the 3rd trochanter. The gluteus medius and gluteus minimus are described by Burmeister. They are little larger than in *Tupaia* (Table I)*.

The **femorococcygeus** muscle claims our next attention.

Burmeister evidently saw this muscle, but, erroneously, as we think, described it as a "pyriformis" muscle; moreover, he apparently regarded the caudofemoralis as a second or deep pyriformis. These two muscles in *Tarsius* so closely resemble the muscles described by Leche in *Tupaia*, and are still more like the muscles of *Lemur*, that we cannot regard them as different structures. Nerve supply is also similar in all these cases. A representative of the femorococcygeus is even found in the anthropoid apes, supplied by a branch from the nerve to biceps (Orang): and these animals possess a separate pyriformis homologous with the human one.

The femorococcygeus of *Tarsius* arises from the deep aspect of the transverse processes of the 1st, 2nd, and 3rd caudal vertebrae and from the intervening intertransverse ligaments. Its origin is therefore quite distinct from that of the superficial gluteus. Below the tail and immediately medial to its place of origin is situated the sacrococcygeus; and it passes out from under cover of the intertransversarius caudae (of Burmeister) and dorsal caudal

* In *Macropus* the gluteus medius is, compared with the rectus femoris, twice as large as in *Tarsius* (data by Haughton, *Pr. R. Ir. Acad.* IX). In cursorials it is also much increased in size.

musculature. It is inserted by muscular fibres along the outer side of the back of the femur for four-sevenths of its total length, covering a broader area above, at the back of the great and 3rd trochanters, than along the shaft. There is a delamination into strips of muscle attached behind the 3rd trochanter, close to the centre of rotation of the hip-joint; the hinder and larger part of the muscle, attached to the femoral shaft, is quite distinct from the remainder.

The nerve-supply of the femorococcygeus is provided by a special branch passing from the great sciatic nerve* caudad to gluteus medius; the nerve enters it behind the great trochanter. The femorococcygeus is superficial except at its insertion, where the bulky vastus externus overlaps it. It covers the great sciatic nerve; also the obturator internus and gemelli, quadratus femoris, adductor magnus and caudofemoralis from above downwards. The muscle is relatively smaller than in *Lemur*, and the insertion is less extensive. It closely resembles the muscle of Simiidae, except in being readily separable from the superficial gluteus. A further difference from Simiidae lies in its (primitive) origin from coccygeal vertebrae. In Simiidae its origin is from the tuber ischii.

The **caudofemoralis** muscle of *Tarsius* arises from the tuber ischii cephalad to the biceps tendon. A flat muscular band, it passes down to the middle of the thigh where it gains insertion to the back of the femur, a rough line marking the place, for a distance equal to one-fifth of the femoral length, equidistant from either end. It is immediately medial to the insertion of the femorococcygeus.

It receives its nerve supply from the nerve to the hamstring muscles (from the great sciatic).

It is situated in its whole length on the deep aspect of the femorococcygeus, except when the great sciatic nerve intervenes between them [*v.* Plate IV, fig. 2].

In *Tupaia* this muscle is a far thinner sheet, with two places of insertion†: (1) to the back of the shaft (one-fourth of the length of the femur); (2) just proximally to the internal condyle. In *Lemur* it is thin and the length of its insertion one-tenth of the length of the femur. In Simiidae it is indistinguishable.

The nerve to the hamstring muscles (*i.e.* to biceps, semitendinosus and semimembranosus) passes deep to the caudofemoralis muscle.

These nerve-relationships are characteristic of the muscle‡ also in *Tupaia* among Insectivora, and in *Lemur* among Primates, in specimens I have dissected. They, together with the attachments of the caudofemoralis to bone, are sufficient to establish the identity of

* As also in *Tupaia* and *Lemur*.

† Cf. the corresponding caudifemoralis of *Reptilia*—Gadow, *loc. cit.* 1882.

‡ Cunningham's figure of the thigh of *Phalangista* shows (as the "ischiofemoral muscle") agreement in that animal also. '*Challenger*' *Report*, 1882, vol. v, Plate III.

the muscle. It appears from the description of Burmeister that his "deep" or "second pyriformis," apparently (*partim*) our caudofemoralis, is represented to be superficial to the great sciatic nerve, not deep to it as we have stated. That error is here a possibility is suggested by the following consideration.

The femorococcygeus and the caudofemoralis muscles are very closely apposed close to their insertions. It is here that the great sciatic nerve passes through, sandwiched between them*; and by its sheath it is so intimately bound up with the muscles that every movement of the muscles must carry the nerve with them. This may possibly be an arrangement for steadying the nerve during rapid and extensive hip extension. In *Lemur*, and in *Tupaia*, such a close association of the caudofemoralis with the femorococcygeus around the great sciatic nerve does not occur.

The tenuissimus† is absent in *Tarsius*, as also in *Lemur*. Klaatsch has shown it to be absent also in other species of Prosimiae. I find, with Leche, that it is present in *Tupaia*.

The quadratus femoris is triangular in shape, as is characteristic of Insectivora and Lemuroidea, and is inserted to the broad space behind and between the lesser (2nd) and 3rd trochanters. It does not occupy the extensive area of insertion found in *Lemur*, and is thus more like *Tupaia* and Simiidae. The bulk of the various muscles, in comparison with *Tupaia* and *Lemur*, forms an instructive study.

As a standard of comparison the weight of the rectus femoris is employed.

All specimens were long preserved in spirit.

Table I.

Tarsius			*Tupaia*	*Lemur*
Muscle	Actual weight in grains	Weight Rectus femoris = 1	Weights Rectus femoris = 1	
Gluteus supfl. and tensor fasciae latae	3·7	1·3	1·2	0·98 ‡
Femorococcygeus	4·0	1·4	1·2	2·18 ‡
Caudofemoralis	3·0	1·0	0·20	0·28
Gluteus medius	3·5	1·2	1·00	1·94
Gluteus minimus	1·8	0·6	0·25	—
Biceps	2·6	0·9	1·05	1·10
Semitendinosus	2·9	1·0	0·58	—
Tenuissimus	—	—	?	—
Semimembranosus, Presemimembranosus	9·8	3·4	2·30	—
Quadratus femoris	0·7	0·2	—	0·71

* See Plate IV, fig. 1.

† The iliofibularis (possibly) of Reptilia; Gadow, *loc. cit.* 1882.

‡ An arbitrary line of division had to be made in this specimen.

The similarity of *Lemur* and *Tupaia* (except as to the gluteus medius) is noteworthy. In *Tarsius* the caudofemoralis is large, while the gluteus medius and quadratus femoralis are little developed.

Table II.

Actual weights of thigh muscles (supplied from lumbar plexus) in a spirit specimen of *Tarsius*—and of certain shank muscles.

	Gracilis	Adductores and pectineus	Obturatores externus and intermedius	Sartorius	Iliacus	Psoas minor	Psoas major	**Rectus femoris**	Vastus externus	Vastus internus	Crurei	Gastrocnemius	Plantaris	Soleus	Popliteus	**Femur** (from spirit)	Femur (dried)
Actual weights in grains	1·6	6·7	1·5	3·7	3·5	1·4	1·8	2·9	51·2	16·5	9·0	5·4	1·8	1·7	1·1	18·0	12·3

It will be noticed that the hip extensors of *Tarsius* are not developed to the huge extent of the quadriceps extensor. Their weight compared with that of the quadriceps is in the ratio 1/5½, viz.:

Gluteus medius
Femorococcygeus
Caudofemoralis
Adductor magnus
} 14·5 grains. Quadriceps extensor 79·6 grains.

Now this ratio shows curiously little variation among Mammalia. It is convenient for the present purpose to employ the same set of four hip-extensors [it would, therefore, prove of very doubtful utility in comparisons with Sauropsida or Monotremes (see also footnote ‡ p. 471)].

In the less specialized forms the ratio as thus defined of $\frac{\text{hip extensors}}{\text{knee extensors}}$ is rather less than unity. It tends to rise somewhat in certain cursorial groups, viz. Artiodactyls and Perissodactyls.

A large leaping animal like the Kangaroo shows no marked disturbance of this ratio. And while the ratio in *Lemur* resembles that of most Mammalia, it is found in *Tarsius* to be extraordinarily small.

Table III.

	Tarsius spectrum	*Lemur*	Average of four Suffolk Sheep, Nos. I, II, IV, V	New born Horse	Rhinoceros*		Kangaroo*	Opossum*	Phalanger*	Badger*	Dog* (dingo)
Hip extensors / Rectus femoris	5·0	5·67	2·74	2·70	4·6		3·17†	2·4	—	1·8	3·0
Quadriceps ext. / Rect. fem.	27·5	7·64	2·91	2·82	3·2		4·0	3·1	—	3·0	4·2
Hip extensors / Quadriceps ext.	0·18	0·74	0·90‡	0·97‡	1·4		0·79	0·77	0·83	0·62	0·73

How then shall we account for the great difference in *Tarsius*? The length of the femur relative to other parts of the body appears to be a matter of great importance. Cursorial animals and jumpers require the action of powerful knee extensors at the moment of springing from a foot, whether in galloping or leaping; for the line of thrust from the foot is placed approximately at a right angle to the long axis of the femur. The quadriceps itself is situated disadvantageously for powerful action when the knee is fully flexed. And the longer the femur, the greater the moment of the animal's weight about the knee-joint. It is as a mitigation of the strain so placed upon the knee extensors that I regard the shortening§ of the femur in cursorial animals; and these muscles are not of disproportionate development in them. In the Macropodidae also, shortening of the femur occurs with a saving of knee-extensor musculature. In the *small* jumpers, however, such as *Pedetes*, *Macroscelides*, *Dipus* and *Tarsius*, this compensatory shortening is not a feature, and a powerful quadriceps is essential. This muscle-group is here shown to be of extraordinary size in *Tarsius*: it would prove of interest to ascertain the muscle-weights in the other saltatory mammals: I am aware of no records.

It appears that in the production of such a form as *Tarsius*, the rectus femoris took no part in the great development of the quadriceps extensor; the remaining muscles (forming the "triceps extensor") alone have been concerned,—most of all the vastus externus.

* Haughton, assuming his identification of muscles to be the same.

† Haughton, *P. R. Ir. Acad.* IX, fails to distinguish a caudofemoralis. It is, however, in *Marsupialia* a distinct muscle from the quadratus femoris as Carlsson (*K. Sv. Vet. Akad. Handl.* 1915) (*Hypsiprymnodon*), Parsons (*Petrogale*), McCormick (*Dasyurus*) have shown. Cunningham, *loc. cit.* 1882, saw both muscles in *Thylacinus* and *Phalangista*.

‡ Inclusion of tensor fasciae latae as a knee extensor would lower this ratio slightly, while admission of semimembranosus as a hip extensor raises this ratio considerably.

§ Appleton, *Proc. Camb. Phil. Soc.* 1921, p. 386.

The musculature of the gluteal region is not only little developed in point of size: it is, moreover, comparatively little specialized as regards muscle attachments.

The changes in *Tarsius* from a *Tupaia*-like form have comprised:

*(1) Attachment of the caudofemoralis (of Leche) to the tuber ischii instead of to the caudal vertebrae†.

*(2) Loss of the condylar attachment of the caudofemoralis (of Leche).

*(3) Increase in size of the remaining part of the caudofemoralis, and concentration of its insertion to the middle of the femur.

(4) Disappearance of the muscular part of the upper part of biceps.

(5) Loss of the tenuissimus.

(6) Close envelopment of the great sciatic nerve by the muscle fibres of femorococcygeus and caudofemoralis.

(7) Concentration of the pull of the tensor fasciae latae on to the 3rd trochanter.

Primitive features, retained in *Tarsius*, but lost in anthropoid Apes comprise the following:

(1) Persistent caudofemoralis muscle (of Leche).

(2) Independent femorococcygeus (more or less fused with gluteus maximus in Apes: Gorilla, Chimpanzee, Orang).

(3) Caudal attachment of femorococcygeus. (Origin is ischial in Apes *mihi* = Duvernoy's ischiofemoralis, and Fick's tubero-femoral muscle).

‡(4) Persistent 3rd trochanter, associated with the attitude of a flexed hip, and the passage of a superficial gluteus muscle-sheet and tensor fasciae latae across a large vastus externus muscle.

In the last four points, *Tarsius* is in agreement with *Lemur*. The most striking difference between these two in the gluteal region is found in the extensive attachment to the femur of quadratus femoris in *Lemur*; the smaller size of the caudofemoralis in *Lemur*, and its "more primitive" origin from coccygeal vertebrae.

The *caudofemoralis* (Leche) has been confused with the *femoro-coccygeus* (Leche). It appears to be merely a verbal confusion, owing partly to the description by Windle and Parsons of the *femorococcygeus* of Carnivora and certain Rodentia under the name of the *caudofemoralis* §. In *Petrogale* a similar muscle is included by Parsons under the term *biceps* ||. He describes what appears to be the *caudofemoralis* of Leche in *Petrogale* as the

* Certain Marsupials resemble *Tarsius* and differ from *Tupaia* in these features; a parallel development.

† *Hypsiprymnodon* is the only Marsupial found by Carlsson, *loc. cit.* 1915, to retain a coccygeal origin for the caudofemoralis.

‡ Appleton, *loc. cit.* p. 380.

§ Parsons, F. G., *Proc. Zool. Soc.* 1898.

|| Parsons, F. G., *op cit.* 1896.

*ischiofemoralis**; Carlsson† identifies the corresponding muscle of *Hypsiprymnodon* as the *caudofemoralis*.

Gregory‡ has fallen into the error of comparing the *caudifemoralis* of Reighard and Jennings (the Cat), in Parsons' meaning, with the *caudofemoralis* of *Hypsiprymnodon* in Leche's sense of the term.

In the present state of the literature, it is indeed not possible to compare muscles of similar name described by different authors without further enquiry into the attachments and nerve-relationships of these muscles. The position of the great sciatic nerve is a ready method of distinguishing the *femorococcygeus* (caudofemoralis of Parsons) from the *caudofemoralis* (ischiofemoralis of Parsons).

The nomenclature of Leche has some claim from usage, and is here employed. Ischiofemoralis is an excellent *descriptive* name in certain more specialised mammals: but in the more primitive animals (some Reptilia and Insectivora)§ a *caudal*‖ attachment rather than *ischial* is found. If it is agreed to use different names for what is practically the same muscle, according to interspecial variations in its attachments, then there must be no verbal homologisation of muscles. The caudofemoralis (or caudifemoralis) muscles of certain authors are no more comparable with the caudofemoralis of Leche and Carlsson than are the various muscles which go by the name of pectineus in various mammalia homologous with one another.

Some correlated changes in bones, associated with muscular peculiarities of *Tarsius*, may be mentioned:

(1) Large transverse processes to the first three caudal vertebrae¶, associated with well developed femorococcygeus.

(2) 3rd trochanter and descending extensor ridge on femur (*v. supra*).

(3) Marking on the posterior aspect of the femoral shaft for attachment of caudofemoralis (paralled in Marsupialia).

(4) Femoral shaft with high pilastric index—associated with the powerful quadriceps extensor.

The peculiar features of *Tarsius* deserve comparison with those of other jumping animals. Some have been already considered.

(1) *Length of femur*. In large jumpers (Macropodidae) shortening occurs for a similar reason in cursorial types**; viz. diminution of the necessary mass of knee-extensors. In small jumpers

* Cunningham, *loc. cit.* 1882, also uses this term.

† Carlsson, A., *Kungl. Svensk. Vetensk. Handl.* 1915.

‡ Gregory, *Bull. Amer. Mus. Nat. Hist.* 1918.

§ Also *Hypsiprymnodon* (Carlsson) and Monotremata (Westling, *Bihang t. K. Sv. Vet.-Akad. Handl.* 1889).

‖ For *M. caudofemoralis* (Leche).

¶ Cf. Duckworth, *Morph. and Anthrop.* vol. 1, 1915, p. 106.

** Appleton, *loc. cit.* 1921

this does not appear to be necessary. For this musculature in jumping animals other than *Tarsius*, and the Kangaroo, verification is needed.

(2) Attachment of muscles tc tuber ischii instead of to tail.

In Reptilia and Insectivora, both femorococcygeus and caudofemoralis (or their precursors) are attached to caudal vertebrae. But in two Insectivora *Macroscelides* and *Erinaceus*), one of them a jumper, the femorococcygeus is recorded as attached to the tuber ischii. A new type of hip-extensor is produced, acting as far from the hip-joint as possible, and therefore exerting the greater leverage.

The same principle is seen in cursorial types where the long vastus (= gluteus maximus + ? femorococcygeus) gains attachment to the tuber ischii, while at the same time the tuber ischii shifts away from the hip-joint downwards and backwards (Artiodactyla).

In *Tarsius* it is the caudofemoralis alone which gains an ischial attachment; but in giant apes we see the femorococcygeus attached here. An ischial attachment for hip-extensors appears to be no prerogative of jumpers, but is shared alike by the active Metatheria and cursorial Eutheria, and also by the giant apes.

To sum up, *Tarsius* exhibits, in the musculature of the gluteal region, primitive features like the genus *Lemur*, recalling *Tupaia* and *Hypsiprymnodon*, and less closely reproducing the conditions of Monotremes and Reptiles. Along with these features we find certain modifications, such as the ischial attachment and development of the caudofemoralis, in which *Tarsius* has progressed beyond the conditions in *Lemur* and is paralleled by many Marsupialia. Its small size is held accountable for a certain lack of parallelism between *Tarsius* and the Kangaroo.

While certain of the primitive features of *Tarsius* and *Lemur* are lost in Simiidae, others, such as a well developed femorococcygeus, are retained in the latter.

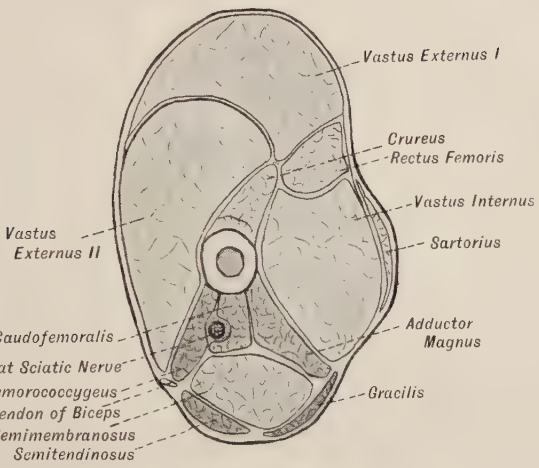


Fig. 1. *Tarsius spectrum.* Cross section of thigh. × 3.
Junction of upper and middle thirds.

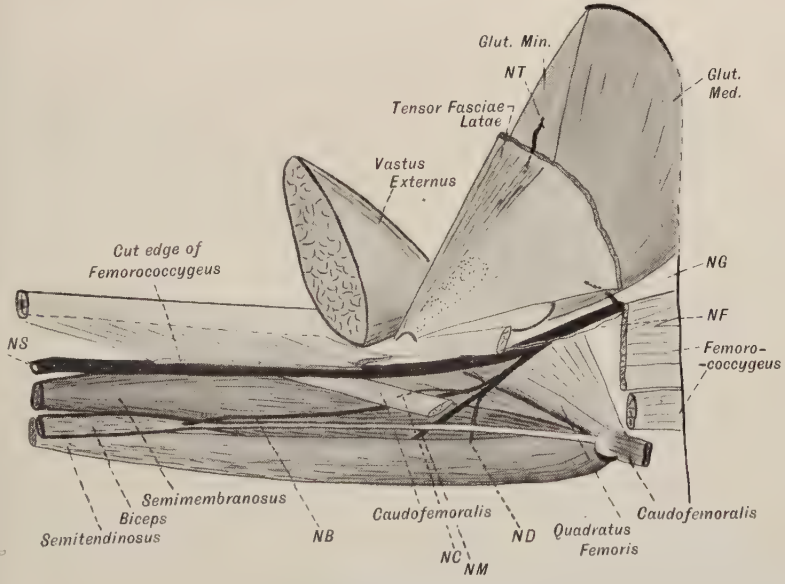


Fig. 2. *Tarsius spectrum.* Outer view left thigh. Parts of Femorococcygeus, Caudofemoralis, Vastus Externus, Tensor Fasciae Latae and Superficial Gluteus removed.

NB=Nerve to Biceps; *NC*=Nerve to Caudofemoralis; *ND*=Nerve to Semitendinosus; *NF*=Nerve to Femorococcygeus; *NG*=Nerve to Gluteus Superficialis (*seu* Maximus); *NM*=Nerve to Semimembranosus; *NS*=Great Sciatic Nerve; *NT*=Nerve to Tensor Fasciae Latae.

An unusual type of male secondary characters in the Diptera. By C. G. Lamb, M.A.

[*Read* 16 May 1921.]

Secondary sexual characters may be classed as persistent or sporadic: the former occur with much regularity throughout a genus or family, the latter crop up suddenly in isolated species of a genus, and are often similar in form in quite unrelated families. Examples of the persistent type are the antennal dimorphism of so many nematocerous and other families, and the male eye holopticism in many families. The sporadic types are such as: (1) frontal processes as in *Ceratitis* sp., (2) very varied types of leg adornment, which is a common form, probably connected with courting habits, (3) replacement of head bristles by hair tufts as on some Chrysosomatinae, (4) costal lumps as in the two quite unrelated species, *Pemphigonotus mirabilis* (Chloropids) and *Ommatius Lema* (Asilids), (5) differences in the palpal structure, as in many Dolichopids and others. The list could be extended, but one thing is apparent, all the characters of either type are peripheral in their position on the insect.

The author has recently completed the study of certain families of flies contained in the collections made by Dr H. Scott in the Seychelles, the descriptions of which will appear in a forthcoming volume of the Percy Sladen Expeditions Reports, *Trans. Linn. Soc.* London, Ser. 2, Zool., vol. 18. Amongst these are some exceedingly interesting forms belonging to the Dolichopodidae. These belong to a new genus, *Craterophorus*, which is probably to be referred to the subfamily Chrysosomatinae; these differ from other known species of fly in having centrally placed secondary characters; further each of the three species constituting the genus bears the same set of characters, one of which is nearly unique, the others being absolutely so. The genus is apparently endemic, being found in the native forests of the Island of Mahé. There are no tarsal or antennal differences between the sexes. The special characters under consideration are four in number: (1) the wing of the male (Fig. 1) is quite different in shape from that of the female (Fig. 2) having a sharply angled posterior margin; this peculiar form of wing is almost if not quite unique. The other three characters are more strictly central; they are shown in the photograph (Fig. 3) and diagram (Fig. 4), these characters are as follows: (2) the Dolichopodidae are usually devoid of alulae, but in these species one is developed though it is of a totally exceptional form.

Fig. 1. *Craterophorus mirus* ♀ wing.

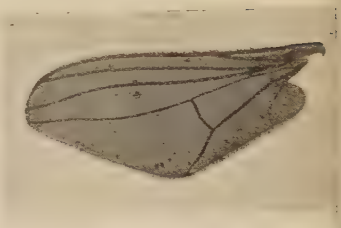


Fig. 2. *Craterophorus mirus* ♂ wing

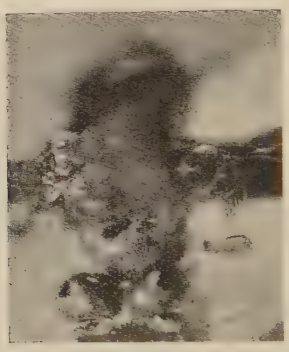


Fig. 3.

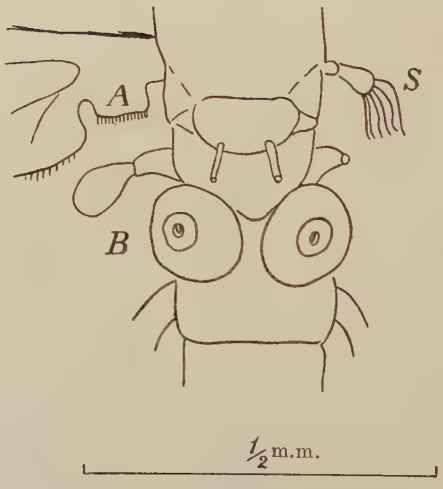


Fig. 4 *Craterophorus mirus* ♂.

It consists of a straight edged lobe (*A*, Fig. 4) which is highly chitinized and has the margin beset with a very close even row of tiny stiff bristles, co-planar with the lobe, the bristles having the utmost regularity of size and shape: they form a perfect comb with the teeth nearly touching. (3) In most dolichopids the squama consists of a more or less pronounced knob bearing a characteristic row of bristles, which are frequently spread out into an elegant fan. In the present species the squama (*S*, Fig. 4) is long stalked, apparently mobile, and instead of the usual fan of long hairs, it carries a set of parallel stout bristles hooked at the tip. The relative position of the last two structures almost irresistibly calls to mind the relation of a hand to a harp. In the figure only one structure is shown on each side to avoid confusion. (4) The fourth structure (*B*, Fig. 4) is even more remarkable: it consists of a pair of large spheroidal bodies, one on each side of the basal segment of the abdomen and confluent below with the hind epimeron. Each is hollow and has a round orifice on the dorsal surface; from the bottom springs a rod which nearly reaches the surface of the sphere. The significance of the whole complex is entirely problematic, though it is certain that the several parts are in some way duly correlated. The general appearance, as said above, is that of some musical instrument, although the function of the bulbs is quite problematic, to suggest 'resonators' is too fanciful, nor can one see the relation of the whole to the remarkable wing form.

It is much to be regretted that the paucity of specimens and the fact that none but pinned individuals were available rendered it impossible to submit the bulbs to a competent histologist, as a suitable examination might have helped to throw some light on this assemblage of mysteries.

A Note on the Mouth-parts of certain Decapod Crustaceans. By L. A. Borradaile, M.A., Fellow and Tutor of Selwyn College, Cambridge, and Lecturer in Zoology in the University.

[*Read* 16 May 1921.]

In that close study of the relation of aquatic organisms to their surroundings which is now becoming an important part of Biology, much attention is given to their modes of feeding; and one of the most interesting branches of the subject concerns those organisms which obtain their food from organic particles, alive or dead, in suspension in the water. Orton has investigated this habit in various animals (Molluscs, Tunicates, etc.), and Potts has shown that it is practised by the Coral-gall Crab *Hapalocarcinus*, and that the mouth-parts of this animal are correspondingly modified, the endopodite of the third maxilliped and the exopodites of the second and first being provided with fringes of bristles which are used for gathering the food, while the inner jaws are reduced in the absence of the need for powerful organs to masticate it. These adaptations are analogous to those that appear in Cirripedia and Branchiopoda, which are also feeders upon suspended matter.

Sundry other crustaceans which live in the mantle-chamber or pharynx of sessile or subsessile organisms must be presumed to get food of the same kind at second hand. Some time ago, in the course of a study of the prawns of the subfamily Pontoniinae, of which most members are commensal and a number live in the mantle cavity or pharynx of bivalve molluscs and ascidians, I endeavoured to discover some correspondence between the habitat of the animals and the structure of their mouth-parts; but was disappointed to find practically no trace of this. The jaws of the crab *Pinnotheres*, which lives between the valves of the shell of Lamellibranchs, show the same absence of specialisation in the direction in which *Hapalocarcinus* is specialised. They have some interesting peculiarities, but not those that might be expected from an animal which was nourished upon finely divided food. Probably these cases are to be explained by the fact that the crustaceans feed, not upon food in suspension, but upon the strings of mucus in which it is entangled by their host. Orton has recently shown that this is done by *Pinnotheres*.

A like explanation, however, cannot be given for *Porcellana platycheles*. This animal is not a commensal, and gathers for itself suspended food, taking it by means of long fringes upon the third maxillipeds, but is provided with well-developed inner mouth-

parts. Possibly it feeds, not, as *Hapalocarcinus* does, only upon the very minute organisms which make up the nannoplankton, but also upon suspended particles of greater size and toughness. Or it may be that it uses the chelae for seizing food, though I have not seen it do this. Another case of the same kind would seem to be presented by the prawn *Paratypton*, which was found by Potts to live in a coral gall somewhat in the same way as *Hapalocarcinus*. Details of its habits are not known, but it does not at present appear likely that it can obtain any but finely divided food. Yet its jaws show neither any provision for gathering such food nor any marked reduction in masticatory structures.

All these organisms deserve a good deal more investigation than they have received.

An Apparatus for Projecting Spectra. By H. Hartridge.

[*Read* 2 May 1921.]

If a Thorpe replica celluloid diffraction grating of about 14,000 lines to the inch be mounted in optical contact with the hypotenuse of a right angled glass prism, and if now a parallel beam of white light be caused to enter normally at one of the other faces, then it is found, while the direct beam is totally internally reflected and occupies the same position as it would if the prism had no grating mounted on it, that the 1st and 2nd order spectra of one side are found approximately in their normal relationship in regard to the direct beam, but that the spectra of the other side pass out through the hypotenuse of the prism without suffering reflection. If the prism be made of crown glass the yellow region of the 1st order spectrum is in a straight line with the incident beam, and the dispersion of this spectrum is about 54 % greater than that of the diffraction grating alone. It is also found that (1) the resolving power for a given aperture is increased about 41 %, (2) the purity factor for a given collimator slit width is also increased by about half, (3) the intensity of this spectrum for a given purity and dispersion is also considerably greater than that of the grating alone. It is to this latter feature that the arrangement owes its value. To make use of this spectrum for projection purposes the green, blue and violet regions of the 2nd order spectrum are screened off by a deep orange colour filter.

The transmission of the spectra through the hypotenuse of the glass prism, in apparent contradiction to the laws of geometrical optics, is almost certainly due to the inclination of the sides of the separate strip elements of the replica, which when the cast was made filled the grooves of the original grating. These sides are then roughly normal to the incident light rays and a number of elementary wavelets would therefore, on Huyghen's hypothesis of diffraction, pass out of the surface, but since they would retain the phase relationships of the original beam they would be in a condition to interfere with one another, and thus form the spectra that are observed.

This prism grating would therefore seem to be in closer theoretical relationship with the echelon of Michelson, or the echelette of R. W. Wood, than with the ordinary diffraction grating.

Note on true and apparent hermaphroditism in sea-urchins. By J. Gray, M.A., Balfour Student, Cambridge University.

[*Read* 16 May 1921.]

During the winter of 1913 I had occasion to examine at Naples the gonads of a large number of specimens of the sea-urchin *Arbacia pustulosa*. It is exceedingly easy to distinguish the two sexes of this animal; the eggs and ovarian tissue of the female contain a dark red pigment, whereas the testes of the male are devoid of any pigment. The gonad of the female is brick-red in colour from the very early stages of its development. That of the male is yellowish white.

On opening one individual, a unique condition of the gonads was observed: four of the gonads were apparently female—being of the usual colour, and of considerable size. Part of the fifth gonad was also in the same condition, but the majority of the gland was typically *male* in appearance. On examining this portion of the abnormal gland, spermatozoa were found, which, though normal in appearance, were either motionless or only capable of feeble movement.

On sectioning the whole of the gonads, I was most surprised to find no trace of ova or ovarian tissue. On the other hand, those parts of the gonads which were female in appearance were found to be full of a mass of degenerating spermatogonial cells, which failed to take up any of the usual stains. The walls of the gonad were perfectly normal.

This animal is of interest as it would appear that a derangement of the sex-cells has been attended by an inversion of the secondary sexual characters. Although no parasite was observed, the condition of the gonad recalls the well-known effects of parasitic castration in the Crustacea. If this view is correct, the male of *Arbacia* appears to be heterozygous for sex.

Owing to the courtesy of Mr H. M. Fox, I have been able to examine the gonads of a true case of hermaphroditism in the urchin *Strongylocentrotus lividus*. In this case three of the gonads were completely female, while the other two contained both ripe eggs and spermatozoa, which were fertile *inter se*. In one of the abnormal gonads most of the tubules were female. It appears that this animal was essentially a female—possibly an "intersex." No abnormality was found in either the male or female portions of the gonads.

As far as I am aware the only other description of an hermaphrodite sea-urchin is that given by Gadd* of an individual of *Strongylocentrotus droebachiensii*, in which one gonad was male and the remaining four female.

* *Zool. Anz.* vol. xxxi, 1906–07, p. 635.

On Certain Simply-Transitive Permutation-Groups. By Dr W. Burnside, Honorary Fellow of Pembroke College.

[*Received* 19 September 1921.]

In 1900 I proved (*Proc. L. M. S.* vol. xxxiii, p. 177) that a simply-transitive permutation-group of prime degree p must contain a self-conjugate subgroup of prime order. In the second edition of my *Theory of Groups* (1911) it was shown that a simply-transitive permutation-group degree p^m, which contains a permutation P of order p^m, necessarily has a self-conjugate subgroup containing $P^{p^{m-1}}$. I ventured then to express an opinion that a similar result was true for any simply-transitive permutation-group which contained a transitive Abelian subgroup. Quite recently I have succeeded, with a single exception, in justifying this expression of opinion in a remarkably simple way.

Denote by $x_{a,b}$ a set of mn variables, where the first suffix takes all values from 0 to $m-1$, and the second all values from 0 to $n-1$.

The permutation $$x'_{a,b} = x_{a+1,b}$$
is a regular permutation M, of order m, in the mn variables, and
$$x'_{a,b} = x_{a,b+1}$$
is a regular permutation N, permutable with M. The two generate a regular Abelian group $\{M, N\}$ simply-transitive in the mn variables.

If ϵ, η are primitive mth and nth roots of unity, and if
$$\xi_{i,j} = \sum_{ab} \epsilon^{-ia}\eta^{-jb}x_{a,b},$$
then $$M\xi_{i,j} = \epsilon^i\xi_{i,j}, \qquad N\xi_{i,j} = \eta^j\xi_{i,j},$$
so that the mn quantities $\xi_{i,j}$ are the reduced variables for the Abelian group $\{M, N\}$ and each gives a distinct representation of the group. Also
$$\sum_{i,j}\epsilon^{ui}\eta^{vj}\xi_{i,j} = \sum_{i,j,\,a,b}\epsilon^{i(u-a)}\eta^{j(v-b)}x_{a,b}$$
$$= mnx_{u,v}.$$

Suppose now that a simply-transitive group G in the mn x's contains $\{M, N\}$. Since no irreducible representation of $\{M, N\}$ occurs more than once, no irreducible representation of G, when it is completely reduced, can occur more than once; and therefore* G_0, the subgroup that leaves $x_{0,0}$ unchanged has just one linear invariant in each irreducible representation of G.

For each irreducible representation of G, the reduced variables must be expressible as a set of ξ's. Let
$$\xi_{\alpha_1,\beta_1}, \xi_{\alpha_2,\beta_2}, \ldots, \xi_{\alpha_\mu,\beta_\mu}$$
be the reduced variables for an irreducible representation Γ.

* *Theory of Groups*, Second Edition, p. 275.

Since $mn\, x_{0,0} = \sum_{i,j} \xi_{i,j}$, the one linear invariant that G_0 has in Γ is

$$\sum_{i=1}^{\mu} \xi_{\alpha_i, \beta_i}.$$

Suppose now that

$$x_{a_1, b_1},\ x_{a_2, b_2},\ \ldots,\ x_{a_p, b_p},$$

is a set of variables which are permuted transitively by G_0. Then

$$\sum_{j=1}^{j=p} x_{a_j, b_j}$$

is a linear invariant for G_0.

Now $$\sum_{i=1}^{\mu} \xi_{\alpha_i, \beta_i} = \sum_{i=1}^{\mu} \sum_{a,b} \epsilon^{-\alpha_i a} \eta^{-\beta_i b} x_{a,b},$$

and since the right-hand side is invariant for G_0 which permutes x_{a_j, b_j} $(j = 1, 2, \ldots p)$, transitively,

$$\sum_{i=1}^{\mu} \epsilon^{-\alpha_i a_j} \eta^{-\beta_i b_j}$$

is independent of j.

Also $$mn \sum_{j=1}^{p} x_{a_j, b_j} = \sum_{j=1}^{p} \epsilon^{a_j u} \eta^{b_j v} \xi_{u,v},$$

and since the only linear invariant that G_0 has in the symbols ξ_{α_i, β_i} $(i = 1, 2, \ldots \mu)$ is their sum

$$\sum_{j=1}^{p} \epsilon^{a_j \alpha_i} \eta^{b_j \beta_i}$$

is independent of i.

Moreover the immediately preceding result may be expressed in the form that

$$\sum_{i=1}^{\mu} \epsilon^{\alpha_i a_j} \eta^{\beta_i b_j}$$

is independent of j.

Hence

$$\sum_{i=1}^{\mu} \sum_{j=1}^{p} \epsilon^{a_j \alpha_i} \eta^{b_j \beta_i} = \mu \sum_{j=1}^{p} \epsilon^{a_j \alpha_i} \eta^{b_j \beta_i} = p \sum_{i=1}^{\mu} \epsilon^{a_j \alpha_i} \eta^{b_j \beta_i}.$$

Now $$\sum_{i=1}^{\mu} \epsilon^{a_j \alpha_i} \eta^{b_j \beta_i}$$

is the sum of the multipliers of the operation, $M^{a_j} N^{b_j}$ is the irreducible representation Γ. Hence in Γ

$$\chi_{M^{a_j} N^{b_j}} = \frac{\mu}{p} \sum_{j=1}^{p} \epsilon^{a_j \alpha_i} \eta^{b_j \beta_i}$$

for each i.

Suppose now that Γ is that irreducible representation to which $\xi_{k,0}$ belongs. Then

$$\chi_M{}^{a_j}{}_N{}^{b_j} = \frac{\mu}{p}\sum_{j=1}^{p}\epsilon^{ka_j}.$$

Now choose the set

$$x_{a_1,b_1}, x_{a_2,b_2}, \ldots, x_{a_p,b_p},$$

so that it contains the symbol $x_{1,0}$. Then

$$\chi_M = \frac{\mu}{p}\sum_{j=1}^{p}\epsilon^{ka_j}.$$

If m is not a prime k may be chosen so that ϵ^k is a primitive qth root of unity, where q is a prime factor of m; and when k is so chosen,

$$\chi_M{}^q = \mu.$$

Hence, if m is not a prime, G has a self-conjugate subgroup containing M^q.

If m and n are different primes, and Γ is that irreducible representation to which $\xi_{1,0}$ belongs, so that

$$\chi_M{}^{a_j}{}_N{}^{b_j} = \frac{\mu}{p}\sum_{j=1}^{p}\epsilon^{a_j},$$

then

$$\frac{\mu}{p}\sum_{j=1}^{p}\epsilon^{a_j} = \sum_{i=1}^{\mu}\epsilon^{a_j\alpha_i}\eta^{b_j\beta_i}.$$

Unless each β_i is zero, in which case the group has a self-conjugate subgroup containing N, this equation actually contains powers of η on the right. Hence when the indices of the powers of η are reduced (mod. n) each power must occur with the same coefficient. This shews that μ must be a multiple of n, and that the reduced variables for Γ must be

$$\xi_{a_1,i}, \xi_{a_2,i}, \ldots, \xi_{a_r,i} \quad (i = 0, 1, 2, \ldots, n-1).$$

The family of representations to which Γ belongs accounts therefore for $(m-1)n$ of the reduced variables. The remaining reduced variables are

$$\xi_{0,0}, \xi_{0,1}, \ldots, \xi_{0,n-1},$$

and in each irreducible representation whose reduced variables belong to this set

$$\chi_M = \chi_E.$$

Hence when m, n are different primes G has a self-conjugate subgroup containing either M or N. It is clear that the same method of proof will apply, when the transitive Abelian subgroup has three or more independent generators. Hence:—

A simply-transitive permutation-group, which contains a regular transitive Abelian subgroup, always has a self-conjugate subgroup, except possibly when the operations of the Abelian subgroup are all of the same prime order.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1920—1921.

ANNUAL GENERAL MEETING.

October 25, 1920.

In the Cavendish Laboratory.

Mr C. T. R. Wilson, President, in the Chair.

The following were elected Officers for the ensuing year:

President:

Prof. Seward.

Vice-Presidents:

Prof. Sir E. Rutherford.
Mr C. T. R. Wilson.
Dr E. H. Griffiths.

Treasurer:

Prof. Hobson.

Secretaries:

Mr H. H. Brindley.
Prof. Baker.
Mr F. W. Aston.

Other Members of Council:

Dr Marshall.
Prof. Newall.
Dr Fenton.
Prof. Inglis.
Mr Rivers.
Mr F. A. Potts.
Prof. Marr.
Mr C. T. Heycock.
Mr H. Lamb.
Prof. Hopkins.
Dr Bennett.
Dr Hartridge.

The following were elected Fellows of the Society:

T. M. Lowry, M.A., Professor of Physical Chemistry.
S. Pollard, M.A., Trinity College.

The following were elected Associates of the Society:

E. S. Bieler, Gonville and Caius College.
T. M. Cherry, Trinity College.
L. Harding, Gonville and Caius College.
W. W. Hurst, Jesus College.
I. Jones, Emmanuel College.
E. B. Ludlam, Trinity College.
J. K. Roberts, Trinity College.

The following Communications were made to the Society:

1. On the stability of the steady motion of viscous liquid contained between two rotating co-axal circular cylinders. By K. Tamaki and W. J. Harrison, M.A., Clare College.

2. Sur le principe de Phragmén-Lindelöf. By M. Marcel Riesz and Professor G. H. Hardy.

3. A note on the nature of the carriers of the Anode Rays. By G. P. Thomson, M.A., Corpus Christi College.

4. On the distribution of primes. By M. H. Cramér. (Communicated by Professor G. H. Hardy.)

5. Note on Ramanujan's trigonometrical function $c_q(n)$, and certain series of arithmetical functions. By Professor G. H. Hardy.

6. On the representation of an algebraic number as a sum of four squares. By L. J. Mordell, M.A., St John's College. (Communicated by Professor H. F. Baker.)

7. The parity of the number which enumerates the Partitions of a number. By Major P. A. MacMahon.

November 8, 1920.

In the Cavendish Laboratory.

Mr C. T. R. Wilson, Vice-President, in the Chair.

The following was elected an Associate of the Society:

M. H. Belz, Gonville and Caius College.

The following communication was made to the Society:

The Inner Structure of Atoms. By Professor Sir Ernest Rutherford.

November 22, 1920.

In the Comparative Anatomy Lecture Room.

Professor Seward, President, in the Chair.

The following were elected Fellows of the Society:

A. B. Appleton, M.A., Downing College.
R. H. Atkinson, M.A., Queens' College.
E. S. Dewing, B.A., Gonville and Caius College.
C. D. Ellis, B.A., Trinity College.
U. R. Evans, M.A., King's College.
J. E. G. Harris, B.A., Jesus College.
L. J. Mordell, M.A., St John's College.
E. K. Rideal, M.A., Trinity Hall.
R. C. Staples-Browne, M.A., Emmanuel College.
B. M. Wilson, B.A., Trinity College.

The following were elected Associates of the Society:

J. Humphries, Queens' College.
A. C. G. Menzies, Christ's College.
Miss Lorna M. Swain, Newnham College.

The following Communications were made to the Society:

1. A note on Vital Staining. By F. A. Potts, M.A., Trinity Hall.

2. Preliminary note on the Superior Vena Cava of the Cat. By W. F. Lanchester, M.A., King's College, and A. G. Thacker.

3. Preliminary note on a cynipid hyperparasite of Aphides. By Miss M. D. Haviland. (Communicated by Mr H. H. Brindley.)

4. The Problem of Soaring Flight. By E. H. Hankin, Sc.D., St John's College. (Communicated by Mr H. H. Brindley.)

5. On the rotation of a non-spinning gyrostat. By Sir George Greenhill.

6. A method of testing Triode Vacuum Tubes. By E. V. Appleton, M.A., St John's College.

January 24, 1921.

In the Cavendish Laboratory.

Professor Seward, President, in the Chair.

The following were elected Fellows of the Society:

E. D. Adrian, M.D., Trinity College.
A. Munro, M.A., Queens' College.
E. G. D. Murray, M.A., Christ's College.
H. W. C. Vines, M.B., Christ's College.

The following were elected Associates of the Society:

Miss M. Barker, Newnham College.
J. Burtt-Davy.
D. A. Keys, Corpus Christi College.

The following Communications were made to the Society:

1. (*a*) Experiments with Rotating Fluids.

(*b*) Tides in the Bristol Channel. By G. I. Taylor, M.A., Trinity College.

2. The deterioration of fabric under the action of Light and its physical explanation. By F. W. Aston, M.A., Trinity College.

3. Note on Constant Volume Explosion Experiments. By S. Lees, M.A., St John's College.

4. On the function $[x]$. By Viggo Brun. (Communicated by Professor G. H. Hardy.)

February 7, 1921.

In the Botany School.

PROFESSOR SEWARD, PRESIDENT, IN THE CHAIR.

The following was re-elected a Fellow of the Society:

E. H. Hankin, Sc.D., St John's College.

The following were elected Associates of the Society:

E. N. Hewitt, Trinity College.
W. Schlundt.

The following Communications were made to the Society:

1. The Development of Photosynthetic Activity during Germination. By G. E. BRIGGS, M.A., St John's College. (Communicated by Professor Seward.)

2. A theorem concerning summable series. By Professor G. H. HARDY.

3. Vectors and Tensors. By E. A. MILNE, M.A., Trinity College.

4. (*a*) Standing waves parallel to a plane beach.

(*b*) A kinetic theory of the Universe. By H. C. POCKLINGTON, M.A., St John's College.

5. (*a*) A configuration in four dimensions.

(*b*) The representation of a cubic surface on a quadric surface.

(*c*) On Delaunay's method in planetary theory.

(*d*) A periodic motion in dynamics. By Professor H. F. BAKER.

February 21, 1921.

In the Anatomy School.

PROFESSOR SEWARD, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

G. S. Adair, B.A., King's College.
H. S. Carslaw, Sc.D., Emmanuel College.
G. N. Nicklin, M.A., St John's College.
D. F. W. Scanlan, B.A., Jesus College.
J. T. Wilson, M.A., St John's College, Professor of Anatomy.

The following was elected an Associate of the Society:

B. N. Banerji, Clare College.

The following Communication was made to the Society:

The present position of the Helmholtz theory of hearing. By W. HARTRIDGE, M.D., King's College.

February 28, 1921.

In the Cavendish Laboratory.

Prof. Sir Ernest Rutherford, Vice-President, in the Chair.

The following were elected Fellows of the Society:

G. E. Briggs, M.A., St John's College.
H. T. H. Piaggio, Sc.D., St John's College.

The following Communications were made to the Society:

1. On the nature of crystal-reflection of X-rays. By Professor Sir Joseph Larmor.

2. An experiment on focal lines formed by a Zone plate. By G. F. C. Searle, Sc.D., Peterhouse.

3. The origin of the disturbances in the initial motion of a shell. By R. H. Fowler, M.A., Trinity College, and C. N. H. Lock, B.A., Gonville and Caius College.

4. On the latent heats of Vaporisation. By E. K. Rideal, M.A., Trinity Hall.

March 7, 1921.

In the Comparative Anatomy Lecture Room.

Professor Seward, President, in the Chair.

The following was elected a Fellow of the Society:

R. E. Holthum, B.A., St John's College.

The following was elected an Associate of the Society:

A. Harrison White.

The following Communications were made to the Society:

1. A peculiar case of heredity in the Sweet Pea. By Professor Punnett.

2. (*a*) Insect Oases.

(*b*) Wing Teratologies in Diptera. By C. G. Lamb, M.A., Clare College.

3. Some Alcyonaria in the Cambridge Museum. By S. J. Hickson, M.A., Downing College. (Communicated by Mr H. H. Brindley.)

4. The Mechanism of Ciliary Movement. By J. Gray, M.A., King's College.

5. The influence of function on the conformation of bones. By A. B. Appleton, M.A., Downing College.

6. A Note on the Hydrogen Ion Concentration of some Natural Waters. By J. T. Saunders, M.A., Christ's College.

7. Animal oecology in deserts. By P. A. Buxton, M.A., Trinity College.

8. The Biology of the Crown Gall Fungus of Lucerne. By J. Line, M.A., Emmanuel College.

May 2, 1921.

In the Cavendish Laboratory.

Professor Seward, President, in the Chair.

The following Communications were made to the Society:

1. On active molecules in physical and chemical reactions. By E. K. Rideal, M.A., Trinity Hall.

2. (*a*) An experiment which favours the resonance theory of hearing.
(*b*) A criticism of Wrightson's theory of hearing.
(*c*) A method of projecting interference bands.
(*d*) A method of projecting absorption spectra.
(*e*) The shift of absorption bands with change of temperature. By H. Hartridge, M.D., King's College.

3. The cooling of a solid sphere with a concentric core of a different material. By H. S. Carslaw, Sc.D., Emmanuel College.

4. An alignment chart for thermodynamical problems. By C. R. G. Cosens, B.A. (Communicated by Professor Inglis.)

5. Symbolical methods in the theory of Conduction of Heat. By T. J. I'a. Bromwich, Sc.D., St John's College.

6. On a property of focal conics and of bicircular quartics. By C. V. Hanumanta-Rao, M.A., Trinity College. (Communicated by Professor H. F. Baker.)

May 16, 1921.

In the Comparative Anatomy Lecture Room.

Professor Seward, President, in the Chair.

The following was elected an Associate of the Society:

A. Pratt, Magdalene College.

The following Communications were made to the Society:

1. The soaring flight of Dragon-flies. By E. H. Hankin, Sc.D., St John's College.

2. An unusual type of secondary male characters in the Diptera. By C. G. Lamb, M.A., Clare College.

3. A note on the mouth-parts of certain Decapod Crustaceans By L. A. Borradaile, M.A., Selwyn College.

4. Hermaphrodite Sea-urchins. By J. Gray, M.A., King's College.

5. (*a*) Preliminary note on the development of muscle, bone and body-weight in Sheep.

(*b*) On the alleged Inheritance of an acquired Character in Man.

(*c*) On the so-called Gluteus Maximus of *Tarsius*. By A. B. Appleton, M.A., Downing College.

6. On the effect of a magnetic field on the intensity of spectrum lines (second paper). By H. P. Waran, M.A., Christ's College. (Communicated by Professor Sir Ernest Rutherford.)

7. The theoretical value of Sutherland's constant in the kinetic theory of Gases. By C. G. F. James, Trinity College. (Communicated by Mr R. H. Fowler.)

8. Orthogonal Systems and the moving trihedral. By T. S. Yang. (Communicated by Professor H. F. Baker.)

INDEX TO THE PROCEEDINGS

with references to the Transactions.

Acetate, Cellulose (Fenton and Berry), 16.
Adair, G. S., Elected Fellow 1921, February 21, 488.
Adam, N. K., Elected Fellow 1920, May 3, 216.
Adrian, E. D., Elected Fellow 1921, January 24, 487.
Alcyonaria in the Cambridge Museum (Hickson), 366.
Algebraic numbers as a sum of four squares (Mordell), 250.
Anode Rays, Nature of the carriers of the (Thomson), 210.
Aphides, Cynipid hyperparasite of (Haviland), 235.
Aphis (*Myzus ribis*, Linn.), Antennal variation in an (Haviland), 35.
Appleton, A. B., Elected Fellow 1920, November 22, 486.
—— The gluteal region of *Tarsius spectrum*, 466.
—— The influence of function on the conformation of bones, 374.
Appleton, E. V., A method of testing Triode Vacuum Tubes, 239.
Aston, F. W., The Mass Spectra of the Chemical Elements, 215.
—— The atomic nature of matter in the light of modern physics, 217.
—— The deterioration of fabric under the action of Light and its physical explanation, 487.
Atkinson, R. H., Elected Fellow 1920, November 22, 486.
Axes, Rectangular, Transformations of (Baker), 147.

Baker, H. F., On the Hart circle of a spherical triangle, 116.
—— On a property of focal conics and of bicircular quartics, 122.
—— On the construction of the ninth point of intersection of two plane cubic curves of which eight points are given, 131.
—— On a proof of the theorem of a double six of lines by projection from four dimensions, 133.
—— On transformations with an absolute quadric, 145.
—— On a set of transformations of rectangular axes, 147.
—— On the reduction of homography to movement in three dimensions, 158.
—— On the transformation of the equations of electrodynamics in the Maxwell and in the Einstein forms, 166.
—— On the stability of periodic motions in general dynamics, 181.
—— On the stability of rotating liquid ellipsoids, 190.
—— On the general theory of the stability of rotating masses of liquids, 198.
Baker, H. F., and Hanumanta, C. V., On the generation of sets of four tetrahedra of which any two are mutually inscribed, 155.
Banerji, A. C., Elected Fellow 1919, October 27, 212.
Banerji, B. N., Elected Associate 1921, February 21, 488.

Barker, M., Elected Associate 1921, January 24, 487.
Belz, M. H., Elected Associate 1920, November 8, 486.
Bennett, G. T., The Rotation of the Non-Spinning Gyrostat, 70.
Bennett, G. T., and Greenhill, G., The Rotation of the Non-Spinning Gyrostat, 243.
Berry, A. J., and Fenton, H. J. H., Studies on Cellulose Acetate, 16.
Berwick, W. E. H., Elected Fellow 1919, October 27, 212.
—— Quintic transformations and singular invariants, 217.
Bhansali, M. D., Elected Fellow 1920, February 9, 214.
Bicircular quartics (Baker), 122.
Bicircular quartics, Property of (Hanumanta Rao), 434.
Bieler, E. S., Elected Associate 1920, October 25, 485.
Bifilar method of measuring the rigidity of wires (Searle), 61.
Bones, Influence of function on the conformation of (Appleton), 374.
Borradaile, L. A., A note of the mouth-parts of certain decapod crustaceans, 478.
Braid, K. W., Elected Associate 1920, May 3, 216.
Briggs, G. E., Elected Fellow 1921, February 28, 489.
—— The Development of Photosynthetic Activity during Germination, 488.
Brindley, H. H., Further notes on the food plants of the Common Earwig (*Forficula auricularia*), 50.
Bristol Channel, Tides in the (Taylor), 320.
Bromwich, T. J. I'A., Symbolical methods in the theory of Conduction of Heat, 411.
Browne, T. A., Elected Associate 1919, October 27, 213.
Brun, V., On the function $[x]$, 299.
Bryan, G. H., Hopkins Prize award 1906–09, 217.
Burnside, W., On the representation of the simple group of order 660 as a group of linear substitutions on 5 symbols, 247.
—— Convex Solids in Higher Space, 437.
—— On Cyclical Octosection. *See Transactions*, xxii.
—— On certain simply-transitive permutation-groups, 482.
—— Hopkins Prize award 1903–06, 217.
Burtt-Davy, J., Elected Associate 1921, January 24, 487.
Buxton, P. A., Elected Fellow 1920, May 3, 216.
—— Animal oecology in deserts, 388.

Carslaw, H. S., Elected Fellow 1921, February 21, 488.
—— The Cooling of a Solid Sphere with a Concentric Core of a Different Material, 399.
Cat, Superior vena cava of the (Lanchester and Thacker), 228.
Cellulose Acetate (Fenton and Berry), 16.
Chadwick, J., Elected Associate 1919, October 27, 213.
Cherry, T. M., Elected Associate 1920, October 25, 485.
Ciliary movement, mechanism of (Gray), 352.
Compton, A. H., Elected Associate 1920, January 26, 214.

Conduction of Heat, Symbolical methods in the theory of (Bromwich), 411.
Conics, focal, Property of (Baker), 122.
Convex Solids in Higher Space (Burnside), 437.
Cooling of a Solid Sphere with a Concentric Core of a Different Material (Carslaw), 399.
Cottingham, E. T., *see* Eddington, A. S.
Craies, L. V., Elected Associate 1919, October 27, 213.
Cramér, H., On the distribution of primes, 272.
Crown-gall fungus of Lucerne, Biology of (Line), 360.
Cubic curves, Construction of the ninth point of intersection of two plane (Baker), 131.
Cunningham, E., Elected Fellow 1920, February 23, 215.
Cynipid hyperparasite of Aphides (Haviland), 235.

Darwin, C. G., Lagrangian Methods for High Speed Motion, 56.
Decapod crustaceans, Mouth parts of certain (Borradaile), 478.
Deserts, Animal oecology in (Buxton), 388.
Dewing, E. S., Elected Fellow 1920, November 22, 486.
Diptera, Type of male secondary characters in the (Lamb), 475.
Diptera, Venational abnormalities in the (Lamb), 393.
Double six of lines by projection from four dimensions (Baker), 133.
Dragon-flies, Soaring flight of (Hankin), 460.

Earwig (*Forficula auricularia*), Food plants of (Brindley), 50.
Eddington, A. S., and Cottingham, E. T., Photographs of a Solar Prominence taken during the eclipse of 1919 May 29, 214.
—— The theory of relativity and recent eclipse observations, 214.
Electrodynamics, Transformation of the equations of, in the Maxwell and in the Einstein forms (Baker), 166.
Electrons, Velocity of X-ray (Whiddington), 442.
Electroscope, Shadow (Whiddington), 109.
Ellipsoids, Stability of rotating liquid (Baker), 190.
Ellis, C. D., Elected Fellow 1920, November 22, 486.
Evans, U. R., Elected Fellow 1920, November 22, 486.
Explosion experiments (Lees), 285.

Fenton, H. J. H., and Berry, A. J., Studies on Cellulose Acetate, 16.
Flight, Soaring, Problem of (Hankin and Page), 219.
Flight, Soaring, of Dragon-flies (Hankin), 460.
Fluids, Rotating (Taylor), 326.
Focal Conics, Property of (Hanumanta Rao), 434.
Focal lines formed by a zone plate (Searle), 330.
Food plants of the Common Earwig (*Forficula auricularia*) (Brindley), 50.
Fowler, R. H., and Lock, C. N. H., The Origin of the Disturbances in the Initial Motion of a Shell, 311.
Function $[x]$ (Brun), 299.

Functions, Ramanujan's trigonometrical function $c_q(n)$, and certain series of arithmetical functions (Hardy), 263.
Fungus, Crown-gall, of Lucerne (Line), 360.

Gaussian Series of Six Elements (Rogers), 257.
Gluteal region of *Tarsius spectrum* (Appleton), 466.
Grating, Plane diffraction (Searle), 88.
Gray, J., The mechanism of ciliary movement, 352.
—— Hermaphrodite Sea-urchins, 481.
Greaves, W. M. H., Elected Fellow 1920, May 17, 217.
Greenhill, G., and Bennett, G. T., The Rotation of the Non-Spinning Gyrostat, 243.
Groups, Representation of the simple group of order 660 as a group of linear substitutions on 5 symbols (Burnside), 247.
Gyrostat, Non-Spinning, Rotation of (Bennett), 70.
Gyrostat, Rotation of the Non-Spinning (Greenhill and Bennett), 243.

Hankin, E. H., Elected Fellow 1921, February 7, 488.
—— The soaring flight of Dragon-flies, 460.
—— The problem of soaring flight, 219.
Hanumanta, C. V., On a property of focal conics and of bicircular quartics, 434.
Hanumanta, C. V., and Baker, H. F., On the generation of sets of four tetrahedra of which any two are mutually inscribed, 155.
Harding, L., Elected Associate 1920, October 25, 485.
Hardy, G. H., Extension of a theorem of Mr Pólya (Landau), 14.
—— Sur le principe de Phragmén-Lindelöf (Riesz), 208.
—— Note on Ramanujan's trigonometrical function $c_q(n)$, and certain series of arithmetrical functions, 263.
—— A theorem concerning summable series, 304.
Harris, J. E. G., Elected Fellow 1920, November 22, 486.
Harrison, W. J., Notes on the Theory of Vibrations, 83.
—— The Hydrodynamical theory of the Lubrication of a Cylindrical Bearing under Variable Load, and of a Pivot Bearing. *See Transactions*, xxii.
—— On the Stability of the steady motion of viscous liquid contained between two rotating coaxal circular cylinders, 455. *See also Transactions*, xxii.
—— *see* Tamaki, K.
Hart circle of a spherical triangle (Baker), 116.
Hartridge, H., The present position of the Helmholtz theory of hearing, 488.
—— An experiment which favours the resonance theory of hearing, 490.
—— A criticism of Wrightson's theory of hearing, 490.
—— A method of projecting interference bands, 490.
—— The shift of absorption bands with change of temperature, 490.
—— An apparatus for projecting spectra, 480.
Haviland, Maud D., Elected Associate 1919, November 24, 214.
—— Preliminary note on antennal variation in an aphis (*Myzus ribis*, Linn.), 35.

Haviland, Maud D., Preliminary note on a cynipid hyperparasite of Aphides, 235.
Henderson, H., Elected Associate 1919, October 27, 213.
Henry, D. C., Elected Fellow 1920, February 9, 214.
Hermaphrodite Sea-urchins (Gray), 481.
Hewitt, E. N., Elected Associate 1921, February 7, 488.
Hickson, S. J., On some Alcyonaria in the Cambridge Museum, 366.
Holthum, R. E., Elected Fellow 1921, March 7, 489.
Homography, Reduction of, to movement in three dimensions (Baker), 158.
Hopkins Prize awards, 1903–1912, 217.
Humphries, J., Elected Associate 1920, November 22, 487.
Hurst, W. W., Elected Associate 1920, October 25, 485.

Infinite integrals (Pollard), 1.
Infinite series (Pollard), 1.
Insect oases (Lamb), 347.
Intensity of spectrum lines (Waran), 45, 428.

James, C. G. F., Theoretical Value of Sutherland's Constant in the Kinetic Theory of Gases, 447.
Jones, B. M., Elected Fellow 1919, October 27, 212.
Jones, I., Elected Associate 1920, October 25, 485.

Keys, D. A., Elected Associate 1921, January 24, 487.
Kienast, A., Proof of the equivalence of different mean values, 74.
Kikuchi, T., Elected Associate 1920, January 26, 214.
Kinetic Theory of Gases, Theoretical Value of Sutherland's Constant in the (James), 447.

Laboratory Valve method for determining the Specific Inductive Capacities of Liquids (Whiddington), 445.
Lagrangian Methods for High Speed Motion (Darwin), 56.
Lamb, C. G., Insect Oases, 347.
—— Venational abnormalities in the Diptera, 393.
—— An unusual type of male secondary characters in the Diptera, 475.
Lanchester, W. F., and Thacker, A. G., Preliminary note on the superior vena cava of the Cat, 228.
Landau, E., Note on Mr Hardy's extension of a theorem of Mr Pólya, 14.
Larmor, J., On the nature of crystal-reflection of X-rays, 489.
Latent Heats of Vaporisation (Rideal), 291.
Lees, S., Note on constant volume explosion experiments, 285.
Line, J., Elected Fellow 1919, November 24, 213.
—— A note on the biology of the Crown-gall fungus of Lucerne, 360.
Liquid, Stability of rotating masses of (Baker), 198.

Liquids, Specific Inductive Capacities of, Laboratory Valve method for determining the (Whiddington), 445.
Lock, C. N. H., and Fowler, R. H., The Origin of the Disturbances in the Initial Motion of a Shell, 311.
Lowry, T. M., Elected Fellow 1920, October 25, 485.
Lucerne, Biology of the Crown-gall fungus of (Line), 360.
Ludlam, E. B., Elected Associate 1920, October 25, 485.

MacAlister, D. A., Elected Associate 1919, November 10, 213.
MacMahon, P. A., Note on the parity of the number which enumerates the partitions of a number, 281.
—— Congruences with respect to composite moduli. *See Transactions*, xxii.
Magnetic field, Effect of, on the Intensity of spectrum lines (Waran), 45, 428.
Malcolm, L. W. G., Elected Associate 1919, November 24, 214.
Male secondary characters in the Diptera (Lamb), 475.
Mean values, Equivalence of different (Kienast), 74.
Menzies, A. C. G., Elected Associate 1920, November 22, 487.
Milne, E. A., Elected Fellow 1920, January 26, 214.
—— The Tensor Form of the Equations of Viscous Motion, 344.
Molin, K., An examination of Searle's method for determining the viscosity of very viscous liquids, 23.
Mordell, L. J., Elected Fellow 1920, November 22, 486.
—— On the representation of algebraic numbers as a sum of four squares, 250, 484.
Motion, High Speed, Lagrangian methods for (Darwin), 56.
Mouth-parts of certain decapod crustaceans (Borradaile), 478.
Munro, A., Elected Fellow 1921, January 24, 487.
Murray, E. G. D., Elected Fellow 1921, January 24, 487.
Myzus ribis, Linn., Antennal variation in (Haviland), 35.

Newman, L. F., Elected Fellow 1919, November 24, 213.
Nicklin, G. N., Elected Fellow 1921, February 21, 488.
Number, Parity of the, which enumerates the partitions of a number (MacMahon), 281.

Octosection, Cyclical, *see* Burnside.
Oecology, Animal, in Deserts (Buxton), 388.

Page, F. Handley, The problem of soaring flight, 219.
Parity of the number which enumerates the partitions of a number (MacMahon), 281.
Periodic motions in general dynamics (Baker), 181.
Permutation-groups, certain simply-transitive (Burnside), 482.
Phragmén-Lindelöf, Principe de (Riesz), 205.
Piaggio, H. T. H., Elected Fellow 1921, February 28, 489.
Plane diffraction grating (Searle), 88.

Pocklington, H. C., Standing waves parallel to a plane beach, 308.
Pollard, S., Elected Fellow 1920, October 25, 485.
—— On the term by term integration of an infinite series over an infinite range and the inversion of the order of integration in repeated infinite integrals, 1.
Pólya, Mr Hardy's extension of a theorem of (Landau), 14.
Potts, F. A., A note on vital staining, 231.
Pratt, A., Elected Associate 1921, May 16, 490.
Primes, Distribution of (Cramér), 272.
Proceedings at the Meetings held during the Session 1919–1920, 212
„ „ „ „ „ „ 1920–1921, 485.
Projecting spectra, Apparatus for (Hartridge), 480.
Prosad, K., Elected Fellow 1919, October 27, 212.
Punnett, R. C., A peculiar case of heredity in the Sweet Pea, 489.

Quadric, absolute, Transformations with (Baker), 145.
Quartics, Bicircular (Baker), 122.

Ramanujan's trigonometrical function $c_q(n)$, and certain series of arithmetical functions (Hardy), 263.
Rao, C. V. Hanumanta, *see* Hanumanta.
Rectangular axes, Transformations of (Baker), 147.
Richmond, H. W., Note on the Diophantine equation $t^3+x^3+y^3+z^3=0$. *See* *Transactions*, xxii.
Rideal, E. K., Elected Fellow 1920, November 22, 486.
—— On the Latent Heats of Vaporisation, 291.
Riesz, M., Sur le principe de Phragmén-Lindelöf, 205.
Ritchie, A. D., Elected Fellow 1919, November 24, 213.
Roberts, J. K., Elected Associate 1920, October 25, 485.
Rogers, L. J., On a Gaussian Series of Six Elements, 257.
Rotating Fluids (Taylor), 326.
Rotating liquid ellipsoids (Baker), 190.
Rotating masses of liquid, Stability of (Baker), 198.
Rotation of the Non-Spinning Gyrostat (Bennett), 70.
Rotation of the Non-Spinning Gyrostat (Greenhill and Bennett), 243.
Rutherford, E., The Inner Structure of Atoms, 486.

Saunders, J. T., A note on the hydrogen ion concentration of some natural waters, 350.
Savoor, S. R. U., On rotating liquid cylinders, 216.
Scanlan, D. F. W., Elected Associate 1920, January 26, 214.
—— Elected Fellow 1921, February 21, 488.
Schlundt, W., Elected Associate 1921, February 7, 488.
Searle, G. F. C., A bifilar method of measuring the rigidity of wires, 61.
—— Experiments with a plane diffraction grating, 88.
—— Experiments on focal lines formed by a zone plate, 330.

Searle, G. F. C. *See* Molin, K.
Sea-urchins, Hermaphrodite (Gray), 481.
Seth, J. B., Elected Fellow 1919, November 24, 213.
Seward, A. C., The origin of the vegetation of the land, 215.
Shah, N. M., Elected Fellow 1920, May 3, 216.
Shearer, G., Elected Associate 1920, January 26, 214.
Shell, Disturbances in the Initial Motion of a (Fowler and Lock), 311.
Shimizu, T., Elected Associate 1920, May 17, 217.
Slater, F. P., Elected Associate 1919, October 27, 213.
Smart, W. M., Elected Fellow 1920, May 17, 217.
Solid Sphere, Cooling of a, with a Concentric Core of a Different Material (Carslaw), 399.
Specific Inductive Capacities of Liquids, Laboratory Valve method for determining the (Whiddington), 445.
Spectra, Projecting, Apparatus for (Hartridge), 480.
Spectrum lines, Effect of Magnetic field on the Intensity of (Waran), 45, 428.
Spherical triangle, Hart circle of a (Baker), 116.
Staining, Vital (Potts), 231
Staples-Browne, R. C., Elected Fellow 1920, November 22, 486.
Summable series, Theorem concerning (Hardy), 304.
Sutherland's Constant in the Kinetic Theory of Gases, Theoretical Value of (James), 447.
Swain, L. M., Elected Associate 1920, November 22, 487.
Symbolical methods in the theory of Conduction of Heat (Bromwich), 411.

Tamaki, K., and Harrison, W. J., On the stability of the steady motion of viscous liquid contained between two rotating coaxal circular cylinders, 455. *See also Transactions*, xxii.
Tarsius spectrum, Gluteal region of (Appleton), 466.
Taylor, G. I., Elected Fellow 1919, November 24, 213.
—— Tides in the Bristol Channel, 320.
—— Experiments with Rotating Fluids, 326.
Tensor Form of the Equations of Viscous Motion (Milne), 344.
Tetrahedra, Generation of sets of four (Hanumanta and Baker), 155.
Thacker, A. G., and Lanchester, W. F., Preliminary note on the superior vena cava of the Cat, 228.
Theorem of Pólya (Landau), 14.
Thomas, T., Elected Fellow 1920, March 8, 215.
Thomson, G. P., Elected Fellow 1919, November 24, 213.
—— A note on the nature of the carriers of the Anode Rays, 210.
Tides in the Bristol Channel (Taylor), 320.
Transformations of rectangular axes (Baker), 147.
Transformations with an absolute quadric (Baker), 145.
Triode Vacuum Tubes (Appleton), 239.

Vacin, E. F., Elected Associate 1919, October 27, 213.

Vacuum Tubes, Triode (Appleton), 239.
Vaporisation, Latent Heats of (Rideal), 291.
Velocity of X-ray Electrons (Whiddington), 442.
Vena Cava of the Cat (Lanchester and Thacker), 228.
Venational abnormalities in the Diptera (Lamb), 393.
Vibrations, Theory of (Harrison), 83.
Vines, H. W. C., Elected Fellow 1921, January 24, 487.
Viscous liquid contained between two rotating coaxal circular cylinders, Stability of the steady motion of (Harrison), 455. *See also Transactions*, xxii.
Viscous liquids, Viscosity of, Searle's method (Molin), 23.
Viscous Motion, Equations of (Milne), 344.
Vital staining (Potts), 231.
Vyvyan, M. C., Elected Fellow 1920, February 23, 215.

Wadham, S. M., Elected Fellow 1919, November 24, 213.
Waran, H. P., On the effect of a magnetic field on the intensity of spectrum lines, 45, 428.
Waters, Hydrogen ion concentration of some natural (Saunders), 350.
Waves, Standing, parallel to a plane beach (Pocklington), 308.
Whiddington, R., The Shadow Electroscope, 109.
—— A Laboratory Valve method for determining the Specific Inductive Capacities of Liquids, 445.
—— Note on the Velocity of X-ray Electrons, 442.
White, A. H., Elected Associate 1921, March 7, 489.
White, F. P., Elected Fellow 1920, May 3, 216.
Wilson, B. M., Elected Fellow 1920, November 22, 486.
Wilson, C. T. R., Hopkins Prize award 1909–12, 217.
Wilson, J. T., Elected Fellow 1921, February 21, 488.
Wires, Bifilar method of measuring the rigidity of (Searle), 61.

X-ray Electrons, Velocity of (Whiddington), 442.

Zone plate, Focal lines formed by (Searle), 330.

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